The radius of univalence for some analytic functions

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1. Introduction. Let (S) denote the class of functions f(z) regular and univalent in E = \{ z \mid |z| < 1 \}, which satisfy f(0) = 0, f'(0) = 1 and which map E onto domain D(f).

We denote by (C), (S*), and (K) the subclass of (S) where D(f) are, respectively, close-to-convex, starlike with respect to the origin, and convex.

In [1], [2] the following theorem was proved.

Theorem A. If f(z) = z + \sum_{n=2}^{\infty} a_n z^n is a member of (K), (S*), or (C), then the function

\[ F(z) = \frac{2}{z} \int_{0}^{z} f(t) \, dt \]

is also a member of (K), (S*), or (C).

Theorem B. If f(z) = z + \sum_{n=2}^{\infty} a_n z^n is a member of (K), (S*), or (C), then the function

\[ F(z) = (c+1)z^{-c} \int_{0}^{z} t^{c-1} f(t) \, dt \]

is also a member of the same class for c = 1, 2, 3, \ldots.

Theorem B represents a generalization of theorem A for the special case c = 1.

Solving the relations (1.1), (1.2) for the inverse function f(z), we have

f(z) = (1/2)(zF(z))', \quad f(z) = (1/(c+1))z^{1-c}(z^c F(z))'.

It is the purpose of this paper (suggesting by the results of Livingston [3] and Benardi [4]) to extend the analytic function F(z) = z + a_n z^{n+1} + \ldots.

2. Proof of the Theorem.

Theorem 1. Let F(z) = z + a_n z^{n+1} + \ldots \in (S*), f(z) = (1/2)(zF(z))'.

Then f(z) is starlike for |z| < r_0 = (1/(n+1))^{1/n}.

This result is sharp.

Proof. Since F(z) \in (S*), Re(zF'(z)/F(z)) > 0 for |z| < 1.

Hence there exists a function \phi(z), with |\phi(z)| \leq 1, regular in |z| < 1 such that
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\[ \frac{zF'(z)}{F(z)} = -\int_0^z f(t) dt - \frac{1-z^n}{1+z^n} \frac{z^n \phi(z)}{z^n \phi(z) + 1} . \]

Solving for \( f(z) \),

\[ f(z) = \frac{z^n \phi(z) - z^n \phi'(z)}{1+z^n \phi(z)} \int_0^z f(t) dt. \]

Therefore

\[ \frac{z f'(z)}{f(z)} = -nz^n \phi(z) - z^n \phi'(z) \frac{z^n \phi(z) - z^n \phi'(z)}{1+z^n \phi(z)} + \frac{z^n \phi(z) - z^n \phi'(z)}{1+z^n \phi(z)} \int_0^z f(t) dt \]

\[ = \frac{1-\{(n+1)z^n \phi(z) - z^{n+1} \phi'(z)\}}{1+z^n \phi(z)}. \]

In order to determine where \( f(z) \) is starlike, we must show that \( \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0 \).

This condition is equivalent to

\[(2.1) \quad \text{Re} \left( 1-\{(n+1)z^n \phi(z) - z^{n+1} \phi'(z)\} \right) > 0.\]

Condition (2.1) is equivalent to

\[(2.2) \quad \text{Re} \left[ z^n \phi'(z) \right] \left[ 1+z^n \phi(z) \right] < 1-\{(n+1)z^n \phi(z) \}^2 - n \text{Re}[z^n \phi(z)]. \]

If we apply the well known result for bounded functions

\[ |\phi'(z)| \leq \frac{1}{1-|z|^2} (1-|\phi(z)|^2) \left( |z|^2 < 1 \right). \]

with our paper, the inequality (2.2) will be satisfied if

\[(2.3) \quad \frac{|z|[n+1]}{1-|z|^2} \left( 1-|\phi(z)|^2 \right) \left( 1+|z|^n |\phi(z)| \right) < \left( 1+|z|^n |\phi(z)| \right) \left( 1-(n+1)|z|^n |\phi(z)| \right). \]

Condition (2.3) is equivalent to

\[(2.4) \quad |z|^n+1 + |z|^2 + (n+1) \left( 1-|z|^2 \right) |z|^n |\phi(z)| - |z|^n+1 |\phi(z)| \leq 1. \]

If in (2.4) we let \( |z| = a \) and \( |\phi(z)| = x \), then it is sufficient to show that for any fixed \( a \), \( 0 \leq a < r_0 \), the function \( h(x) = a^n+1+a^2+(n+1) \left( 1-a^2 \right) a^n x-a^n+1x^2 \) is bounded above by one for \( 0 \leq x \leq 1 \).

It is easily seen that \( h'(x) > 0 \) for \( 0 \leq x \leq 1 \), provided that

\[ a < \frac{-1+(1+(n+1)a^2)^{1/2}}{n+1} \] and therefore \( a < r_0 \). Thus, if \( 0 \leq a < r_0 \), the maximum value of \( h(x) \), \( 0 \leq x \leq 1 \), is given by \( q(a) = h(1) = a^2+(n+1)a^n -(n+1)a^{n+2} \).

Since the function \( q(a) \) increases for \( 0 < a < r_0 \), \( q(a) < q(r_0) = 1 \).

Therefore, (2.4) is satisfied for every function \( |\phi(z)| \) where \( |\phi(z)| \leq 1 \) if \( |z| < r_0 \).

To see that the result is sharp, let

\[ F(z) = z \exp \left\{ 2 \int_0^z \frac{t^{n-1}}{1-t^n} dt \right\} \text{ which is in } (S^n). \]
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Then,
\[ f(z) = \frac{z}{1 - z^n} \exp \left\{ \frac{2}{\int_0^z t^n \frac{1}{1 - t^n} \, dt} \right\}, \]

\[ zf'(z) = \frac{z}{(1 - z^n)^2} \left( 1 + (n + 1)z^n \right) \exp \left\{ \frac{2}{\int_0^z t^n \frac{1}{1 - t^n} \, dt} \right\}. \]

Therefore,
\[ \frac{zf'(z)}{f(z)} = \frac{1 + (n + 1)z^n}{(1 - z^n)}. \]

Thus, \( f(z) \) is not starlike in any circle \( |z| < r \), if \( r > r_0 \).

**Theorem 2.** If \( F(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{K} \), then
\[ f(z) = \frac{1}{2} (zF(z))' \]
is univalent in \( |z| < 1 \) and is convex for \( |z| < r_0 = \frac{1}{(n+1)^{1/n}} \). This result is sharp.

**Proof.** We have \( 2f'(z) = 2F'(z) + zF''(z) \). Thus
\[ \text{Re} \left( \frac{f'(z)}{F'(z)} \right) = 1 + \text{Re} \left( \frac{zF''(z)}{F'(z)} \right) > 0 \]
for \( |z| > 1 \).

Thus, \( f(z) \) is close-to-convex relative to \( F(z) \) and therefore is univalent in \( |z| < 1 \). To show that \( f(z) \) is convex for \( |z| < r_0 \), we notice that \( zf'(z) = 1/2 \left( 2zF'(z) + z^2F''(z) \right) \), \( \in \mathcal{K} \).

Since \( F(z) \) is in \( \mathcal{K} \), \( zF'(z) \) is in \( \mathcal{S}^* \). Therefore, by Theorem 1, \( zf'(z) \) is starlike for \( |z| < r_0 \) and hence \( f(z) \) is convex for \( |z| < r_0 \).

To see that the result is sharp, let \( F(z) = \int_0^z \frac{1}{(1 - t^n)^{1/n}} \, dt \) for \( |z| < 1 \) which is in \( \mathcal{K} \).

Since \( f(z) = 1/2 \left( F(z) + zF'(z) \right) \),
\[ f(z) = \frac{1}{2} \left\{ \int_0^z \frac{1}{(1 - t^n)^{1/n}} \, dt + z \exp \left\{ 2 \int_0^z t^n \frac{1}{1 - t^n} \, dt \right\} \right\}, \]
then we have
\[ f'(z) = \frac{1}{1 - z^n} \exp \left\{ 2 \int_0^z t^n \frac{1}{1 - t^n} \, dt \right\}, \]
\[ f''(z) = (n+2) \frac{z^{n-1}}{(1 - z^n)^2} \exp \left\{ 2 \int_0^z t^n \frac{1}{1 - t^n} \, dt \right\}, \]
and therefore
\[ 1 + \frac{zf''(z)}{f'(z)} = \frac{(n+1)z^n+1}{1 - z^n}. \]

Thus \( f(z) \) is not convex in any circle \( |z| < r \), if \( r > r_0 \).

**Theorem 3.** Let \( F(z) + z + a_{n+1}z^{n+1} + \cdots \) be close-to-convex with respect to \( G(z) \), \( f(z) = 1/2 \left( zF(z) \right)' \), \( g(z) = 1/2 \left( zG(z) \right)' \).

Then \( f(z) \) is close-to-convex with respect to \( g(z) \) for \( |z| < r_0 = \frac{1}{(n+1)^{1/n}} \). This result is sharp.
Proof. By definition of close-to-convexity [5], there exists $G$ in $(S^*)$ such that

\[(2.7) \quad \text{Re}[zf'(z)/G(z)] > 0 \quad (|z| < 1).\]

By Theorem 1, $g(z)$ is starlike for $|z| < r_0$.

Only showing that $\text{Re}[zf'(z)/g(z)] > 0$ for $|z| < r_0$ suffices to prove the theorem.

We have

\[
\frac{zf'(z)}{G(z)} = \frac{zf(z) - \int_0^z f(t) \, dt}{\int_0^z g(t) \, dt}.
\]

Thus we may set

\[(2.8) \quad \frac{zf(z) - \int_0^z f(t) \, dt}{\int_0^z g(t) \, dt} = P(z)\]

where $P(z)$ is regular in $|z| < 1$ and satisfies $P(0) = 1$ and by (2.7), $\text{Re} \ P(z) > 0$ for $|z| < 1$.

By the equality (2.8), our computation gives

\[(2.9) \quad zf'(z) = P(z)g(z) + P'(z)\int_0^z g(t) \, dt.\]

Therefore

\[(2.10) \quad \frac{zf'(z)}{g(z)} = P(z) + \frac{P'(z)}{g(z)}\int_0^z g(t) \, dt.\]

If we apply the known result

\[(2.11) \quad |P'(z)| \leq \frac{2n}{1 - |z|^{2n}} \text{Re} \ P(z) \quad (|z| < 1),\]

with our paper, we have from (2.10), taking real parts,

\[(2.12) \quad \text{Re} \left[ \frac{zf'(z)}{g(z)} \right] \geq \text{Re} \ P(z) \left( 1 - \frac{2n}{1 - |z|^{2n}} \right) \left( \frac{1}{\int_0^z g(t) \, dt} \right).\]

We note that

\[(2.13) \quad g^*(z) = zg(z) / \int_0^z g(t) \, dt = (G(z) + zG'(z)) / G(z) = 1 + zG'(z) / G(z), \text{ and} \]

\[\text{Re} \ (zG'(z)/G(z)) > 0 \quad (|z| < 1).\]

Thus we have from (2.13),

\[\text{Re} \ (g^*(z)) > 1 \quad \text{for} \ |z| < 1.\]

Hence, there exists $\phi(z)$, analytic and satisfying $|\phi(z)| \leq 1$ for $|z| < 1$, such that $g^*(z) = 2/(1 + z^n \phi(z))$. 

Therefore,

\[(2.14) \quad \left| \int_{0}^{z} g(t) dt / g(z) \right| = \left| (z + z^{n+1} \phi(z)) / 2 \right| \leq 1/2(|z| + |z|^{n+1}).\]

If we apply (2.14) with our paper, the inequality (2.12) gives

\[(2.15) \quad \Re \left[ \frac{zf'(z)}{g(z)} \right] = \Re P(z) \left[ \frac{1 - n|z|^{n-1}}{1 - |z|^{2n}} \right] \leq \Re P(z) \left[ \frac{1 + |z|^{n}}{1 - |z|^{2n}} \right].\]

The right side of (2.15) is positive, if |z| < r0.

To see that the result is sharp, let F(z) = G(z) = z exp \{2t/1 - t^n\} which belongs to the class (S*) and therefore to the class (c). Moreover \((zf'(z))/g(z)\) is the same expression as the right side of (2.5).

Therefore, f(z) is not univalent and not close-to-convex in |z| < r, if r > r0.

Theorem 4. Let F(z) = z + a_{n+1} z^{n+1} + \cdots be such that \(\Re (F(z)/z) > 0\) for z in E and let \(f(z) = 1/(c+i)z^c(zc(F(z))^c)\) for \(c = 1, 2, \ldots,\), then \(\Re (f(z)/z) > 0\) for |z| < r1 = ((\sqrt{n^2 + (1+c)^2} - n)/(1+c))^1/n.\]

This result is sharp.

Proof. Since \(\Re (F(z)/z) > 0\), we can deduce that F(z) cannot vanish in |z| < 1 except for a simple zero at z = 0.

Let F(z)/z = P(z) where P(0) = 1 and \(\Re P(z) > 0\) for z in E.

Then we have

\[(1+c) \frac{f(z)}{z} = (1+c)P(z) + zP'(z).\]

If we apply (2.11) with our paper, we have

\[(2.16) \quad (1+c) \Re \frac{f(z)}{z} \geq \Re P(z) \left[ (1+c) - \frac{2n|z|^n}{1 - |z|^{2n}} \right] = \Re P(z) \left[ \frac{(1+c) - 2n|z|^n - (1+c)|z|^{2n}}{1 - |z|^{2n}} \right].\]

The right hand side of (2.16) is positive for |z| < r1.

The result is sharp, for each c, for the function \(F(z) = \frac{z(1+z^n)}{1-z^n}\).

For this function we have \(\Re \frac{F(z)}{z} > 0\) for |z| < 1, and

\[\frac{f(z)}{z} = \frac{1}{1+c} \frac{(1+c) + 2nz^n - (1+c)z^{2n}}{(1-z^n)^2}.\]

Then we have \(\frac{f(z)}{z} = 0\), when \((1+c) + 2nz^n - (1+c)z^{2n} = 0\), or \(z = -r_1\).

Thus \(\Re (f(z)/z) > 0\) in any circle |z| < r, if r > r1.
References


