On the radius of starlikeness and convexity
for some univalent functions

By

Toshiaki YOSHIIKAI

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Abstract

In this paper we have shown that if $F(z) = \sum_{k=1}^{\infty} \frac{c+1}{c+k} a_k z^k$ then $f(z)$ is also the same class for $|z| < r_0$ indicating the proof of the theorems. And our results are the generalization of the theorems of Bajpai and Srivastava [1].

1. Introduction

By $S$ we denote the class of normalized functions $f(z)$ which are regular and univalent in the unit open disk $E \{ |z| < 1 \}$, while $S^*$ denotes the class of functions in $S$ which map $E$ onto a starlike domain with respect to the origin. By $S_\alpha^*$ we denote the class of functions $f(z)$ in $S^*$ having the additional property $\Re \{ z f'(z)/f(z) \} > \alpha$, $z \in E$, $0 \leq \alpha < 1$. The class of functions $f(z)$, which are in $S$ and map $E$ onto a convex domain, is denoted by $K$, while $K_\alpha$ denote the class of convex functions of order $\alpha$, if $\Re \{ 1 + z f''(z)/f'(z) \} > \alpha$, $z \in E$, $0 \leq \alpha < 1$. If $f(z) \in S$ and $g(z) \in S_\alpha^*$ satisfy the condition $\Re \{ z f'(z)/g(z) \} > \beta$, $z \in E$, $0 \leq \beta < 1$, then $f(z)$ is said to be a close-to-convex functions of order $\beta$ and type $\alpha$. We denote this class by $C(\beta, \alpha)$. This concept of close-to-convex functions on $C(0, 0)$ is due to Kaplan [5]. Libera [6] proved that if $f(z)$ belongs to $S_0^*$, $K$ or $C(0, 0)$ then the function $F(z) = (2/z) \int_0^z f(t)dt$ also belongs to $S_0^*$, $K$ or $C(0, 0)$ respectively. Bernardi [2] further proved that if $f(z)$ belongs to $S_0^*$, $K$ or $C(0, 0)$ then the function $F(z) = (c+1) z^{-c} \int_0^z t^{c-1} f(t)dt$ also belongs to $S_0^*$, $K$ or $C(0, 0)$, $C = 1, 2, \ldots$. Livingston [7] has studied the converse
problem and obtained sharp results. He has shown that if \( F(z) \) is a member of \( S^*, K \) or \( C(0, 0) \) then \( f(z) = (zF(z))'/2 \) is starlike, convex or close-to-convex respectively for \( |z| < \frac{1}{2} \). Furthermore YoshiKai [9] generalized the Livingstone's results. Bernardi [3] again considered the converse problem and obtained sharp results. He has shown that if \( F(z) = \sum_{n=1}^{\infty} \left( \frac{c+1}{c+n} \right) a_n z^n = (c+1) z - c \int_0^z t^{c-1} f(t) dt \) \( (a_1=1) \) is a member of \( S^*, K \) or \( C(0, 0) \) then \( f(z) \) is starlike, convex or close-to-convex respectively for \( |z| < \frac{1}{2} \) for \( c=1 \). Padmanabhan [8] considered the converse problem of Libera for the class \( S_n^*, K_\alpha \) and \( C(\beta, \alpha) \), \( 0 \leq \beta < \frac{1}{2} \). Recently Bajpai and Srivastava [1] generalized the Bernardi's results. They proved that if \( F(z) = \sum_{n=1}^{\infty} \left( \frac{c+1}{c+n} \right) a_n z^n \) \( (a_1=1) \) is a member of \( S_n^*, K_\alpha \) and \( C(\beta, \alpha) \) in which \( 0 \leq \alpha < 1, 0 \leq \beta < 1 \), then \( f(z) \) is \( S_n^*, K_\alpha \) and \( C(\beta, \alpha) \) respectively for

\[
|z| < \frac{(2-\alpha) + (3+\alpha^2+c^2+2c\alpha-2\alpha)^{\frac{1}{2}}}{c+2\alpha-1}, \quad \text{for } c=2, 3, \ldots ,
\]

\[
< \frac{1}{2}, \quad \text{for } c=1 \quad \text{and} \quad \alpha=0,
\]

\[
< \frac{(2-\alpha) + (4+\alpha^2)^{\frac{1}{2}}}{2\alpha}, \quad \text{for } c=1 \quad \text{and} \quad 0 < \alpha < 1.
\]

In this paper we generalize the above theorems for functions with whose power series begins \( F(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{c+1}{c+k} \right) a_k z^k \), \( c=1, 2, \ldots \).

2. Proof of the Theorems

First we state the following lemma which we shall need.

**Lemma.** If \( p(z) = 1 + a_n z^n + \ldots \) is analytic and satisfies \( \text{Re } p(z) > \beta, 0 \leq \beta < 1 \), for \( |z| < 1 \), then

\[
|p'(z)| \leq \frac{2n |z|^{n-1}}{1 - |z|^{2n}} \text{Re} \{p(z) - \beta\}.
\]

**Proof.** Let \( \phi(z) = \frac{p(z) - 1}{p(z) + 1 - 2\beta} \), where \( \phi(z) \) is analytic and \( |\phi(z)| < 1 \). For such functions it is known that

\[
|\phi'(z)| \leq \frac{n |z|^{n-1}}{1 - |z|^{2n}} \left( 1 - |\phi(z)|^2 \right). \quad (4).
\]

Also

\[
|\phi'(z)| = \frac{2(1-\beta)}{1-2\beta} |p'(z)|.
\]

Eliminating \( \phi(z) \) and \( \phi'(z) \) from the preceding equations, we reached the following result
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\[ |p'(z)| \leq \frac{1}{2(1-\beta)} \frac{n}{1-|z|^{2n}} \left( 1 - 2\beta + \frac{p(z)^2 - p(z) - 1^2}{1 - |z|^{2n}} \right) \]

\[ = \frac{2n}{1 - |z|^{2n}} \cdot \text{Re}(p(z) - \beta). \]

Thus the proof is completed.

**Theorem 1.** Let \( F(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{c+1}{c+k} \right) a_k z^k \in S^*_n, \)

Then \( f(z) = \left( \frac{1}{1+c} \right) z^{1-c} [z^c F(z)'] \) is starlike of order \( \alpha \) for

\[ |z| < r_0 = \left\{ \frac{-(1+n-\alpha) + \left( \frac{n^2 + 2n + \alpha^2 + c^2 + 2c\alpha - 2n\alpha}{c + 2\alpha - 1} \right)^{1/2}}{(1+n-\alpha) + \left( \frac{n^2 + 2n + \alpha^2 + c^2 - 2n\alpha}{2\alpha} \right)^{1/2}} \right\}, \]

if \( c = 2, 3, \ldots, \)

\[ = \left( \frac{1}{n+1} \right)^{1/n} \]

if \( c = 1 \) and \( \alpha = 0, \)

\[ = \left\{ \frac{-(1+n-\alpha) + \left( \frac{n^2 + 2n + \alpha^2}{2\alpha} \right)^{1/2}}{(1+n-\alpha) + \left( \frac{n^2 + 2n + \alpha^2}{2\alpha} \right)^{1/2}} \right\}^{1/n}, \]

if \( c = 1 \) and \( 0 < \alpha < 1. \)

This result is sharp.

**Proof.** Since \( F(z) \in S^*_n, \) \( \text{Re}\{zF'(z)/F(z)\} > \alpha \) for \( |z| < 1. \)

Hence there exists a function \( \omega(z) \) regular in \( |z| < 1, \)

with \( \omega(0) = 0, |\omega(z)| \leq |z|^n. \) In this case,

\[ \frac{z^c F'(z)}{F(z)} = \frac{z^c f(z) - c \int_0^z t^{c-1} f(t)dt}{\int_0^z t^{c-1} f(t)dt} = \frac{1 - (1-2\alpha) \omega(z)}{1 + \omega(z)} \]

for \( |z| < 1. \) Solving for \( f(z), \)

\[ f(z) = \frac{c+1 + (c-1)\omega(z)}{(1+\omega(z))z^c} \left\{ \int_0^3 t^{c-1} f(t)dt \right\}. \]

For this function we have

\[ \frac{zf'(z)}{f(z)} - \alpha = (1-\alpha) \left\{ \frac{1-\omega(z)}{1+\omega(z)} - \frac{2\omega(z)}{(1+\omega(z))(1+c+(c+2\alpha-1)\omega(z))} \right\}. \]

To show that \( f(z) \) is starlike of order \( \alpha \) for \( |z| < r_0, \) we must show that

\[ \text{Re}\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > 0 \text{ for } |z| < r_0. \] This condition is equivalent to the following inequality
(2) \[ \text{Re} \left\{ \frac{2z\omega'(z)}{(1+\omega(z))(1+c+(c+2\alpha-1)\omega(z))} \right\} < \text{Re} \left\{ \frac{1-\omega(z)}{1+\omega(z)} \right\} \]
for \( |z| < r_0 \).

Here \( \text{Re} \left\{ \frac{1-\omega(z)}{1+\omega(z)} \right\} = \text{Re} \left\{ \frac{(1-\omega(z))(1+\omega(z))}{|1+\omega(z)|^2} \right\} \]
\[ = \frac{1-|\omega(z)|^2}{|1+\omega(z)|^2}, \]
and the left hand side of (2)

(3) \[ \leq \frac{2|z||\omega'(z)|}{1+\omega(z)\left|1+c+(c+2\alpha-1)\omega(z)\right|}. \]

Applying (1), the right hand side of (3) does not exceed the expression

\[ \frac{2n|z|^n(1-|\omega(z)|^2)}{(1-|z|^{2n})|1+\omega(z)|^2|1+c+(c+2\alpha-1)\omega(z)|}. \]

Therefore the inequality (2) will be satisfied if

(4) \[ \frac{2nz^n}{(1-|z|^{2n})} < \frac{|1+c+(c+2\alpha-1)\omega(z)|}{1+\omega(z)}. \]

Noticing
\[ \frac{1+c+(c+2\alpha-1)\omega(z)}{1+\omega(z)} \geq \frac{1+c+(c+2\alpha-1)z^n}{1+|z|^n}, \]
(4) will be satisfied if.
\[ \frac{1+c+(c+2\alpha-1)z^n}{1+|z|^n} > \frac{2nz^n}{1-|z|^{2n}}. \]

Putting \( |z| = r < 1 \), this reduced to
\[ 1 + c - 2(1+n-\alpha) r^n - (c+2\alpha-1) r^{2n} > 0 \]
which gives the required root \( r_0 \) of Theorem 1.

To show that the result is sharp for each \( c \), consider the function
\[ F(z) = z \exp \left\{ 2(1-\alpha) \int_0^{\frac{z}{1-t^n}} \frac{t^{n-1}}{1-t^n} \right\} \in S_{\alpha}^* \]

For this function we have
\[ f(z) = \frac{1}{1+c} \left( z \left( 1+c+2(1-\alpha) \right) \frac{z^n}{1-z^n} \right) \exp \left\{ 2(1-\alpha) \int_0^{\frac{z}{1-t^n}} \frac{t^{n-1}}{1-t^n} \right\} \]
\[ \frac{zf'(z)}{f(z)} = \frac{(1-\alpha)}{1-z^n} \frac{(c+1+2(1-\alpha+n)z^n-(c+2\alpha-1)z^{2n})}{(c+1-(c+2\alpha)z^n)}, \]
so that \( \frac{zf'(z)}{f(z)} - \alpha = 0 \) for \( z = -r_0 \). Thus \( f(z) \) is not starlike in any circle
\( |z| < r \ (r > r_0) \). Thus the proof is completed.
Theorem 2. Let $F(z) \in k_{\alpha}$, $f(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)'$, $c = 1, 2, 3, \ldots$. Then $f(z)$ is convex of order $\alpha$ for $|z| < r_0$, where $r_0$ is defined as all the same as in Theorem 1. This result is sharp.

Proof. Since $F(z) \in k_{\alpha}$, $zF'(z) \in S^*_{\alpha}$. Therefore, by Theorem I, $zf'(z)$ is starlike of order $\alpha$ for $|z| < r_0$ and hence $f(z)$ is convex of order $\alpha$ for $|z| < r_0$.

This result is sharp for each $c$ applying the function

$$F(z) = \int_0^z \frac{1}{(1-t^n)^{2(1-\alpha)/n}} \, dt \in K_{\alpha}.$$ 

For this function we have

$$f(z) = \frac{1}{1+c} \left\{ c \int_0^z \frac{1}{(1-t^n)^{2(1-\alpha)/n}} \, dt + \left( -\frac{z}{(1-z^n)^{2(1-\alpha)/n}} \right) \right\}.$$ 

Therefore a computation yields,

$$f(z) = \frac{z f'(z)}{f''(z)} - (1-\alpha) \frac{1+c+2(1-\alpha+n)z^n-(c+2\alpha-1)z^n}{(1-z^n)(1+cz^n+2(1-\alpha)nz^n)}.$$ 

As the expression $(1-\alpha) + \frac{z f''(z)}{f'(z)}$ vanishes for $z = -r_0$, $f(z)$ is not convex of order $\alpha$ in any circle $|z| < r$, $r > r_0$. Thus the proof is completed.

Theorem 3. Let $F(z) \in C(\beta, \alpha)$ with respect to $G(z) \in S^*_{\alpha}$, $f(z) = \left( \frac{1}{1+c} \right) z^{1-c} F(z)'$, $g(z) = \left( \frac{1}{1+c} \right) z^{1-c} G(z)'$, $c = 1, 2, \ldots$. Then $f(z) \in C(\beta, \alpha)$ with respect to $g(z) \in S^*_{\alpha}$ for $|z| < r_0$, where $r_0$ is the same as in Theorem 1. This result is sharp.

Proof. Since $F(z) \in C(\beta, \alpha)$, there exists a $G(z) \in S^*_{\alpha}$ for $|z| < 1$. we have

$$\text{Re} \left\{ \frac{zF'(z)}{G(z)} \right\} > \beta.$$ 

Also, from Theorem 1, $\text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > \alpha$ for $|z| < r_0$.

we have

$$\frac{zF'(z)}{G(z)} = \left\{ z^c f(z) - c \int_0^z t^{c-1} f(t) \, dt \right\} / \int_0^z t^{c-1} g(t) \, dt.$$ 

By virtue of (5),

$$\frac{zF'(z)}{G(z)} = (1 - (1-2\beta) z^n \phi(z)) / (1+z^n \phi(z)).$$
In this case, \( \phi(z) \) is analytic and satisfying \( |\phi(z)| \leq 1 \) for \( |z| < 1 \). Therefore, we can write

\[
(6) \quad p(z) \int_{0}^{z} t^{c-1} g(t) dt = z^c f(z) - c \int_{0}^{z} t^{c-1} f(t) dt.
\]

In this case,

\[
p(z) = z F'(z) / G(z) = (1 - (1 - 2\beta) z^n \phi(z)) / (1 + z^n \phi(z)).
\]

By differentiating (6) with respect to \( z \), we obtain

\[
\frac{z f'(z)}{g(z)} = p(z) + \frac{p'(z)}{g(z)} z^{1-c} \int_{0}^{z} t^{c-1} g(t) dt.
\]

Hence we have

\[
(7) \quad \Re \left\{ \frac{z f'(z)}{g(z)} - \beta \right\} = \Re \left\{ p(z) - \beta \right\} + \Re \left\{ \frac{z \int_{0}^{z} t^{c-1} g(t) dt}{z^c g(z)} \right\}.
\]

Also, we can write

\[
(8) \quad \left| \int_{0}^{z} t^{c-1} g(t) dt \right| \leq \frac{(1 + |z|^n)}{(1 + (c + 2\alpha - 1) |z|^n)}.
\]

From (7) and (8), using Lemme, we get

\[
(9) \quad \Re \left\{ \frac{z f'(z)}{g(z)} - \beta \right\} \geq \Re \left\{ p(z) - \beta \right\} - \left| p'(z) \right| \frac{z \int_{0}^{z} t^{c-1} g(t) dt}{z^c g(z)} \leq \left\{ \frac{1 + c - 2(1 + n - \alpha) z^n - (c + 2\alpha - 1) |z|^n}{(1 - |z|^n)(1 + c + (c + 2\alpha - 1) |z|^n)} \right\} \Re \left\{ p(z) - \beta \right\}.
\]

The right hand side of (9) is positive, if \( |z| < r_0 \). This implies that \( f(z) \in C(\beta, \alpha) \) with respect to the function \( g(z) \). Sharpness of the theorem follows from Theorem 2. The proof of the theorem is complete.

If we take \( n=1 \) in Theorem 1, 2, 3 then the Theorems of Bajpai and Srivastava
(1) is obtained as a corollary to Theorem 1, 2, 3.

References


