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SOME IMPORTANT PROPERTIES OF 
HIGH-ORDER FINITE-DIFFERENCE SCHEMES 
FOR LAPLACE EQUATION¹

by Xiping Yu²

An inherent contradiction in the finite-difference methods for the Laplace equation is demonstrated. The grid size in a practical computation must, on the one hand, be fine enough to ensure the numerical accuracy, but, on the other hand, be necessarily large to avoid an ill-conditioned difference equation system. Higher-order schemes are shown to be advantageous in terms of both the accuracy and the condition of the difference equation system.

Introduction

Development of high-order schemes for the Laplace equation has been very active. The "best" nine-point scheme for square grid elements, of the eighth-order accuracy, was obtained by Bickley (1948), Greenspan (1957), Kantorovich and Krylov (1958) and Fox (1962). Derivation of the relevant scheme for rectangular grid elements with arbitrary width-to-length ratio has also been paid attention since 1980s. Manohar and Stepheson (1982) presented a scheme of the sixth-order by assuming that the local solution of the Laplace equation can be approximated by a polynomial of degree four in both independent variables. To avoid the negative coefficients in Manohar and Stepheson's (1982) scheme that arise when the width-to-length ratio of grid element is large, Chwang and Chen (1987) proposed the "optimal" scheme, which is essentially identical to Manohar and Stepheson's (1982) scheme but adopts a lower-order algorithm as the width-to-length ratio of the grid elements becomes relatively large or small. With a rather different approach, Chen et al. (1980) obtained the so-called finite-analytic scheme. The finite-analytic scheme seems to be equivalent to a fourth-order scheme in general, but its coefficients are always positive.

It is well understood that the accuracy is an important but not the only criterion in evaluating a finite-difference scheme. Among the other important items for schemes of elliptic partial differential equations is the condition of the finite-difference equation system resulted from the application of the relevant scheme. In the present study, we examine some important properties of high-order schemes of the Laplace equation. Emphasis is paid on an inherent contradiction in pursuing a not only accurate but also well-conditioned scheme.

Typical Schemes

Consider the Laplace equation:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]  

(1)

We assume that the finite-difference scheme of interest is in general a nine-point one and correlates the values of the independent variable at the nine nodes of the rectangular grid element (Fig. 1) in the following manner:

\[
\phi_0 = a_n(\phi_{ne} + \phi_{we}) + a_v(\phi_{ne} + \phi_{se}) + \beta(\phi_{ne} + \phi_{se} + \phi_{sw} + \phi_{sw})
\]  

(2)

where \(a_n\), \(a_v\) and \(\beta\) are coefficients depending on the shape of the grid element.

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Fig. 1 Definition sketch of a grid element.

The five-point scheme (Lapidus and Pinder 1982) is probably the most classical finite-difference scheme for the Laplace equation. When represented by (2), this scheme possesses the following coefficients:

\[ a_H = \frac{-r^2}{2 + 2r^2} \]  \hspace{1cm} (3)
\[ a_V = \frac{1}{2 + 2r^2} \]  \hspace{1cm} (4)
\[ \beta = 0 \]  \hspace{1cm} (5)

where \( r = \frac{\Delta y}{\Delta x} \) is the width-to-length ratio of grid elements. Since \( \beta \) vanishes, the scheme involves only five nodes of the grid element. Variation of \( a_H \) and \( a_V \) of the five-point scheme versus \( r \) is demonstrated in Fig. 2. \( a_H \) increases while \( a_V \) decreases as \( r \) increases. At \( r = 1 \), that is, for square grid, \( a_H = a_V = 1/4 \).

Chen et al. (1980) showed that the finite-analytic method can also be employed to derive a finite-difference scheme for the Laplace equation. Following Chwang and Chen (1987), the coefficients of the finite-analytic scheme read

\[ a_H = \frac{1}{2} \frac{1}{\cosh \frac{\pi}{2r}} \]  \hspace{1cm} (6)
\[ a_V = \frac{1}{2} \frac{1}{\cosh \frac{\pi r}{2}} \]  \hspace{1cm} (7)
\[ \beta = \frac{1}{4} \left( 1 - \frac{1}{\cosh \frac{\pi}{2r}} - \frac{\pi r}{2} \right) \]  \hspace{1cm} (8)

At \( r = 1 \), \( a_H = a_V = 0.1992584 \) and \( \beta = 0.0507316 \). These values are very close to those of the “best” nine-point scheme mentioned in the following.

Fig. 3 demonstrates the variation of \( a_H \), \( a_V \) and \( \beta \) for the finite-analytic scheme versus \( r \). \( a_H \) is shown to increase while \( a_V \) decrease as \( r \) increases; \( \beta \), however, has a peak value at \( r = 1 \).

The “best” nine-point scheme with the sixth-order accuracy in general is represented by the following coefficients:

\[ a_H = \frac{10r^3 - 2}{20 + 20r^3} \]  \hspace{1cm} (9)
\[ a_V = \frac{10 - 2r^2}{20 + 20r^3} \]  \hspace{1cm} (10)
\[ \beta = \frac{1}{20} \]  \hspace{1cm} (11)

At \( r = 1 \), \( a_H = a_V = 0.2 \) and \( \beta = 0.05 \). Variation of \( a_H \), \( a_V \) and \( \beta \) for the “best” scheme versus \( r \) is shown in Fig. 4. It is noticed that the scheme has the numerically unpopular negative coefficients at relatively large or small values of \( r \) (\( r > 5 \) or \( r < 1/\sqrt{5} \)). This seems to be the cost of the possibly highest-order accuracy.

It should be pointed out that the above schemes for the Laplace equation all satisfies the following
Fig. 4 Coefficients of the "best" nine-point scheme.

relation:

\[ 2a_0 + 2a_v + 4\beta = 1 \]  

This implies that the value of the dependent variable at the central node of a grid element is a weighted average of its values at the ambient nodes. This seems to be a necessary condition of any scheme for the Laplace equation, because any constant should be automatically a solution.

**Condition of Finite-Difference Equation System**

Consider the finite-difference solution of a Dirichlet problem defined over a \( a \times a \) square domain. We discretize the domain into square grid elements with \( \Delta x = \Delta y = \alpha_a \), where \( \nu = 1/N \) and \( N \geq 2 \) is an integer. For square grid, we denote \( a_x = a_y = a \). Application of a particular scheme at each inner node of the domain then gives rise to the following difference equation system:

\[ A\phi = b \quad \text{(13)} \]

where

\[ A = \begin{bmatrix} A & E \\ E & A \end{bmatrix} \quad \text{(14)} \]

\[ A = \begin{bmatrix} 1 & a \\ a & 1 & a \\ & \ddots & \ddots & \ddots \\ & & a & 1 & a \\ & & & a & 1 \end{bmatrix}^{(N-1) \times (N-1)} \]

\[ E = \begin{bmatrix} a & \beta \\ \beta & a & \beta \\ & \ddots & \ddots & \ddots \\ & & \beta & a & \beta \\ & & & \beta & a \end{bmatrix}^{(N-1) \times (N-1)} \]

is a tri-diagonal block matrix with each block expressed by one of the following exact tri-diagonal matrices:

The unknown vector

\[ \phi = (\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,N-1}, \phi_{2,1}, \ldots, \phi_{N-1,N-1})^T \]

is formed by the nodal values of the dependent variable and the right hand side vector

\[ b = (b_{1,1}, b_{1,2}, \ldots, b_{1,N-1}, b_{2,1}, \ldots, b_{N-1,N-1})^T \]

is determined by the boundary condition of the problem.

Whether matrix \( A \) is ill-conditioned can be determined by the condition number \( \mu \), which, for a real and symmetric matrix, is expressed by (Young and Gregory 1988):

\[ \mu = \frac{|\lambda_{\text{max}}|}{|\lambda_{\text{min}}|} \quad \text{(19)} \]

where \( |\lambda_{\text{max}}| \) and \( |\lambda_{\text{min}}| \) are the eigenvalues of \( A \) with maximum and minimum absolute values, respectively. When \( \mu \) is very large, the matrix \( A \) and, consequently, the finite-difference method is ill-conditioned.

It is known that any eigenvalue of \( A \), denoted by \( \lambda \), satisfies

\[ A\phi = \lambda\phi \quad \text{(20)} \]

where

\[ \phi = (\phi_{1,1}, \phi_{1,2}, \ldots, \phi_{1,N-1}, \phi_{2,1}, \ldots, \phi_{N-1,N-1})^T \]

is the eigenvector of \( A \) corresponding to \( \lambda \). Obviously, (20) is equivalent to the following difference equation

\[ \phi_{i,j} - a(\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1}) - \beta(\phi_{i-1,j-1} + \phi_{i+1,j-1} + \phi_{i,j+1} + \phi_{i+1,j+1}) = \lambda\phi_{i,j} \quad \text{(22)} \]

\( (i, j = 1, 2, \ldots, N-1) \)
with $\phi_{N,j} = \phi_{i,j} = \phi_{i,0} = \phi_{i,N} = 0$. The solution of $\phi$ and $\lambda$ can then be represented by

$$
\phi_{m,n} = \sin im\pi \sin jn\pi \\
\lambda_{m,n} = 1 - 2a(\cos m\pi \nu + \cos n\pi \nu) - 4\beta \cos m\pi \cos n\pi \nu
$$

(23)

$$(m, n = 1, 2, \ldots, N - 1)$$

When the five-point scheme is applied, we have $\alpha = 1/4$ and $\beta = 0$. (24) then becomes

$$
\lambda_{m,n} = 1 - \frac{1}{2}(\cos m\pi \nu + \cos n\pi \nu)
$$

(25)

This implies that all the eigenvalues of $A$ are positive. Recalling that $1/N \leq \mu, \nu \leq 1 - 1/N$, we readily find

$$
|\lambda_{\max}| = \lambda^{(N-1,N-1)} = 1 + \cos \pi \nu
$$

(26)

$$
|\lambda_{\min}| = \lambda^{1,1} = 1 - \cos \pi \nu
$$

(27)

Therefore,

$$
\mu = \frac{1 + \cos \pi \nu}{1 - \cos \pi \nu}
$$

(28)

For very fine grid, i.e., at

$$
\nu = \frac{1}{N} \rightarrow 0
$$

(29)

leads to

$$
\mu \rightarrow \frac{4}{\pi \nu} \rightarrow \infty
$$

(30)

This means that the five-point finite-difference scheme is ill-conditioned if the grid adopted is fine. We should also note that the finer the grid, the worse the condition of the difference equation system.

Now behold the “best” nine-point scheme. For square grid, $a = 1/5$ and $\beta = 1/20$. (24), therefore, can be written as

$$
\lambda_{m,n} = 1 - \frac{2}{5}(\cos m\pi \nu + \cos n\pi \nu)
$$

$$
= \frac{1}{5}\cos m\pi \nu \cos n\pi \nu
$$

$$
= \frac{9}{5} - \frac{1}{5}(2 + \cos m\pi \nu)(2 + \cos n\pi \nu)
$$

(31)

Again, we note that $\lambda_{m,n}$ are positive for all possible values of $m$ and $n$. It can also be shown, from (31), that

$$
|\lambda_{\max}| = \lambda^{(N-1,N-1)} = 1 + \frac{4}{5}\cos \pi \nu - \frac{1}{5}\cos^2 \pi \nu
$$

(32)

$$
|\lambda_{\min}| = \lambda^{1,1} = 1 - \frac{4}{5}\cos \pi \nu - \frac{1}{5}\cos^2 \pi \nu
$$

(33)

Hence,

$$
\mu = \frac{5 + 4\cos \pi \nu - \cos^2 \pi \nu}{5 - 4\cos \pi \nu - \cos^2 \pi \nu}
$$

(34)

For very fine grid,

$$
\mu \rightarrow \frac{8}{3\pi \nu} \rightarrow \infty
$$

(35)

Similar to the five-point scheme, the “best” nine-point scheme is also ill-conditioned if the grid adopted is fine, and refining the grid leads to worse condition of the difference equation system. This property of the finite-difference equation system seems to be an inherent contradiction in the application of finite-difference method to the Laplace equation. On the one hand, the grid should be fine enough so that the truncation error of the scheme can be controlled. On the other hand, the grid should not be too fine so the system of the difference equation becomes ill-conditioned.

Comparing (35) with (30) we find that the condition number of the “best” nine-point scheme is only 2/3 of that of the five-point scheme when the grid is fine. This simply means that the “best” nine-point scheme is advantageous in terms of not only the accuracy but also the condition of the finite-difference equation system.

**Numerical Performance**

To examine the behavior of various finite-difference schemes for the Laplace equation, we apply them to the computation of a Dirichlet problem. The domain of the problem is the interior region bounded by $x = 0, y = 0, x = 1$ and $y = 1$. The boundary condition of the problem is given in accordance to

$$
\phi = e^{-\pi \nu} \cos \pi \nu
$$

(36)

Since $\phi$ satisfies the Laplace equation and the solution of a Dirichlet problem of the Laplace equation is unique, (36) must also express the analytic solution of the relevant problem. For numerical solutions, the domain is discretized into square
grid elements with $\Delta x = \Delta y = 1/N$, where $N$ is an integer. The overall estimation of the numerical error is given by

$$E = \frac{1}{N+1} \sqrt{\sum_{m=0}^{N} \sum_{n=0}^{N} (\phi_m,n - \bar{\phi}_m,n)^2}$$

(37)

Fig. 5 shows the relation between $E$ and $N$. We note from this figure that, for the five-point scheme and the "best" nine-point scheme, the overall numerical error decreases continuously as the grid refined. The accuracy of the "best" nine-point scheme, however, is much higher than that of the five-point scheme at any fixed grid size and, moreover, it increases much more rapidly as the grid size is reduced. In other words, to get a solution of the same accuracy, the five-point scheme needs a much finer grid than the "best" nine-point scheme. We also note that the finite-analytic scheme is comparable to the "best" nine-point scheme only when a relatively rough grid is under consideration. The overall numerical error of the finite-analytic scheme does not show a monotonically reducing tendency as $N$ increases. It decreases at a rate equivalent to or even more rapid than that of the "best" nine-point scheme as $N$ increases up to a certain value (25 in the present case), but it decreases at a rate equivalent to that of the five-point scheme as $N$ becomes large. Since it is understood that the accuracy of the finite-analytic scheme depends on the representation of the variation of the dependent variable along the boundary of grid element, we reckon that the formula for this representation adopted by Chwang and Chen (1987) is not necessarily good for a problem with fine grid.

Fig. 6 shows the relation of the condition number in the above application of various schemes. Just as a confirmation to the analysis in the previous section, the five-point scheme is shown to be disadvantageous in terms of not only the accuracy but also the condition of difference equation system.

**Conclusions**

We have studied the finite-difference schemes, including the classical five-point scheme, the finite-analytic scheme, and the "best" nine-point scheme, for numerical solution of the Laplace equation. The schemes, based on fairly different considerations, were all formally represented by a formula that relates the nodal values of the dependent variable in a grid element. The coefficients of the schemes were expressed by different functions of the width-to-length ratio of the grid element. We demonstrated that there is an inherent contradiction in the finite-difference methods for the La-
place equation. The grid size in an application must, on the one hand, be fine enough to ensure the numerical accuracy, but, on the other hand, be necessarily large to avoid an ill-conditioned difference equation system. Higher-order schemes are shown to be advantageous in terms of both the accuracy and the condition of the difference equation system.

**References**


