METHOD OF CAUCHY’S POLYGONS AND
PARTIAL DIFFERENTIAL EQUATIONS I

By

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Introduction

Professor M. Hukuhara* has recently shown the possibility of applying the well-known method of Cauchy’s polygons which constitutes a powerful tool for solving ordinary differential equations to the proof of the existence of solutions for some partial differential equations.

The purpose of this note is to give an outline of such possibility, generalizing slightly Prof. Hukuhara’s result; namely, we shall consider the first boundary value problem for a parabolic partial differential equation of the form

\[
\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left[ p(x) \frac{\partial u}{\partial x} \right] + q(x)
\]

and apply the method of polygons to prove the existence of the solution in the rectangular domain

\[ a \leq x \leq b, \quad 0 \leq t \leq T \]

satisfying the boundary conditions

\[
\begin{align*}
|t=0: & \quad u|_{t=0} = 0, \quad u|_{x=a} = u|_{x=b} = 0. \\
|u|_{x=0:} & \quad u|_{x=a} = u|_{x=b} = 0.
\end{align*}
\]

Following Prof. Hukuhara’s idea we firstly interpret the equation (A) as a differential equation in the Banach space of continuous functions and secondly construct Cauchy’s polygons in an appropriate and natural manner. The last step is to select from thus constructed family of polygons a uniformly convergent subsequence, the limit of which is to be the desired solution.

In our argument we shall make frequent use of some results in the

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theory of ordinary differential equations; in particular, some inequalities will play a basic role.

Though our consideration is restricted to a simple case it seems to us that our point of view will still be available for a wider class of partial differential equations such as non-linear equations in two independent variables and linear or non-linear equations in more than two independent variables. We shall investigate those general cases in our forthcoming papers.

1. Cauchy’s Polygons

We rewrite the equation (A) in the form

\[ (A') \quad D_t u = (pu')' + q \]

and interpret it as a differential equation for functions \( u \) in the Banach space \( C[a, b] \) of continuous functions. The prime indicates the differentiation with respect to \( x \). \( u \) is also a function of \( t \) and the differentiation with respect to \( t \) is denoted by \( D_t \). Let us suppose further that \( p \in C^1[a, b], q \in C^2[a, b] \) and \( \min_{x \in [a, b]} p(x) > 0 \).

In order to construct Cauchy’s polygons we divide the time interval \( 0 \leq t \leq T \) into \( n \) equal subintervals:

\[ (A_n) \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T. \]

Setting \( u_0 = 0 \in C^2[a, b] \) we propose to define \( u_1, \ldots, u_n \) by the formulas

\[
\begin{align*}
(1.1) \quad u_{j+1} &= u_j + (t_{j+1} - t_j) f_j, \\
(1.2) \quad f_j &= (pu_j)' + q, \\
(1.3) \quad u_{j+1}|_{x=a} = u_{j+1}|_{x=b} = 0, \quad j = 0, 1, \ldots, n-1. 
\end{align*}
\]

Such a mechanism of determination of \( u_j \) is clearly seen not to be absurd under the hypothesis that \( \lambda = 0 \) is not an eigenvalue for the boundary value problem

\[ \tau \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) - u = 0, \quad u|_{x=a} = u|_{x=b} = 0, \]

where \( \tau \) is any positive constant; for, (1.1), (1.2), (1.3) form a group of boundary value problems for ordinary differential equations of the second order and \( u_1, \ldots, u_n \) can be determined as elements of \( C^2[a, b] \).

Let us set for \( t \in [t_j, t_{j+1}], j = 0, 1, \ldots, n-1 \)
(1.4) \[ \varphi_n(t) = \frac{t-t_j}{t_{j+1}-t_j} u_{j+1} + \frac{t_{j+1}-t}{t_{j+1}-t_j} u_j = \frac{t-t_j}{\tau} u_{j+1} + \frac{t_{j+1}-t}{\tau} u_j \]

where \( \tau \) denotes the common length of subintervals in the division \( (\Delta_n) \).

\( \varphi_n(t) \) whose values are in \( C[a, b] \) as a function of \( t \) can be regarded as a polygon in the space \([0, T] \times C[a, b]\) joining the vertices \((t_j, u_j), j=0, 1, \ldots, n\), and is therefore termed Cauchy's polygon corresponding to the division \( (\Delta_n) \).

We have for \( t \in [t_j, t_{j+1}) \) that

\[ D^i \varphi_n(t) = f_j = (pu_j)' + q. \]

Attention must be paid to the fact that in defining the quantities \( f_j \) we adopted (1.2) instead of (1.6)

\[ f_j = (pu_j)' + q. \]

If we were to take (1.6) in place of (1.2), we would have obtained according to (1.1) that

\[ u_1 = q, \quad u_2 = 2r + r^2 (pq'), \quad \ldots \]

and the degree of smoothness of \( u_j \) would have gradually been lowered. It will be shown in the last section that the choice (1.2) serves our purpose.

Once the sequence of Cauchy's polygon \( \{\varphi_n(t)\} \) has been constructed it remains to prove the normality of such a sequence, i.e., the possibility of singling out from the \( \{\varphi_n(t)\} \) a uniformly convergent subsequence and to verify that the solution of our problem is established as the uniform limit of the above mentioned subsequence. We need for this knowledge of some elementary results concerning the theory of ordinary differential equations of the second order.

2. Basic Inequalities

If \( u \in C^2[a, b] \) satisfies the boundary conditions

\[ u|_{x=a} = u|_{x=b} = 0, \]

we obtain

\[ u(x) = - \int_a^b G_1(x, y) Lu(y) \, dy \]

where \( G_1(x, y) \) denotes the Green's function corresponding to the differential
operator $L u = \frac{d}{dx} \left( p(x) \frac{du}{dx} \right)$ with the boundary conditions (2.1).

By virtue of the explicit expression of $G_{x,y}$:

$$G_{x,y} = \begin{cases} \frac{1}{A} \int_a^b \int_a^y \frac{d\xi}{p(\xi)} \int_y^b \frac{d\eta}{p(\eta)} & (x \leq y) \\ \frac{1}{A} \int_b^x \int_b^y \frac{d\xi}{p(\xi)} \int_y^x \frac{d\eta}{p(\eta)} & (x \geq y) \end{cases}$$

$A$ being the definite integral $\int_a^b \frac{dx}{p(x)} > 0$, we have

(2.2) $A u(x) = \int_a^b \frac{dx}{p(x)} \int_a^y \left( \int_a^y \frac{d\xi}{p(\xi)} \right) L u(y) \, dy + \int_a^x \frac{dx}{p(x)} \int_x^y \left( \int_x^y \frac{d\eta}{p(\eta)} \right) L u(y) \, dy$

and further

(2.3) $A u'(x) = -\frac{1}{p(x)} \left[ \int_a^y \left( \int_a^y \frac{d\xi}{p(\xi)} \right) L u(y) \, dy + \int_y^b \left( \int_y^b \frac{d\eta}{p(\eta)} \right) L u(y) \, dy \right]$.

The following inequalities are easily derived from (2.2), (2.3) in view of $p - mm p(x) > 0$.

(2.4) $|u(x)| \leq \frac{2(b-x)(x-a)}{p} \|L u\|$,

(2.5) $\|u\| \leq \frac{(b-a)^2 \|L u\|}{2p}$,

(2.6) $\|u'\| \leq \frac{(b-a)^2 \|L u\|}{Ap^2}$,

where we employed the customary notation $\|v\| = \max_{a \leq x \leq b} |v(x)|$, $v \in C[a, b]$.

Let us consider the differential equation

(2.7) $\tau L u = \tau \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) = u - g(x)$.

Since for $v = u - g$ we have

$\tau L v = v - \tau L g$,

there holds for a solution $u$ of (2.7) an estimate of the form

(2.8) $\tau \|L u\| \leq \max \{\tau \|L g\|, |u(a) - g(a)|, |u(b) - g(b)|\}$.

For sufficiently small $\delta > 0$ the function
\[ \Delta u(x) = u(x+\delta) - u(x) \]
is defined in \([a, b-\delta]\) and satisfies there the ordinary differential equation
\[ \tau L \Delta u = \Delta u - L \Delta g. \]

Hence we have in view of (2.8)
\[ (2.9) \quad \tau \| L \Delta u \| \leq \max \{ \tau \| L \Delta g \| , \| \Delta u(a) - \Delta g(a) \| , \| \Delta u(b-\delta) - \Delta g(b-\delta) \| \} \]
where \( \| w \| = \max_{a \leq x \leq b-\delta} |w(x)|. \)

3. Existence Proof

Rewriting the equations (1.1), (1.2) into the form
\[ (3.1) \quad Lu_{j+1} = u_{j+1} - u_j - \tau q \]
and subtracting from (3.1) the same equation with \( j \) replaced by \( j-1 \) we have the equation
\[ (3.2) \quad L(u_{j+1} - u_j) = (u_{j+1} - u_j) - (u_j - u_{j-1}). \]

Inequality (2.8) applied to (3.2) yields by virtue of (1.3)
\[ (3.3) \quad \| L(u_{j+1} - u_j) \| \leq \| L(u_j - u_{j-1}) \| \]
and consequently for \( j = 1, 2, \ldots, n-1 \)
\[ (3.4) \quad \| L(u_{j+1} - u_j) \| \leq \| Lu_1 \|. \]

On the other hand, \( u_1 \) is a solution of the equation
\[ \tau Lu_1 = u_1 - \tau q \]
and hence applying again (2.8) we have
\[ (3.5) \quad \| Lu_1 \| \leq \tau Q, \quad Q = \max \{ \| Lq \| , \| q(a) \| , \| q(b) \| \}. \]

It is shown by induction that
\[ (3.6) \quad \| Lu_j \| \leq j\tau Q, \quad j = 0, 1, \ldots, n. \]

Combining (2.5), (2.6) and (3.6) we can immediately verify the inequalities below.
\[ (3.7) \quad \| u_j \| \leq \frac{(b-a)^2 j\tau Q}{2p}, \]
If we recall the definition (1.4) of the polygon $\varphi_n(t)$ and note further that

$$\varphi_n(t) = \frac{t-t_j}{\tau} w_{j+1} + \frac{t_{j+1}-t}{\tau} w_j$$

then, we have by means of (3.7) and (3.8)

$$\|\varphi_n(t)\| \leq \frac{(b-a)^2 Q t}{2p} \leq \frac{(b-a)^2 QT}{2p}$$

and

$$\|\varphi_n(t)\| \leq \frac{(b-a)^2 Q t}{Ap^2} \leq \frac{(b-a)^2 QT}{Ap^2}.$$ 

Thus we can conclude that the $\varphi_n(t)$, for every $n$ and $t$, are contained in a fixed compact subset of $C[a, b]$.

Let us apply (2.9) to (3.1). Then, there holds the estimate

$$\tau \| L \Delta u_{j+1} \| \leq \max \{ \tau \| L(u_j + \tau q) \|, \ |u_{j+1}(a)-u_j(a)|+\tau |\Delta q(a)|, \ |u_{j+1}(b-\delta)-u_j(b-\delta)|+\tau |\Delta q(b-\delta)| \}$$

On the other hand we see that

$$|u_{j+1}(x)-u_j(x)| \leq \frac{2(b-a) \tau \delta Q}{p}$$

for $x=a, b-\delta$, combining (2.4), (3.4) and (3.5). Consequently

$$\| L \Delta u_{j+1} \| \leq \max \left\{ \| L(u_j + \tau q) \|, \| \Delta q \| + \frac{2(b-a) \tau \delta Q}{p} \right\}$$

and by induction on $j$

$$\| L \Delta u_j \| \leq j \tau \|L \Delta q\| + \| \Delta q \| + \frac{2(b-a) \tau \delta Q}{p}.$$ 

As the right-hand side of the last inequality can be made as small as possible by choosing $\delta>0$ sufficiently small, the $L u_j$ also lie in a fixed compact subset of $C[a, b]$ irrespective of the division of the interval $0 \leq t \leq T$.

Let us note moreover that (1.5) and (3.6) yield

$$\| D^s \varphi_n(t) \| \leq TQ + \| q \|,$$
which implies the equi-continuity of the sequence \( \{\varphi_{n}(t)\} \) as functions of \( t \in [0, T] \). Since, as we have just seen, the values of \( \varphi_{n}(t) \) belong to a compact set, it is possible to pick out from \( \{\varphi_{n}(t)\} \) a subsequence \( \{\varphi_{n}(t)\} \) which converges uniformly on \([0, T]\). This proves the normality of \( \{\varphi_{n}(t)\} \).

The normality of the sequence \( \{D_{t}^{+} \varphi_{n}(t)\} \) is verified as follows.

It is obvious that
\[
\tau L(u_{j+1} - u_{k+1}) = (u_{j+1} - u_{k+1}) - (u_{j} - u_{k}),
\]
which yields in view of (2.8)
\[
\|L(u_{j+1} - u_{k+1})\| \leq\|L(u_{j} - u_{k})\|
\]
and hence if \( 0 < k < j \leq n \)
\[
\|L(u_{j} - u_{k})\| \leq \|L(u_{j-k+1} - u_{k})\|
\]
From the obvious relation
\[
\tau L(u_{j-k+1} - u_{k}) = (u_{j-k+1} - u_{k}) - (u_{j-k})
\]
We have by using (3.6)
\[
\|L(u_{j-k+1} - u_{k})\| \leq \|L(u_{j-k+2} - u_{k})\| \leq (j-k)\tau Q
\]
or we have established that
\[
\|L(u_{j} - u_{k})\| \leq (j-k)\tau Q.
\]

For any \( t, t+\delta \in [0, T] \), if we take natural numbers \( j, k \) such that \( t \in [t_{j}, t_{j+1}] \), \( t+\delta \in [t_{j}, t_{j+1}] \), we obtain
\[
\|D_{t}^{+} \varphi_{n}(t+\delta) - D_{t}^{+} \varphi_{n}(t)\| = \|f_{j} - f_{j}\|
\]
\[
= \|L(u_{j} - u_{k})\| \leq (j-k)\tau Q \leq (\delta + 2\tau)Q.
\]

This is nothing but the equi-continuity of the sequence \( \{D_{t}^{+} \varphi_{n}(t)\} \) as functions of \( t \). The values of \( D_{t}^{+} \varphi_{n}(t) \) being situated in a compact set, we can single out from \( \{D_{t}^{+} \varphi_{n}(t)\} \) a uniformly convergent subsequence and thus the sequence \( \{D_{t}^{+} \varphi_{n}(t)\} \) turns out to be normal.

Let \( \{\varphi_{n}(t)\} \) be a subsequence of \( \{\varphi_{n}(t)\} \) such that both \( \{\varphi_{n}(t)\} \) and \( \{D_{t}^{+} \varphi_{n}(t)\} \) are uniformly convergent on \([0, T]\). It is clear that if we set \( \varphi(t) = \lim \varphi_{n}(t) \), then \( D_{t} \varphi(t) = \lim D_{t}^{+} \varphi_{n}(t) \).

It now remains to show that the function \( \varphi(t) \) is the solution of (A).

For this purpose let us set
\[ f(t, u) = Lu + q \]

which is a closed mapping \([0, T] \times C[a, b] \to C[a, b] \).

For any \( t \in [t_j, t_{j+1}) \) we have
\[
\| D^+ \psi_n(t) - f(t, u_j) \| = \| f_j - f(t, u_j) \| = \| L(u_{j+1} - u_j) \| \leq TQ/n
\]

and therefore we can find for any \( t \in [0, T) \) a sequence \( \{s_n\} \subset [0, T) \) such that
\[
s_n \to t \quad \text{and} \quad \| D^+ \psi_n(t) - f(s_n, \psi_n(s_n)) \| \leq TQ/n.
\]

Let \( n \) tend to infinity, then, we obtain finally
\[
D^+ \psi(t) = f(t, \psi(t)) = L\psi(t) + q
\]
or \( \psi(t) \) is the desired solution of (A) satisfying the boundary condition (B).