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On the radius of starlikeness and convexity
for some univalent functions

By

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Abstract

In this paper we have shown that if \( F(z) = \sum_{k=0}^{\infty} \frac{c+k}{c+k} - a_k z^k \)

\[ = (c+1) z - c \int_0^z t^{c-1} f(t) dt, \quad c=1, 2, \ldots, \in \mathcal{S}_\alpha^* \], \( \mathcal{K}_\alpha \), or \( \mathcal{C}(\beta, \alpha) \) then \( f(z) \) is also the

same class for \( |z| < r_0 \) indicating the proof of the theorems. And our results are the
generalization of the theorems of Bajpai and Srivastava [1].

1. Introduction

By \( \mathcal{S} \) we denote the class of normalized functions \( f(z) \) which are regular and
univalent in the unit open disk \( \mathbb{E} \{ |z| < 1 \} \), while \( \mathcal{S}^* \) denotes the class of
functions in \( \mathcal{S} \) which map \( \mathbb{E} \) onto a starlike domain with respect to the origin.
By \( \mathcal{S}_\alpha^* \) we denote the class of functions \( f(z) \) in \( \mathcal{S}^* \) having the additional property
\( \Re\{ z f'(z)/f(z) \} > \alpha \), \( z \in \mathbb{E} \), \( 0 \leq \alpha < 1 \). The class of functions \( f(z) \), which are
in \( \mathcal{S} \) and map \( \mathbb{E} \) onto a convex domain, is denoted by \( \mathcal{K} \), while \( \mathcal{K}_\alpha \) denote the
class of convex functions of order \( \alpha \), if \( \Re\{1 + z f''(z)/f'(z)\} > \alpha \), \( z \in \mathbb{E} \), \( 0 \leq \alpha < 1 \).
If \( f(z) \in \mathcal{S} \) and \( g(z) \in \mathcal{S}_\alpha^* \) satisfy the condition \( \Re\{ z f'(z)/g(z) \} > \beta \), \( z \in \mathbb{E} \), \( 0 \leq \beta < 1 \), then \( f(z) \) is said to be a close-to-convex functions of order \( \beta \) and
type \( \alpha \). We denote this class by \( \mathcal{C}(\beta, \alpha) \). This concept of close-to-convex
functions on \( \mathcal{C}(0, 0) \) is due to Kaplan [5]. Libera [6] proved that if \( f(z) \) belongs
to \( \mathcal{S}^* \), \( \mathcal{K} \) or \( \mathcal{C}(0, 0) \) then the function \( F(z) = (2/z) \int_0^z f(t) dt \) also belongs to \( \mathcal{S}^* \),
\( \mathcal{K} \) or \( \mathcal{C}(0, 0) \) respectively. Bernardi [2] further proved that if \( f(z) \) belongs to
\( \mathcal{S}^* \), \( \mathcal{K} \) or \( \mathcal{C}(0, 0) \) then the function \( F(z) = (c+1) z - c \int_0^z t^{c-1} f(t) dt \) also belongs
to \( \mathcal{S}^* \), \( \mathcal{K} \) or \( \mathcal{C}(0, 0) \), \( C = 1, 2, \ldots \). Livingston [7] has studied the converse
problem and obtained sharp results. He has shown that if $F(z)$ is a member of $S^*$, K or $C(o, o)$ then $f(z) = (zF(z))'/2$ is starlike, convex or close-to-convex respectively for $|z| < \frac{1}{2}$. Furthermore YoshiKai [9] generalized the Livingstone’s results. Bernardi [3] again considered the converse problem and obtained sharp results. He has shown that if $F(z) = \sum_{n=1}^{\infty} \frac{c+1}{c+n} a_n z^n = (c+1) z^{-c} \int_0^z t^{c-1} f(t) dt$ $(a_1=1)$ is a member of $S^*$, K or $C(o, o)$ then $f(z)$ is starlike, convex or close-to-convex respectively for $|z| < (2+(3+c^2)^{1/2})/(c-1)$ for $c=2, 3, \ldots$ and $|z| < \frac{1}{2}$ for $c=1$. Padmanabhan [8] considered the converse problem of Libera for the class $S_\beta^*$, $K_\alpha$ and $C(\beta, \alpha)$, $0 \leq \beta < \frac{1}{2}$. Recently Bajpai and Srivastava [1] generalized the Bernardi’s results. They proved that if $F(z) = \sum_{n=1}^{\infty} \frac{c+1}{c+n} a_n z^n (a_1=1)$ is a member of $S_\beta^*$, $K_\alpha$ and $C(\beta, \alpha)$ in which $0 \leq \alpha < 1, 0 \leq \beta < 1$, then $f(z)$ is $S_\beta^*$, $K_\alpha$ and $C(\beta, \alpha)$ respectively for $|z| < -\frac{(2-\alpha)+(3+2c^2+2c\alpha^2-2\alpha\beta)}{c+2\alpha-1}$, for $c=2, 3, \ldots$.

In this paper we generalize the above theorems for functions with whose power series begins $F(z) = \sum_{k=n+1}^{\infty} \frac{c+1}{c+k} a_k z^k, c=1, 2, \ldots$.

2. Proof of the Theorems

First we state the following lemma which we shall need.

**LEMMA.** If $p(z) = 1 + a_n z^n + \ldots$ is analytic and satisfies $\text{Re} \ p(z) > \beta$, $0 \leq \beta < 1$, for $|z| < 1$, then

$$|p'(z)| \leq \frac{2n \cdot |z|^{n-1}}{1-|z|^2} \text{Re} \ {p(z) - \beta}.$$  

**PROOF.** Let $\phi(z) = \frac{p(z)-1}{p(z)+1-2\beta}$, where $\phi(z)$ is analytic and $|\phi(z)| < 1$. For such functions it is known that

$$|\phi'(z)| \leq \frac{n |z|^{n-1}}{1-|z|^2} \ (1-|\phi(z)|^2). \ [4].$$

Also

$$|\phi'(z)| = \frac{2(1-\beta) |p'(z)|}{1-2\beta + p(z)^2}.$$  

Eliminating $\phi(z)$ and $\phi'(z)$ from the preceding equations, we reached the following result
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\[ \left| p'(z) \right| \leq \frac{1}{2(1-\beta)} \frac{n}{1-|z|^2n} \left( 1 - 2\beta + p(z)^2 - |p(z) - 1|^2 \right) \]

\[ = \frac{2n}{1-|z|^2n} \cdot \text{Re}(p(z) - \beta). \]

Thus the proof is completed.

**Theorem 1.** Let \( F(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{c+1}{c+k} \right) a_k z^k \in S^a \).

Then \( f(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)' \) is starlike of order \( \alpha \) for

\[ |z| < r_0 = \left\{ \frac{-\left( 1 + n - \alpha \right) + (n^2 + 2n + \alpha^2 + c^2 + 2c\alpha - 2n\alpha)^{1/2}}{c + 2\alpha - 1} \right\}^{1/n}, \]

if \( c = 2, 3, \ldots \),

\[ = \left( \frac{1}{n+1} \right)^{1/n} \]

if \( c = 1 \) and \( \alpha = 0 \),

\[ = \left\{ \frac{-\left( 1 + n - \alpha \right) + \left( \frac{(1 + n - \alpha)^2 + 4\alpha^2}{2\alpha} \right)^{1/2}}{2\alpha} \right\}^{1/n}, \]

if \( c = 1 \) and \( 0 < \alpha < 1 \).

This result is sharp.

**Proof.** Since \( F(z) \in S^a \), \( \text{Re} \left\{ zF'(z)/F(z) \right\} > \alpha \) for \( |z| < 1 \).

Hence there exists a function \( \omega(z) \) regular in \( |z| < 1 \),

with \( \omega(0) = 0 \), \( |\omega(z)| \leq |z|^n \). In this case,

\[ z^c f(z) - c \int_0^z t^{c-1} f(t) dt \]

\[ = \int_0^z t^{c-1} f(t) dt \]

for \( |z| < 1 \). Solving for \( f(z) \),

\[ f(z) = \frac{c+1+(c-1)\omega(z)}{(1+\omega(z))^2} \left\{ \int_0^z t^{c-1} f(t) dt \right\}. \]

For this function we have

\[ \frac{zF'(z)}{f(z)} - \alpha = (1-\alpha) \left\{ \frac{1-\omega(z)}{1+\omega(z)} \right\} \frac{2z\omega'(z)}{(1+\omega(z))(1+c+(c+2\alpha-1)\omega(z))}. \]

To show that \( f(z) \) is starlike of order \( \alpha \) for \( |z| < r_0 \), we must show that \( \text{Re} \left\{ \frac{zF'(z)}{f(z)} - \alpha \right\} > 0 \) for \( |z| < r_0 \). This condition is equivalent to the following inequality.
(2) \[ \text{Re} \left\{ \frac{2z\omega'(z)}{(1+\omega(z))(1+c+(c+2\alpha-1)\omega(z))} \right\} < \text{Re} \left\{ \frac{1-\omega(z)}{1+\omega(z)} \right\} \]

for \(|z|<r_0\).

Here \( \text{Re} \left\{ \frac{1-\omega(z)}{1+\omega(z)} \right\} = \text{Re} \left\{ \frac{(1-\omega(z))(1+\bar{\omega}(z))}{|1+\omega(z)|^2} \right\} \)

\[ = \frac{1-|\omega(z)|^2}{1+|\omega(z)|^2}, \]

and the left hand side of (2)

(3) \[ \frac{2|z| |\omega'(z)|}{1+\omega(z)|1+c+(c+2\alpha-1)\omega(z)|}. \]

Applying (1), the right hand side of (3) does not exceed the expression

\[ \frac{2n|z|^n(1-|\omega(z)|^2)}{(1-|z|^{2n}) |1+\omega(z)| |1+c+(c+2\alpha-1)\omega(z)|}. \]

Therefore the inequality (2) will be satisfied if

(4) \[ \frac{2n|z|^n}{(1-|z|^{2n})} < \frac{|1+c+(c+2\alpha-1)\omega(z)|}{1+\omega(z)}. \]

Noticing

\[ \frac{1+c+(c+2\alpha-1)\omega(z)}{1+\omega(z)} \geq \frac{1+c+(c+2\alpha-1)|z|^n}{1+|z|^n}, \]

(4) will be satisfied if.

\[ \frac{1+c+(c+2\alpha-1)|z|^n}{1+|z|^n} > \frac{2n|z|^n}{1-|z|^{2n}}. \]

Putting \(|z| = r < 1\), this reduced to

\[ 1 + c - 2(1+n-\alpha) r^n - (c+2\alpha-1) r^{2n} > 0 \]

which gives the required root \( r_0 \) of Theorem 1.

To show that the result is sharp for each \( c \), consider the function

\[ F(z) = z \exp \left\{ 2(1-\alpha) \int_0^z \frac{t^{n-1}}{1-t^n} \, dt \right\} \in S_\alpha^* \]

For this function we have

\[ f(z) = \frac{1}{1+c} \left[ z \left( 1+c+2(1-\alpha)\frac{z^n}{1-z^n} \right) \right] \exp \left\{ 2(1-\alpha) \int_0^z \frac{t^{n-1}}{1-t^n} \, dt \right\} \]

\[ \frac{zf'(z)}{f(z)} - \alpha = \frac{(1-\alpha) z^n - (c+1+2(1-\alpha+n)z^n-(c+2\alpha-1)z^{2n})}{c+1-(c+2\alpha)z^n}, \]

so that \( \frac{zf'(z)}{f(z)} - \alpha = 0 \) for \( z = -r_0 \). Thus \( f(z) \) is not starlike in any circle

\(|z|<r (r>r_0)\). Thus the proof is completed.
Theorem 2. Let \( F(z) \in k_\alpha, f(z) = \left( -\frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)' \), \( c = 1, 2, 3, \ldots \). Then \( f(z) \) is convex of order \( \alpha \) for \( |z| < r_o \), where \( r_o \) is defined as all the same as in Theorem 1. This result is sharp.

Proof. Since \( F(z) \in k_\alpha, zF'(z) \in S_{\alpha}^* \). Therefore, by Theorem I, \( zf'(z) \) is starlike of order \( \alpha \) for \( |z| < r_o \) and hence \( f(z) \) is convex of order \( \alpha \) for \( |z| < r_o \).

This result is sharp for each \( c \) applying the function

\[
F(z) = \int_0^z \frac{1}{(1-t^n)^{2(1-\alpha)/n}} \, dt \in K_\alpha.
\]

For this function we have

\[
f(z) = \frac{1}{1+c} \left\{ c \int_0^z \frac{1}{(1-t^n)^{2(1-\alpha)/n}} \, dt + \frac{z}{(1-z^n)^{2(1-\alpha)/n}} \right\}.
\]

Therefore a computation yields,

\[
(1-\alpha) + \frac{zf'(z)}{f'(z)} = (1-\alpha) - \frac{1+c+2(1-\alpha+n)z^n-(c+2\alpha-1)z^2n}{(1-z^n)((1+c)(1-z^n)+2(1-\alpha)z^n)}.
\]

As the expression \( (1-\alpha) + \frac{zf'(z)}{f'(z)} \) vanishes for \( z = -r_o \), \( f(z) \) is not convex of order \( \alpha \) in any circle \( |z| < r, r > r_o \). Thus the proof is completed.

Theorem 3. Let \( F(z) \in C(\beta, \alpha) \) with respect to \( G(z) \in S_{\alpha}^* \),

\[
f(z) = \left( -\frac{1}{1+c} \right) z^{1-c} [z^c F(z)]', \quad g(z) = \left( -\frac{1}{1+c} \right) z^{1-c} [z^c G(z)]', \quad c = 1, 2, \ldots \quad \text{Then}
\]

\( f(z) \in C(\beta, \alpha) \) with respect to \( g(z) \in S_{\alpha}^* \) for \( |z| < r_o \), where \( r_o \) is the same as in Theorem 1. This result is sharp.

Proof. Since \( F(z) \in C(\beta, \alpha) \), there exists a \( G(z) \in S_{\alpha}^* \) for \( |z| < 1 \). we have

\[
\text{Re} \left\{ zF'(z)/G(z) \right\} > \beta.
\]

Also, from Theorem 1, \( \text{Re} \left\{ zg'(z)/g(z) \right\} > \alpha \) for \( |z| < r_o \).

we have

\[
zF'(z)/G(z) = \left\{ z^c f(z) - c \int_0^z t^{c-1} f(t) \, dt \right\} / \int_0^z t^{c-1} g(t) \, dt.
\]

By virtue of (5),

\[
zF'(z)/G(z) = (1 - (1-2\beta) z^n \phi(z)) / (1+z^n \phi(z)).
\]
In this case, $\phi(z)$ is analytic and satisfying $|\phi(z)| \leq 1$
for $|z| < 1$. Therefore, we can write

$$p(z) \int_0^z t^{c-1} g(t) dt = z^c f(z) - c \int_0^z t^{c-1} f(t) dt.$$  

In this case,

$$p(z) = z F'(z)/G(z) = (1-(1-2\beta)z^n \phi(z))/(1+z^n \phi(z)).$$

By differentiating (6) with respect to $z$, we obtain

$$\frac{zf'(z)}{g(z)} = p(z) + \frac{p'(z)}{g(z)} z^{1-c} \int_0^z t^{c-1} g(t) dt.$$  

Hence we have

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} - \beta \right\} = \text{Re}\{p(z) - \beta\} + \text{Re}\left\{ \frac{z \int_0^z t^{c-1} g(t) dt}{z^c g(z)} \right\}.$$  

Also, we can write

$$z \int_0^z t^{c-1} g(t) dt \leq \frac{z(1+z^n \phi(z))}{c+1+(c+2\alpha-1) z^n \phi(z)}.$$  

From (7) and (8), using Lemma, we get

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} - \beta \right\} \geq \text{Re}\{p(z) - \beta\} - \left| p'(z) \right| \frac{z \int_0^z t^{c-1} g(t) dt}{z^c g(z)}.$$  

The right hand side of (9) is positive, if $|z| < r_0$. This implies that $f(z) \in C(\beta, \alpha)$ with respect to the function $g(z)$. Sharpness of the theorem follows from Theorem 2. The proof of the theorem is complete.

If we take $n=1$ in Theorem 1, 2, 3 then the Theorems of Bajpai and Srivastava
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(1) is obtained as a corollary to Theorem 1, 2, 3.

References