<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>補1</td>
<td>長崎大学教養部紀要 (自然科学) に掲載された内容</td>
</tr>
<tr>
<td>補2</td>
<td>補3</td>
</tr>
</tbody>
</table>

補1節では、長崎大学教養部紀要（自然科学）に掲載された内容について説明しています。
On the radius of starlikeness and convexity
for some univalent functions

By

Toshiaki YOSHIKAI

(Received September 25, 1972)

Abstract

In this paper we have shown that if \( F(z) = \sum_{k=n+1}^{\infty} \frac{c+1}{c+k} a_k z^k \)

\[ = (c+1) z - \int_0^z t^{c-1} \int f(t) \, dt \, dz, \quad c=1, 2, \ldots, \in S^*_\alpha, \quad K \alpha, \quad \text{or} \quad C(\beta, \alpha) \]

then \( f(z) \) is also in the same class for \( |z| < r_0 \) indicating the proof of the theorems. And our results are the generalization of the theorems of Bajpai and Srivastava [1].

1. Introduction

By \( S \) we denote the class of normalized functions \( f(z) \) which are regular and univalent in the unit open disk \( E \{ |z| < 1 \} \), while \( S^* \) denotes the class of functions in \( S \) which map \( E \) onto a starlike domain with respect to the origin. By \( S^*_\alpha \) we denote the class of functions \( f(z) \) in \( S^* \) having the additional property \( \Re \{ z f'(z) / g(z) \} > \alpha, \quad z \in E, \quad 0 \leq \alpha < 1 \). The class of functions \( f(z) \), which are in \( S \) and map \( E \) onto a convex domain, is denoted by \( K \), while \( K \alpha \) denote the class of convex functions of order \( \alpha \), if \( \Re \{ 1 + z f''(z) / f'(z) \} > \alpha, \quad z \in E, \quad 0 \leq \alpha < 1 \). If \( f(z) \in S \) and \( g(z) \in S^*_\alpha \) satisfy the condition \( \Re \{ z f'(z) / g(z) \} > \beta, \quad z \in E, \quad 0 \leq \beta < 1 \), then \( f(z) \) is said to be a close-to-convex functions of order \( \beta \) and type \( \alpha \). We denote this class by \( C(\beta, \alpha) \). This concept of close-to-convex functions on \( C(0, 0) \) is due to Kaplan [5]. Libera [6] proved that if \( f(z) \) belongs to \( S^* \), \( K \) or \( C(0, 0) \) then the function \( F(z) = \frac{2}{z} \int f(t) \, dt \) also belongs to \( S^*, \quad K \) or \( C(0, 0) \) respectively. Bernardi [2] further proved that if \( f(z) \) belongs to \( S^*, \quad K \) or \( C(0, 0) \) then the function \( F(z) = (c+1) z - c \int f(t) \, dt \) also belongs to \( S^*, \quad K \) or \( C(0, 0), \quad C = 1, 2, \ldots \). Livingston [7] has studied the converse
problem and obtained sharp results. He has shown that if \( F(z) \) is a member of \( S^* \), \( K \) or \( C(0,0) \) then \( f(z) = (zF(z))'/2 \) is starlike, convex or close-to-convex respectively for \( |z| < \frac{1}{2} \). Furthermore YoshiKai [9] generalized the Livingstone’s results. Bernardi [3] again considered the converse problem and obtained sharp results. He has shown that if \( F(z) = \sum_{n=0}^{\infty} \left( \frac{c+1}{c+n} \right) a_n z^n = (c+1) z^{-c} \int_0^z f(t) dt \) \( (a_1 = 1) \) is a member of \( S^* \), \( K \) or \( C(0,0) \) then \( f(z) \) is starlike, convex or close-to-convex respectively for \( |z| < (2 + (3 + c^2)^{1/2})/(c-1) \) for \( c = 2, 3, \ldots \) and \( |z| < \frac{c}{2} \) for \( c = 1 \). Padmanabhan [8] considered the converse problem of Libera for the class \( S^*_\beta \), \( K_\alpha \) and \( C(\beta, \alpha) \), \( 0 \leq \beta < \frac{1}{2} \). Recently Bajpai and Srivastava [1] generalized the Bernardi’s results. They proved that if \( F(z) = \sum_{n=1}^{\infty} \left( \frac{c+1}{c+n} \right) a_n z^n \) \( (a_1 = 1) \) is a member of \( S^*_\beta \), \( K_\alpha \) and \( C(\beta, \alpha) \) in which \( 0 \leq \alpha < 1, 0 \leq \beta < 1 \), then \( f(z) \) is \( S^*_\beta \), \( K_\alpha \) and \( C(\beta, \alpha) \) respectively for

\[
|z| < \frac{-(2-\alpha) + (3 + \alpha^2 + c^2 + 2c\alpha - 2\alpha)^{1/2}}{c + 2\alpha - 1}, \quad \text{for } c = 2, 3, \ldots,
\]

\[
< \frac{1}{2}, \quad \text{for } c = 1 \text{ and } \alpha = 0,
\]

\[
< \frac{-(2-\alpha) + (4 + \alpha^2)^{1/2}}{2\alpha}, \quad \text{for } c = 1 \text{ and } 0 < \alpha < 1.
\]

In this paper we generalize the above theorems for functions with whose power series begins \( F(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{c+1}{c+k} \right) a_k z_k, c = 1, 2, \ldots \).

2. Proof of the Theorems

First we state the following lemma which we shall need.

**Lemma.** If \( p(z) = 1 + a_n z^n + \ldots \) is analytic and satisfies \( \Re p(z) > \beta, 0 \leq \beta < 1 \), for \( |z| < 1 \), then

\[
|p'(z)| \leq \frac{2n |z|^{n-1}}{1 - |z|^2} \Re \{p(z) - \beta\}.
\]

**Proof.** Let \( \phi(z) = \frac{p(z) - 1}{p(z) + 1 - 2\beta} \), where \( \phi(z) \) is analytic and \( |\phi(z)| < 1 \). For such functions it is known that

\[
|\phi'(z)| \leq \frac{n |z|^{n-1}}{1 - |z|^2} \Re \{1 - |\phi(z)|^2\}. \quad (4).
\]

Also

\[
|\phi'(z)| = \frac{2(1-\beta) |p'(z)|}{1 - 2\beta + |p(z)|^2}.
\]

Eliminating \( \phi(z) \) and \( \phi'(z) \) from the preceding equations, we reached the following result
On the radius of starlikeness and convexity for some univalent functions

\[ |p'(z)| \leq \frac{1}{2(1-\beta)} \cdot \frac{n \cdot z^{n-1}}{1-|z|^{2n}} \cdot \left( 1 - \frac{2\beta + |p(z)|^2 - |p(z)-1|^2}{1-|z|^{2n}} \right) \]

Thus the proof is completed.

**Theorem 1.** Let \( F(z) = z + \sum_{k=n+1}^{\infty} \left( \frac{c+1}{c+k} \right) a_k z^k \in S^*_n \).

Then \( f(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)' \) is starlike of order \( \alpha \) for

\[ |z| < r_0 = \left\{ \left( \frac{1 + n - \alpha + (n^2 + 2n + \alpha^2 + c^2 + 2c\alpha - 2n\alpha)^{1/2}}{c + 2\alpha - 1} \right)^{1/n} \right. \]

if \( c = 2, 3, \ldots, \)

\[ = \left( \frac{1}{n+1} \right)^{1/n} \quad \text{if } c = 1 \text{ and } \alpha = 0, \]

\[ = \left( \frac{1}{n+1} \right)^{1/n} \left( \frac{1 + n - \alpha + (1 + n - \alpha)^2 + 4\alpha}{2\alpha} \right)^{1/2} \]

if \( c = 1 \) and \( 0 < \alpha < 1. \)

This result is sharp.

**Proof.** Since \( F(z) \in S^*_n \), \( \Re \left\{ zF'(z)/F(z) \right\} > \alpha \) for \( |z| < 1. \)

Hence there exists a function \( \omega(z) \) regular in \( |z| < 1, \)

with \( \omega(0) = 0, \quad |\omega(z)| \leq |z|^n. \) In this case,

\[ \frac{z^c f(z) - c \int_0^z t^{c-1} f(t) dt}{z^c F(z)} = \frac{z^c f(z) - c \int_0^z t^{c-1} f(t) dt}{z^c F(z)} = \frac{1 - (1-2\alpha)\omega(z)}{1+\omega(z)} \]

for \( |z| < 1. \) Solving for \( f(z), \)

\[ f(z) = \frac{c+1+c-1\omega(z)}{(1+\omega(z))z^c} \left\{ \int_0^z t^{c-1} f(t) dt \right\}. \]

For this function we have

\[ \frac{zf'(z)}{f(z)} = \alpha \left\{ 1 - \alpha \right\} \frac{1-\omega(z)}{1+\omega(z)} - \frac{2\omega'(z)}{(1+\omega(z))(1+c+(c+2\alpha-1)\omega(z))}. \]

To show that \( f(z) \) is starlike of order \( \alpha \) for \( |z| < r_0, \) we must show that

\[ \Re \left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > 0 \text{ for } |z| < r_0. \] This condition is equivalent to the following inequality
(2) \( \text{Re} \left\{ \frac{2z\omega'(z)}{(1 + \omega(z))(1 + c + (c + 2\alpha - 1)\omega(z))} \right\} < \text{Re} \left\{ \frac{1 - \omega(z)}{1 + \omega(z)} \right\} \)

for \( |z| < r_0 \).

Here \( \text{Re} \left\{ \frac{1 - \omega(z)}{1 + \omega(z)} \right\} = \text{Re} \left\{ \frac{(1 - \omega(z))(1 + \bar{\omega}(z))}{|1 + \omega(z)|^2} \right\} \)

\[ = \frac{1 - |\omega(z)|^2}{1 + |\omega(z)|^2}, \]

and the left hand side of (2)

(3) \[ \leq \frac{2|z|\omega'(z)}{1 + \omega(z)\|1 + c + (c + 2\alpha - 1)\omega(z)\|}. \]

Applying (1), the right hand side of (3) does not exceed the expression

\[ \frac{2n|z|^n(1 - |\omega(z)|^2)}{(1 - |z|^{2n})\|1 + \omega(z)\|\|1 + c + (c + 2\alpha - 1)\omega(z)\|}. \]

Therefore the inequality (2) will be satisfied if

(4) \[ \frac{2n|z|^n}{(1 - |z|^{2n})} < \frac{1 + c + (c + 2\alpha - 1)\omega(z)}{1 + \omega(z)}. \]

Noticing

\[ \frac{1 + c + (c + 2\alpha - 1)\omega(z)}{1 + \omega(z)} \geq \frac{1 + c + (c + 2\alpha - 1)|z|^n}{1 + |z|^n}, \]

(4) will be satisfied if.

\[ - \frac{1 + c + (c + 2\alpha - 1)z^n}{1 + |z|^n} > \frac{2n|z|^n}{1 - |z|^{2n}}. \]

Putting \( |z| = r < 1 \), this reduced to

\[ 1 + c - 2(1 + n - \alpha)rn - (c + 2\alpha - 1)r^{2n} > 0 \]

which gives the required root \( r_0 \) of Theorem 1.

To show that the result is sharp for each \( c \), consider the function

\[ F(z) = z \exp \left\{ 2(1 - \alpha) \int_0^z \frac{t^{n-1}}{1 - t^n} \, dt \right\} \subseteq S_\alpha. \]

For this function we have

\[ f(z) = \frac{1}{1 + c} \left\{ z \left( (1 + c + 2(1 - \alpha)) - \frac{z^n}{1 - z^n} \right) \exp \left\{ 2(1 - \alpha) \int_0^z \frac{t^{n-1}}{1 - t^n} \, dt \right\} \right\} \]

\[ \frac{zf'(z)}{f(z)} - \alpha = \frac{(1 - \alpha) - (c + 1 + 2(1 - \alpha + n)z^n - (c + 2\alpha - 1)z^{2n})}{1 - z^n}, \]

so that \( \frac{zf'(z)}{f(z)} - \alpha = 0 \) for \( z = -r_0 \). Thus \( f(z) \) is not starlike in any circle \( |z| < r \) (\( r > r_0 \)). Thus the proof is completed.
On the radius of starlikeness and convexity for some univalent functions

Theorem 2. Let \( F(z) \in \mathcal{K}_a \), \( f(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)' \), \( c = 1, 2, 3, \ldots \). Then \( f(z) \) is convex of order \( \alpha \) for \( |z| < r_0 \), where \( r_0 \) is defined as all the same as in Theorem 1. This result is sharp.

Proof. Since \( F(z) \in \mathcal{K}_a \), \( zF'(z) \in \mathcal{S}_a^* \). Therefore, by Theorem I, \( zf'(z) \) is starlike of order \( \alpha \) for \( |z| < r_0 \) and hence \( f(z) \) is convex of order \( \alpha \) for \( |z| < r_0 \).

This result is sharp for each \( c \) applying the function

\[
F(z) = \int_0^z \frac{1}{(1-t^n)^{2(1-a)/n}} \, dt \in \mathcal{K}_a.
\]

For this function we have

\[
f(z) = \frac{1}{1+c} \left\{ c \int_0^z \frac{1}{(1-t^n)^{2(1-a)/n}} \, dt + \left( \frac{-z}{1-z^n} \right)^{2(1-a)/n} \right\}.
\]

Therefore a computation yields,

\[
(1-\alpha) + \frac{zf'(z)}{f'(z)} = \frac{1+c+2(1-\alpha+n)z^n - (c+2\alpha-1)z^n}{(1-z^n)((1+c)(1-z^n)+2(1-\alpha)z^n)}.
\]

As the expression \( (1-\alpha) + \frac{zf'(z)}{f'(z)} \) vanishes for \( z = -r_0 \), \( f(z) \) is not convex of order \( \alpha \) in any circle \( |z| < r, r > r_0 \). Thus the proof is completed.

Theorem 3. Let \( F(z) \in C(\beta, \alpha) \) with respect to \( G(z) \in \mathcal{S}_a^* \).

\[
f(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c F(z) \right)' \), \( g(z) = \left( \frac{1}{1+c} \right) z^{1-c} \left( z^c G(z) \right)' \), \( c = 1, 2, \ldots \). Then \( f(z) \in C(\beta, \alpha) \) with respect to \( g(z) \in \mathcal{S}_a^* \) for \( |z| < r_0 \), where \( r_0 \) is the same as in Theorem 1. This result is sharp.

Proof. Since \( F(z) \in C(\beta, \alpha) \), there exists a \( G(z) \in \mathcal{S}_a^* \) for \( |z| < 1 \). we have

\[
\text{Re} \left\{ \frac{zF'(z)}{G(z)} \right\} > \beta.
\]

Also, from Theorem 1, \( \text{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} > \alpha \) for \( |z| < r_0 \).

we have

\[
zF'(z)/G(z) = \left\{ z^c f(z) - c \int_0^z t^{c-1} f(t) \, dt \right\} / \int_0^z t^{c-1} g(t) \, dt.
\]

By virtue of (5),

\[
zF'(z)/G(z) = (1 - (1-2\beta) z^n \phi(z) ) / (1 + z^n \phi(z)).
\]
In this case, $\phi(z)$ is analytic and satisfying $|\phi(z)| \leq 1$ for $|z| < 1$. Therefore, we can write

$$(6) \quad p(z) \int_0^z t^{c-1} g(t) \, dt = z^c f(z) - c \int_0^z t^{c-1} f(t) \, dt.$$ 

In this case,

$$p(z) = zF'(z)/G(z) = (1 - (1 - 2\beta)z^n \phi(z))/(1 + z^n \phi(z)).$$

By differentiating (6) with respect to $z$, we obtain

$$\frac{zf'(z)}{g(z)} = p(z) + \frac{p'(z)}{g(z)} z^{-c} \int_0^z t^{c-1} g(t) \, dt.$$ 

Hence we have

$$(7) \quad \text{Re} \left\{ \frac{zf'(z)}{g(z)} - \beta \right\}$$

$$= \text{Re} \{ p(z) - \beta \} + \text{Re} \left\{ \frac{p'(z)}{z^c g(z)} \right\}.$$ 

Also, we can write

$$\frac{z \int_0^z t^{c-1} g(t) \, dt}{z^c g(z)} = \frac{z(1 + z^n \phi(z))}{c + 1 + (c + 2\alpha - 1) z^n \phi(z)},$$

$$(8) \quad \left| \frac{\int_0^z t^{c-1} g(t) \, dt}{z^c g(z)} \right| \leq \frac{(1 + |z|^n)}{c + 1 + (c + 2\alpha - 1) |z|^n}.$$ 

From (7) and (8), using Lemme, we get

$$(9) \quad \text{Re} \left\{ \frac{zf'(z)}{g(z)} - \beta \right\}$$

$$\geq \text{Re} \{ p(z) - \beta \} - \left| \text{Re} \left\{ \frac{p'(z)}{z^c g(z)} \right\} \right|$$

$$\geq \left\{ \frac{1 + c - 2(1 + n - \alpha) \cdot z^n - (c + 2\alpha - 1) z^{2n}}{(1 - |z|^n)(1 + c + (c + 2\alpha - 1)|z|^n)} \right\} \text{Re} \{ p(z) - \beta \}.$$ 

The right hand side of (9) is positive, if $|z| < r_0$. This implies that $f(z) \in C(\beta, \alpha)$ with respect to the function $g(z)$. Sharpness of the theorem follows from Theorem 2. The proof of the theorem is complete.

If we take $n=1$ in Theorem 1, 2, 3 then the Theorems of Bajpai and Srivastava
On the radius of starlikeness and convexity for some univalent functions

(1) is obtained as a corollary to Theorem 1, 2, 3.

References