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On the Primäreideal of Rings*

By

Toshio EGUCHI

(昭和47年9月29日受理)

This paper extends the concept of Starker primäreideal for finite Ideal to Commutative Rings and develops methods for computing the Starker primäreideal in terms of certain homomorphism series of the Rings.

In this paper we assume the reader is acquainted with the terminology and results of Schwache primäreideal. All Rings considered herein are finite unless otherwise stated.

Lemma

Starker primäreideal \( \pi \in \mathcal{O} \) is uniquely \( \phi \)-sectorial operator if and only if \( \pi \) is either a prime or \( \pi = \pi_1 \pi_2 \) where \( \pi_1 \) is left \( \phi \)-sectorial and \( \pi_2 \) is right \( \phi \)-sectorial.

Proof

First assume \( \pi \) is uniquely \( \phi \)-sectorial operator. We may assume that \( [\alpha \pi, \pi] \) does not intersect \( \mathcal{O} \) \( (\alpha \pi, \pi) \). Let \( \pi = L_1 (\alpha \pi) \pi_1 \mathcal{O}_1 (\alpha \pi) \), where \( L_1 = A(\pi) A(\alpha) \), \( R_1 = \mathcal{O}(\pi) \mathcal{O}(\alpha) \). Suppose \( || \pi_1 || = || \alpha \pi || \). \( \pi_1 \) is sectorized into the product of elements of \( \rho \), \( P \) of \( F \), \( (\alpha) \):

\[
\pi_1 = P_1 P_2 \quad || P_1 ||, || P_2 || = || \alpha \pi ||
\]

such that \( L_1 P_1 \) is a prime. Also \( P_2 \mathcal{O}_1 \in F \) and \( P_2 \mathcal{O}_1 \) can be sectorized into the product of primes, say \( P_2 \mathcal{O}_1 = \pi_1 \pi_2 \ldots \pi_n \),

where \( \pi_i \ (i = 1, 2, \ldots, n) \) are primes Accordingly we have a prime sectorization of \( \pi \):

* Some of these results had been reported on the following Algebraic branch of Mathematical Society;

T. Eguchi, : On the Ideal of Multipliative Ring. 1964. 10. 16 In Fukuoka
(1) \[ \pi = (L_1 P_1) \pi_1 \pi_2 \ldots \pi_n \]

On the other hand, by a prime sectorization of \( \pi_1 \Omega_1 \) we have another prime sectorization of \( \pi \):

(2) \[ \pi = L_1 \pi'_1 \pi'_2 \ldots \pi'_n \quad \text{where} \quad \pi' R_1 = \pi'_1 \pi'_2 \ldots \pi'_n \]

Thus (1) and (2) are different prime sectorization of \( \pi \), a contradiction to the assumption. Therefore \( \pi'_{r_1} = 0 \), that is, \( \pi = L_1 R_1 \), \( L_1 \) is left \( \phi \) - sectorial, \( R_1 \) is right \( \phi \) - sectorial.

Next we will prove the converse. If \( \pi \) is a prime, it is obvious. Assume \( \pi = \pi_1 \pi_2 \), \( L(\pi, \alpha) = \pi_1 \), \( R(\pi, \alpha) = \pi_2 \).

Suppose \( \pi \) has another sectorization - \( \phi \) - \( L \) - \( R \).

\[ \pi = \pi_1 \pi_2 = \pi'_1 \pi'_2 \quad \pi'_1, \pi'_2 \in \Omega(\alpha, \pi) \]

in which \( \pi'_1 \) and \( \pi'_2 \) are not assumed to be prime. Suppose \( \pi_1 \cong \pi'_1 \). Then since \( \pi_1 = L(\pi_1, \alpha) = L(\pi'_1, \alpha) \)

\[ \pi'_1 = \pi_1 P \]

and

\[ \pi_2 = P \pi'_2, \quad \pi_2 = R(\pi_2, \alpha) = R(\pi'_2, \alpha) \]

This is a contradiction to the assumption that \( \pi_2 \) is right \( \phi \) - sectorial.

Therefore \( \pi_1 = \pi'_1 \), hence \( \pi_2 = \pi'_2 \), \( \pi \) is uniquely \( \phi \) - sectorizable.

Theorem

\( \Omega^* (\Omega, \pi, e, P, A) \) is a starker primère ideal which is a one-parameter holomorphie extention of \( \alpha \) by \( \Omega(\alpha, \pi) \).

Proof

We first show that \( \Omega^* \) is a ideal. Let \( P_1, P_2, P_3 \in P \) and \( \alpha, \beta, A, \in \Omega(\alpha, \pi) \).

Then

\[ [ (P_1, \alpha) (P_2, \beta) ] (P_3, A) = [ (P_1 J_p P_2 J_q) J_r P_3 J_s, \alpha \beta A ] \]

\[ = (P_1 J_{r-p} P_2 J_{r-q} P_3 J_s, \alpha \beta A) \quad (1) \]
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Where
\[ \pi = (\alpha \beta), \alpha_1^{-1}, P = (\alpha \beta)_1 \alpha \beta_1^{-1}, \gamma = (\alpha \beta A)_1 (\alpha \beta)_1^{-1} \]
\[ s = (\alpha \beta A)_1 \alpha \beta A_1^{-1}. \]

Also
\[ (P_1, \alpha) ((p^* \beta) (P_3, A)) = (P_1, \alpha) ((P_2, \alpha) (P_3, A)_1 \gamma, \alpha \beta A) \]
\[ = (P_1, \alpha) ((P_2, \alpha) (P_3, \gamma) A_w, \alpha \beta A), \]

where
\[ \pi_1 = (\beta A)_1 \beta_1^{-1}, \pi_2 = (\beta A)_1 A A_1^{-1}, \pi_3 = (\alpha \beta A)_1 \alpha_1^{-1} \]
\[ \pi_4 = (\alpha \beta A)_1 \alpha (\beta A)_1^{-1}. \]

To establish associativity, it is enough to show that
\[ \gamma \pi = \pi_3 \quad \gamma P = \pi_4 \pi_1 \quad s = \pi_4 \pi_2 \]

By part (1) of lemma, we have that \( \gamma (\alpha \beta)_1 = (\alpha \beta A)_1; \)

hence
\[ \gamma \pi = \gamma (\alpha \beta)_1 \alpha_1^{-1} = (\alpha \beta A)_1 \alpha_1^{-1} = \pi_3 \]

Similarly
\[ \gamma \rho = \gamma (\alpha \beta)_1 \alpha \beta_1^{-1} = (\alpha \beta A)_1 \alpha \beta_1^{-1}, \]
\[ \pi_4 \pi_1 = \pi_4 (\beta A)_1 \beta_1^{-1} = (\alpha \beta A)_1 \alpha \beta_1^{-1}, \]

Hence \( \pi_4 \pi_2 = \pi_4 (\beta A)_1 \beta A_1^{-1} = (\alpha \beta A)_1 \alpha \beta A_1^{-1} = s \)

Thus \( \mathcal{Q}^* \) is a ideal

Next we show that \( \mathcal{Q}^* \) is a primärdeal. Let \( (P, \alpha) \in \mathcal{Q}_1^* \)

Taking \( \beta = \alpha^{-1}, A = \alpha, \) and \( P_2 = P_1, \) we find that
\[ (P_1, \alpha) (P_2, \alpha^{-1}) (P_3, \alpha) = (P_1, \alpha) (P_2, \alpha^{-1}) (P_1, \alpha) \]
\[ = (P_1) P_2 A_1, P_1) (P_1, \alpha) \]
(2)

Hence \( x^{-1} \in \mathcal{Q}_e \) Also \( x x^{-1} = e, \) since \( x \in \pi_e. \)

\[ P_1 (P_2, A) P_1 = P_1 (P_1^{-1}, A x^{-1}) P_1 = P_1 (P_1^{-1} A_e) P_1 = P_1 P_1 P_1 = P_1 \]

Thus, from Eq. (2)
\[ (P_1, \alpha) (P_2, \alpha^{-1}) (P_1, \alpha) = (P_1, \alpha) \]

We now examine the indempotents of \( \mathcal{Q}^*. \) Let \( (P_1, \alpha) \in \mathcal{Q}^* \) and suppose that
\[ (P_1, \alpha)^2 = (P_1, \alpha). \] Then \( \alpha^2 = \alpha \) and \( P_1, \alpha P_1 P_1, \alpha = P, \)

where
\[ \pi = (\alpha^2)_1 \alpha_1 \alpha_1^{-1} = e, \]
\[ P = (\alpha^2)_1 \alpha \alpha_1^{-1} = \alpha \alpha \alpha^{-1} \alpha_1^{-1} = \alpha_1 \alpha_1^{-1} = e \]

Hence \( P^2 = P \) and so \( P^1 = 1, \) the identity of \( P. \) Conversely, if \( \alpha^2 = \alpha, \)

then \( (1, \alpha)^2 = (1, \alpha). \) Thus the set of idempotents of \( \mathcal{Q}^* \) is \{((1, \alpha) \in \mathcal{Q}^*; \alpha^2 = \alpha\}

Since the idempotents of \( \mathcal{Q} \) commute, it follows that \( \alpha = \pi_1 \pi_2 \ldots \pi_r, \)
\[ \mathfrak{d}^* \supset (\mathfrak{d}^*, \pi), \pi = P_1 P_2 \ldots P_s \]

Thus \( \mathfrak{d}^* \) is a primär ideal. To see that \( \mathfrak{d}^* \) is starker, consider any two
\[ \pi_1, \pi_2 \in \alpha. \]
Since \( \mathfrak{d} \) is primär, there exists
\[ P_1, P_2 \text{ and } \pi_1 \pi_2 \ldots = \alpha = (\pi_1 P_1) \ldots (\pi_r P_r) \]
\[ \pi_1 = \pi_1, P_1, \pi_2 = \pi_2. \]

Hence
\[ \pi^n \subseteq (\pi_1, \pi_2 \ldots, \pi_s, \alpha) \subseteq \pi \]
\[ \alpha: \pi_1 \subset \alpha; \pi_1 \pi_2 \subset \ldots \subset \alpha; \pi_1 \ldots \pi_1 \subset \alpha; \pi_1 \pi_2 \ldots \pi_{i+1} \subset \ldots \]
It follows that \( \mathfrak{d}^* \) is starker. Define an equivalence or on \( \mathfrak{d}^* \) by the rule that
\[ \alpha \subset \alpha; \alpha_1 \subset \alpha; \alpha_1 \alpha_2 \subset \ldots \]
It is immediate that \( \alpha \) is a congruence on \( \mathfrak{d}^* \). Further, from the form of the starker primär ideal in \( \mathfrak{d}^* \) it is clear that \( \alpha \) is prime. \( \beta = P_1 \cap P_2 \cap \ldots \cap P_n. P_1: \]
Starker Primär Ideal. Then \( \beta \) is a \( P(\alpha) \in \mathfrak{d} \) and is a holomorphic of by \( \mathfrak{d} \).
Thus \( \mathfrak{d}^* \) is starker over \( \alpha \) and \( \mathfrak{d}^*/\alpha \cong \mathfrak{d} \).

Theorem 2

Let \( \mathfrak{d}^* \) be \( \pi \) solvable and consider all starker homomorphism series of the form
\[ \mathfrak{d}^*: (\pi_1) \supset \mathfrak{d}^*: (\pi_1^2) \supset \ldots \supset \mathfrak{d}^*: (\pi_1^n) = \mathfrak{d}^*: (\pi_1^{n+1}) = \ldots \]
where the \( \pi_1^m \) is \( \pi \) - prime-ideal. Let the length of such a series be \( n \). Let
\[ \alpha' \pi (\mathfrak{d}^*) = \max \{ \text{length of all such series for } \mathfrak{d} \}. \]

Then \( l_\pi (\mathfrak{d}^*) = \alpha' \pi (\mathfrak{d}^*). \)

Proof

By virtue of lemma, every epimorphism in these series is proper except possibly the first and last. Furthermore, every series of type \( (1) \) can be replaced by the following series of the same length:
\[ \mathfrak{d}^*: \pi_1 \subset \mathfrak{d}^*: \pi_1 \pi_2 \subset \mathfrak{d}^*: \pi_1 \pi_2 \pi_3 \subset \ldots \subset \mathfrak{d}^*: \pi_1 \pi_2 \pi_3 \ldots \pi_n = \pi; \pi_1 \pi_2 \ldots \pi_{n+1} = \ldots \]
To prove this, let \( T \) be a \( \mathfrak{d}: \pi_1 \pi_2 \pi_3 \ldots \pi_i \text{ prim and consider } \mathfrak{d} \rightarrow P \)
Now it is easy to see that \( \alpha' \pi (\mathfrak{d}^*) \leq l_\pi (\mathfrak{d}^*), \) for by the definition of \( \mathfrak{d}^* P \mathfrak{d}, \) the \( \pi \) length must drop exactly one from \( \pi P \mathfrak{d}^* \) to \( \mathfrak{d}^* \pi_1. \) We prove the reverse inequality, \( l_\pi (\mathfrak{d}^*) \leq \alpha' \pi (\mathfrak{d}^*), \) by induction on the order of \( \mathfrak{d}^*. \)
Assume the inequality true for all prim ideal of order less than \( |\mathfrak{d}^*|. \) Let
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If \( \mathcal{Q}^* \) is not subdirectly indecomposable, then by Lemma, there exists a proper starken map \( T \) of \( \mathcal{Q}^* \) with \( l_\pi (\mathcal{Q}^*) = l_\pi (T) \). But by induction \( l_\pi (T) = \alpha ' \pi (T) \), so \( T \), hence \( \mathcal{Q}^* \), has a chain of type (1) of length \( n \), so \( \alpha ' \pi (\mathcal{Q}^*) \geq l_\pi (\mathcal{Q}^*) \).

Let \( \mathcal{Q}^* \) be subdirectly indecomposable; then \( \mathcal{Q}^* \) is either a \( \mathcal{Q} \mathcal{Q}^* T, P \pi T, \pi_1 \pi_2 \ldots \pi_n \) primär or a primär with a null ideal. In the last three cases, \( \mathcal{Q}^* \) has a proper \( \mathcal{Q} \) -schwächen image where \( \mathcal{Q} \) length is the same as so \( l_\pi (\mathcal{Q}) \), so again by induction, \( \alpha ' \pi (\pi) \geq n = l_\pi (\mathcal{Q}^*) \).

Let \( \mathcal{Q}^* \) be a non- \( \pi \mathcal{Q}^* P \) primär. then \( l_\pi (\mathcal{Q}^*) = l_\pi (\pi \mathcal{Q}^* P) \), so using the same argument as above, \( \alpha ' \pi (\mathcal{Q}^*) \geq l_\pi (\mathcal{Q}^*) \). If \( \mathcal{Q}^* \) is a \( \mathcal{Q} \mathcal{Q}^* P \), then \( l_\pi (\pi_1 \mathcal{Q}^* P_1) = n-1 \). If \( n = 1 \), then \( \mathcal{Q}^* \to \pi_1 \pi_2 \ldots \pi_n \) is a series of type (1) of length 1, so \( \alpha ' \pi (\mathcal{Q}^*) \geq 1 = l_\pi (\mathcal{Q}^*) \). If \( n > 1 \), then \( \pi_1 \mathcal{Q}^* P_j \) has a series of length \( n-1 \), so the series \( \mathcal{Q}^* \to \pi_1 \pi_2 \ldots \pi_1 \) followed by this longest series for \( \pi_1 \mathcal{Q}^* P_j \) has length \( n \), because \( \pi_1 \mathcal{Q}^* P_j \) is not a \( \pi_1 \pi_2 \ldots \pi_n \mathcal{Q}^* P_1 \ldots P_n \) ideal.

Thus again \( \alpha ' \pi (\mathcal{Q}^*) \geq l_\pi (\mathcal{Q}^*) \), and then \( \alpha ' \pi (\mathcal{Q}^*) = l_\pi (\mathcal{Q}^*) \).

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Summary

I had been reported some results about the ideal-theory in which the division-chain-condition is assumed.

That is, by a new concept of primaryideal which is called

Starker Primärideal or Schwache Primärideal, we had been proved the following theorem.

Arbitrary ideal of a ring is showed as intersection of finite numer primaryideals

Now, we introduce a new concept of $\Omega^\ast (\Omega, \pi, e, P, \Delta)$ and extends the concept of Starker primärideal, further we develops methods for computing the Starker Primärideal in terms of certain series of the Rings.

1. $\Omega^\ast (\Omega, \pi, e, P, \Delta)$ is a starker primärideal which is a one-parameter holomorphic extension of $\alpha$ by $\Omega (\alpha, \pi)$.

2. Let $\Omega^\ast$ be $\pi$ solvable and consider all starker homomorphism series of the form

   $\Omega^\ast; (\pi_1) \subset \Omega^\ast; (\pi_1^2) \subset \ldots \subset \Omega^\ast; (\pi_1^n) = \Omega^\ast; (\pi_1^n+1) = \ldots$.

   where the $\sigma^n$ is $\pi$-prime-ideal. Let the length of such a series be $n$. Let $\alpha^\ast (\Omega^\ast) = \max \{\text{length of all such series for } \Omega^\ast\}$. Then $l_{\pi} (\Omega^\ast) = \alpha^\ast_{\pi} (\Omega^\ast)$. 