NUMERICAL METHODS FOR THE DISPERSION EQUATION OF WAVES RIDING ON A STEADY CURRENT

by Xiping Yu

Waves riding on a steady current are categorized into P waves and S waves. The P waves are such kind that exist no matter whether the current is in presence or not. They are usually generated by oscillatory sources or by disturbances to so generated waves. The S waves on the other hand are the water surface response to the interaction of the current with a stationary or a steadily moving obstacle. Numerical methods for evaluating the wavenumber of both P waves and S waves for given values of the Airy number and the Froud number are described in details. Numerical solutions show that the S waves are usually very short waves. They become relatively influential to a phenomenon only when the Froud number is close to a certain value for a known frequency of oscillation of the water. A P wave is stretched by following currents and shrunk by opposing currents. If a preexisting current is strong enough, neither P waves nor S waves opposing the current can exist.

1. INTRODUCTION

It is well known that a steady current has appreciable effects on both the kinematics and dynamics of a wave riding on it. The kinematic effects include the shrink or stretch of the wavelength and the diffraction of the wave ray if the current is spatially non-uniform. The dynamic effects on the other hand are mainly represented by the change of the wave height. Although both the kinematic and dynamic effects of a current on waves had been observed since a long time ago, a full understanding to the mechanism of these phenomena has, however, not been satisfactorily achieved until Longuet-Higgins and Steward, who studied the change of the wavelength and the wave height in a spatially varying current.

The kinematic relation between the wavelength and the current velocity, which is a projection of the dispersion relation for waves riding on the current, was first investigated by Longuet-Higgin and Steward for deep-water waves. The shallow-water approximation of this relation can be found in an earlier paper by Burns. For waves in arbitrary water depth, the dispersion relation in the complete form was discussed in details by Jonsson et al. Jonsson et al. found that, different from the waves propagating over a still water, where the wavelength is uniquely determined by the water depth if the wave period is specified, waves on a current could appear locally in, at most, four different wavelengths for fixed water depth, current velocity and wave period. This multi-solution property of the dispersion equation for waves on a current was also shown by Peregrine and Jonsson. It was elucidated by Peregrine and Jonsson that the four possible solutions represent four waves with different physical properties. Although these waves may not always be equally important in practical problems, none of them can be proved to be generally less significant than the others. Nevertheless, incorporation of all these waves in a problem must be avoided whenever possible, because a complete formulation lead to great difficulties in obtaining the solution of an even very simplified problem. The conditions for a particular type of wave to be considerably significant or negligibly insignificant is thus of interest to coastal engineers. However, satisfactory studies of such con-

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Department of Civil Engineering
conditions does not seem to have been carried out.

The present study is to explore the characteristics of waves riding on a steady current and to present the numerical methods for evaluating the wavenumber. In the following sections, we first show the mathematical process from which the dispersion equation arises. Then we discuss the properties of the P waves and the S waves based on a graphic method for the dispersion equation. Numerical methods for both the P waves and S waves are studied in details. Finally, we show the numerical results of the wavenumber obtained under various conditions.

2. DISPERSION EQUATION FOR WAVES ON CURRENTS

Consider a long-crested wave train riding on a steady current with constant depth as shown in Fig. 1. For the mathematical description we employ a Cartesian-coordinate system with the xoy-plane horizontal on the still water level and the z-axis vertically upwards. The steady current is represented by its velocity vector \( \mathbf{U} = (U, V) \). By the conventional assumption that the fluid is inviscid and incompressible and its motion irrotational, we can define the velocity potential \( \phi = \phi(x, y, z, t) \) to depict the wave-current coexistent flow field. From the conservation of mass, the velocity potential is known to satisfy the following Laplace equation:

\[
\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = 0
\]

where \( \nabla = (\partial/\partial x, \partial/\partial y) \) is the horizontal gradient operator. On the free surface, the following kinematic and dynamic boundary conditions must hold:

\[
\frac{\partial \phi}{\partial t} + \nabla \phi \cdot \nabla \eta + \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \ z = \eta
\]

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \nabla \phi \cdot \nabla \phi + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] + g\eta = 0 \quad \text{at} \ z = \eta
\]

where \( g \) is the gravity acceleration and \( \eta \) is the free surface elevation measured from the still water level. At the impermeable bottom, the vertical velocity component vanishes. Therefore,

\[
\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \ z = -h
\]

where \( h \) (=const) is the still water depth.

We express the wave–current coexistent field by the superposition of a steady part and a purely oscillatory part, representing the effect of the current and the wave, respectively. The velocity potential \( \phi \) and the water surface elevation \( \eta \) can then be written as

\[
\phi = \mathbf{U} \cdot \mathbf{x} + \phi' \quad \text{(5)}
\]

\[
\eta = \Delta + \eta' \quad \text{(6)}
\]

where \( \mathbf{x} = (x, y) \), \( \Delta \) is the elevation of the mean water level and \( \phi' \) and \( \eta' \) denote the potential and the water surface elevation associated with the wave.

Inserting (5) and (6) into (1) to (4) and taking into consideration that \( U \) and \( \Delta \) are independent of \( x \), we have the following conditions to be approximately satisfied by a small amplitude wave:

\[
\nabla^2 \phi' + \frac{\partial^2 \phi'}{\partial z^2} = 0
\]

\[
\frac{D \phi'}{Dt} - \frac{\partial \phi'}{\partial z} = 0 \quad \text{at} \ z = \Delta
\]

\[
\frac{D \phi'}{Dt} + g\eta' = -g\left( \Delta + \frac{1}{2} \frac{U^2}{g} \right) \quad \text{at} \ z = \Delta
\]

\[
\frac{\partial \phi'}{\partial z} = 0 \quad \text{at} \ z = -h
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla
\]

denotes the rate of change observed in a reference moving with the current. The right hand side of (9) can be set to zero upon redefining the velocity potential \( \phi' \) without causing any change of the velocity field\(^8\). Anyhow, a combination of (8) and (9) gives

\[
\frac{D^2 \phi'}{Dt^2} + g \frac{\partial \phi'}{\partial z} = 0 \quad \text{at} \ z = \Delta
\]
We denote by $\omega$ the frequency of the water surface oscillation observed in the fixed coordinate, namely, the absolute or apparent frequency of the wave–current coexistent system, and by $\omega'$ the frequency of the water surface oscillation observed in a coordinate moving with the current, i.e., the relative or intrinsic frequency of the wave–current coexistent system. $\omega$ and $\omega'$ satisfy the following Doppler relation:

$$\omega' = \omega - k \cdot V$$

(1)

where $k$ is the wavenumber vector, which is invariant with the reference coordinate.

Consider a wave with the water surface elevation expressed by

$$\eta' = ae^{i(S - \omega t)}$$

(2)

where $a$ is the wave amplitude and $S = S(x, y)$ the spatial phase function. The spatial phase function is related to the wavenumber vector through

$$\nabla S = k$$

(3)

The velocity potential which represents this wave can be assumed to have the following form:

$$\phi' = -iFe^{i(S - \omega t)}$$

(4)

where $F$ is a real function of the wave amplitude. The inclusion of $-i$ in (4) is to account for the fixed phase shift between the velocity potential and the free surface oscillation. Substituting (4) and (3) into (7), (8), (3) and (2) we have

$$F = \frac{ga}{\omega - k \cdot U} \frac{\cosh k(h + x)}{\cosh kh'}$$

(5)

$$\omega^2 - (k \cdot U)^2 = gk \tanh kh'$$

(6)

where $k = |k|$ is the wavenumber and $h' = h + \Delta$ is the effective (mean) water depth. Eq. (6) is the dispersion equation for waves on a current.

3. P WAVE AND S WAVE

We introduce the following nondimensional parameters:

$$\xi = kh'$$

(7)

$$\delta = \frac{U \cdot I}{\sqrt{gh'}}$$

(8)

$$A = \omega \sqrt{\frac{h'}{g}}$$

(9)

where $\xi$ is the nondimensional wavenumber; $F$ is the Froude number, which represents the effect of the gravity in a steady flow with free surface and has been one of the most important parameters in open channel hydraulics; $A$ is called the Airy number, which is so named because of its important role in the linear wave theory; $I$ is the unit vector in the direction of the wavenumber vector $k$. With the nondimensional wavenumber, the Froude number and the Airy number introduced, $\delta$, the dispersion equation for waves on a current, can be written as

$$(A - F) = \xi \tanh \xi$$

(10)

When $F = 0$, (10) reduces to

$$A = \xi \tanh \xi$$

(11)

which is the dispersion equation for progressive waves over a still water. When $A = 0$, (11) becomes

$$F^2 = \frac{\tanh \xi}{\xi}$$

(12)

which is a relation satisfied by the stationary waves around a moving obstacle in still water and has been paid attention mainly by the naval engineers who are concerned with the ship waves.

There is no question that the nondimensional wavenumber $\xi$ can be known upon solving (12) for a given pair of $A$ and $F$. Even so, the solution method has not been well established because of the complex properties of this equation. Explicit solutions can only be obtained for extremely deep-water or shallow-water waves. In general, numerical methods are needed. Before any detailed discussion of the numerical methods, however, it is worthwhile to have a qualitative understanding of the solution. The graphic method described by Jonsson et al. 4) and Peregrine 7) can be used for this purpose. Following Peregrine 7), we define the following two functions:

$$m = m_1(\xi) = \pm \sqrt{\tanh \xi}$$

(13)

$$m = m_2(\xi) = A - F\xi$$

(14)

It is then obvious that the intersections of $m_1(\xi)$ and $m_2(\xi)$ on the $m - \xi$ plane give the solutions of $\xi$. As it is indicated in Fig. 2, there could have at most four
solutions of $\xi$ for a given pair of $A$ and $F$. We denote these solutions, in the ascending order of the value of $\xi$, by $S^-$, $P^-$, $P^+$ and $S^+$.

The $P$ waves are obviously such kind that exist no matter whether the current exists or not. However, they disappear if $A = 0$. This implies that the $P$ waves are generated by oscillatory sources or by disturbances to so generated waves. Therefore, they correspond to the conventional progressive waves we deal with in many coastal engineering context. $P^+$ is the wave following the current and $P^-$ that opposing the current. The $P^+$ wave and $P^-$ wave have the same wavenumber only when $F = 0$, i.e., when the current vanishes. Otherwise, the magnitude of $\xi$ of the $P^+$ wave is larger but that of the $P^-$ wave is smaller compared to what they should be when no current is in presence. This means that the following current has an effect to stretch the wave but the opposing current to shrink it.

The $S$ waves are the kind which can exist when $A = 0$, i.e., when there is no oscillation of the water and the phenomenon is completely steady. They disappear when $F = 0$. This indicates that the $S$ waves are essentially a phenomenon associated with the current. Under the special case with $A = 0$, these waves have been known as the stationary waves and have been discussed by many authors who are concerned with ship waves. When the current coexists with a wave, it becomes now obvious that the $S$ waves are no longer stationary. They propagate with a finite velocity. The wavenumber is also changed.

Fig. 2 also indicates the condition for each kind of wave to exist. The $P^+$ wave and the $S^+$ wave can always exist as long as $A$ is non-zero. When $A = 0$, the $P^+$ wave vanishes and the $S^+$ wave appears only if $F$ is less than unity. The existence of the opposing current $P^-$ wave and $S^-$ wave is subjected to the following condition:

$$F = -\frac{\tanh \xi + \xi \text{sech}^2 \xi}{2 \sqrt{\tanh \xi}}$$

$$A = \frac{\tanh \xi + \xi \text{sech}^2 \xi}{2 \sqrt{\tanh \xi}}$$

$\xi$ and $A$ give a monotonic relation between $A$ and $F$, which, for a given $F$ (or $A$), determines a critical value of $A$ (or $F$) beyond which the opposing current waves can not exist.

4. NUMERICAL METHODS FOR THE DISPERSION EQUATION

4.1 The Critical Condition

The critical condition for the opposing current waves to vanish can be obtained by eliminating $\xi$ from (27) and (28). For short waves, this condition can be approximated by

$$A = \frac{1}{4F}$$

and for long waves by

$$A = \frac{2 \sqrt{2}}{3} \left(1 - F\right)^{\frac{3}{2}}$$

To ensure the absolute error of $A$ or $F$ obtained by the above approximations to be less than $10^{-5}$ (this criterion is valid throughout the present section), $F$ should be limited to $0.2$ or $A \geq 1.25$ and $F$ to $0.998 \leq F \leq 1$ or $A \leq 0.0005$. In general, an explicit expression of the critical condition is difficult to derive and for a given value of $F$ or $A$, we need to apply Newton's iteration method to the due equation of (27) and (28) for the critical value of $\xi$ and to obtain the critical value of $A$ or $F$ by the other equation. The initial value of $\xi$ for the iteration may be reckoned from $27$ for $F \leq 0.5$ or $A \geq 0.5$ and from $28$ for $F > 0.5$ or $A < 0.5$. Fig. 3 shows the numerical results of the critical condition along with the short-wave and long-wave approximations.
The P+ wavenumber should be solved from
\[ m = \sqrt{\xi \tanh \xi} = A - F\xi \] (57)
with \( \xi \geq 0 \). For short waves \([A \geq 2.5(1 + 2.5F)]\) we have
\[ \xi = \frac{1 + 2AF - \sqrt{1 + 4AF}}{2F^2} \] (58)
while for long waves \([A \leq 0.005(1 + F)]\) we have
\[ \xi = \frac{A}{1 + F} \] (59)
For intermediate waves, we adopt the following iteration formula:
\[ \xi_{n+1} = \frac{A^2 - \xi_n \tanh \xi_n}{\tanh \xi_n + \xi_n \text{sech}^2 \xi_n} \]
\[ \xi_0 = \begin{cases} A^2 & (A < 1) \\ A & (A \geq 1) \end{cases} \] (60)
Fig. 4 shows the variation of the P+ wavenumber versus the Airy number for various values of the Froud number. The figure indicates that \( \xi \) decreases
as \( F \) increases, i.e., the following current has an effect to stretch the wave.

### 4.5 \( P^- \) Wave

The \( P^- \) wavenumber is determined by

\[
m = \sqrt{\xi} \tanh \xi = A - F\xi
\]

with \( \xi \leq 0 \). Equation (40) reduces to

\[
\xi = -\frac{1 - 2 AF + \sqrt{1 - 4 AF}}{2F^2}
\]

for short waves \([A \geq 2.5(1 - 2.5F)]\) and to

\[
\xi = -\frac{A}{1 - F}
\]

for long waves \([0.005(1 - F) \leq A \leq A_c]\). The intermediate wavenumber can be determined by the following iteration formula:

\[
\xi_{n+1} = \frac{2A\sqrt{\xi_n \tanh \xi_n - \xi_n \tanh \xi_n + \xi_n^2 sech^2 \xi_n}}{2F \sqrt{\xi_n \tanh \xi_n + \tanh \xi_n + \xi_n^2 sech^2 \xi_n}},
\]

\[A^2 = \xi_n \tanh \xi_n \quad (\xi_n \leq 0)
\]

### 4.6 \( S^+ \) Wave

The \( S^+ \) wavenumber satisfies

\[
m = -\sqrt{\xi} \tanh \xi = A - F\xi
\]

with \( \xi > 0 \). For short waves \([F \leq 0.4(1 + 0.4A)]\) it can be approximated by

\[
\xi = \frac{1 + 2AF + \sqrt{1 + 4AF}}{2F^2}
\]

and for long waves \([F \geq 1 + 200A]\) by

\[
\xi = \frac{A}{F - 1}
\]

For intermediate waves, it can be determined by the following iteration formula:

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**Fig. 5** Normalized \( P^+ \)-wavenumber versus Airy number and Froud number.

**Fig. 6** Normalized \( S^+ \)-wavenumber versus Froud number and Airy number.
The variation of the $S^+$ wavenumber versus the Froud number for various values of the Airy number is shown in Fig. 6. From this figure we note that the $S^+$ wavenumber is rather large when the current velocity is relatively small, or more precisely, when the Froud number is smaller than (but not very close to) unity. Such short waves are impossible to develop appreciable wave height and are usually insignificant in practice.

4.7 $S^-$ Wave

The $S^-$ wavenumber is governed by

$$m = \sqrt{\xi} \tan \xi = A - F^2$$

with $\xi > 0$. (69) can be approximated by

$$\xi = \frac{1 - 2AF - \sqrt{1 - 4AF}}{2F^2}$$

for short waves [$F \leq 0.4(1 - 0.4A)$] and by

for long waves [$(1 - 200A) \leq F \leq F_c$]. For intermediate waves, the following iteration formula is effective:

$$\xi_{n+1} = \frac{A - \sqrt{\xi_n} \tan \xi_n}{F}, \quad F^2 = \frac{\tan \frac{\xi_0}{\xi_0}}{\xi_0} (\xi_0 > 0)$$

The variation of the $S^-$ wavenumber versus the Froud number and the Airy number is shown in Fig. 7. It can be noted from this figure that relatively small wavenumber appears only when the Froud number is less than while close to unity and the Airy number is small. Under more general conditions, the $S^-$ wavenumber is rather large and the corresponding waves can usually be neglected in practice.

5. CONCLUSIONS

We have studied the kinematics of the waves riding on a steady current. The waves have been categorized into P waves, which exist no matter whether the current is in presence or not, and S waves, which are essentially the water surface response to stationary or steadily moving disturbance. Numerical methods for evaluating the wavenumber of both the P waves and S waves at given values of the Airy number and the Froud number were described in details. Numerical results showed that the S waves are usually of rather large wavenumber. They become relatively influential to a subcritical phenomenon only when the Froud number is close to unity while the Airy number is small. A P wave is stretched by following currents and shrunk by opposing currents. If a preexisting current is strong enough, neither P waves nor S waves opposing the current can exist.

REFERENCES

4) Jonsson, I. G., Skougaard, C. and Wang, J. D.,


