<table>
<thead>
<tr>
<th>Title</th>
<th>On certain classes of analytic functions in the unit disk*</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
</tbody>
</table>
On certain classes of analytic functions
in the unit disk

By

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Abstract

In this note we shall consider the problem of finding the radius of convexity in some
univalent region for functions \( f(z) = z + a_2 z^2 + \cdots \) which are analytic and satisfy
\[
\text{Re} \left\{ \frac{f'(z)}{(\lambda f'(z) + (1-\lambda)\phi'(z))} \right\} > \beta \quad \text{for } |z| < 1, \; 0 \leq \beta < 1, \; 0 \leq \lambda < 1
\]
where \( \phi(z) = z + b_2 z^2 + \cdots \) is analytic, univalent and convex of order \( \alpha, \; 0 \leq \alpha < 1. \)
Finally a distortion theorem for \( f(z) \) is shown.

1. Introduction

Let \( k(\alpha) \) denote the class of functions analytic in \( |z| < 1 \) and of the form
\[
\phi(z) = z + b_2 z^2 + \cdots,
\]
such that \( \text{Re} \left\{ z\phi''(z) / \phi'(z) + 1 \right\} > \alpha \) for \( |z| < 1 \) and \( 0 \leq \alpha < 1. \)
Then \( \phi(z) \) is said to be convex of order \( \alpha. \)
we say that an analytic function \( f(z) = z + a_2 z^2 + \cdots \) is in the class \( C(\alpha, \beta) \) if there
exists a function \( \phi(z) \in k(\alpha) \) such that
\[
\text{Re} \left\{ f'(z) / \phi'(z) \right\} > \beta, \; 0 \leq \beta < 1.
\]
Kaplan [2] proved that \( C(0, 0) \), the class of close-to-convex functions, is univalent in
\( |z| < 1 \).
In this paper we shall consider the radius of convexity of \( f(z) \) under the conditions
which are pointed out in abstract. And finally we state on the distortion theorems for
\( f(z) \).

2. Proof of the theorems

In proving the theorems we will make use of the following lemmas.

* Some of these results had been reported in the Kyushu branch of Mathematical
Lemma 1. (5). If \( h(z) = 1 + d_1 z + \cdots \) is analytic for \( |z| < 1 \) and \( \text{Re} \ h(z) > \alpha \), \( 0 \leq \alpha < 1 \), then

\[
\frac{1 - (1 - 2\alpha) |z|}{1 + |z|} \leq \text{Re} \ h(z) \leq \frac{1 + (1 - 2\alpha) |z|}{1 - |z|}.
\]

Lemma 2. (4). Let \( p(z) \) be analytic for \( |z| < 1 \), \( p(0) = 1 \). Then \( \text{Re} \ p(z) > \beta \), \( 0 \leq \beta < 1 \), if and only if

\[ p(z) = \frac{1 + (1 - 2\beta) \omega(z)}{1 - \omega(z)}, \]

where \( \omega(z) \) is analytic, \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) for \( |z| < 1 \).

Lemma 3. Let \( p(z) = 1 + c_1 z + \cdots \) is analytic and \( \text{Re} \ p(z) > \beta \), then \( 0 \leq \beta < 1 \), for \( |z| < 1 \).

\[
(1 - \lambda) |p(z)|^{-1} \leq \frac{1 - |z|}{1 - \lambda - (1 + (1 - 2\beta)\lambda) |z|},
\]

for \( |z| < \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} < 1 \), where \( 0 \leq \lambda < 1 \).

Proof. Using Lemma 1,

\[
\frac{1}{1 - \lambda} |p(z)| \leq \frac{1}{1 - \lambda} \frac{1}{1 + (1 - 2\beta) |z|} = \frac{1 - |z|}{1 - \lambda - (1 + (1 - 2\beta)\lambda) |z|}
\]

for \( |z| < \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} < 1 \).

Lemma 4. (3). With the same hypothesis as in Lemma 3, we have

\[
\left| \frac{z p'(z)}{p(z)} \right| \leq \frac{2\gamma}{(1 - \gamma) \left\{ 1 + \frac{\beta}{1 - \beta} (1 - \gamma) \right\}}
\]

for \( |z| = \gamma < 1 \).

Lemma 5. (1). Let \( \phi(z) \in k(\alpha) \). Then

\[
|\phi'(z)| \leq \frac{1}{(1 - \gamma)^2 (1 - \alpha)}, \quad |\phi'(z)| \geq \frac{1}{(1 + \gamma)^2 (1 - \alpha)}
\]

for \( |z| = \gamma < 1 \).

Theorem 1. Let \( f(z) = z + a_2 z^2 + \cdots \) be analytic for \( |z| < 1 \) and \( \phi(z) \in k(\alpha) \). If \( \text{Re} \left\{ f'(z) \left/ \left( \alpha f'(z) + (1 - \lambda) \phi'(z) \right) \right. \right\} > \beta \),

\[
0 \leq \beta < 1, \quad 0 \leq \lambda < 1 \quad \text{for} \ |z| < 1, \quad \text{then} \ \text{Re} \ \frac{f'(z)}{\phi'(z)} > 0
\]

for \( |z| < R^* = \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} \).

Proof. Let

\[
(1) \quad p(z) = f'(z) \left/ \left( \alpha f'(z) + (1 - \lambda) \phi'(z) \right) \right. = 1 + c_1 z + \cdots,
\]

then \( p(z) \) is analytic and \( \text{Re} \ p(z) < \beta \) for \( |z| < 1 \).

Now from (1),

\[
(2) \quad \frac{f'(z)}{\phi'(z)} = \frac{(1 - \lambda) p(z)}{1 - \lambda p(z)}.
\]
The expression (2) is valid for those $z$ for which $1-\lambda p(z) \leqslant 0$ for $|z| < 1$.

Since $|p(z)| \leqslant \frac{1+(1-2\beta)|z|}{1-|z|}$, $1-\lambda p(z) \leqslant 0$ in particular if $|z| < \frac{1-\lambda}{1+(1-2\beta)\lambda}$.

From Lemma 2, by Schwarz's lemma,

(3) $|\omega(z)| \leqslant |z|$.

Using Lemma 2 and (3),

$$\Re \frac{f'(z)}{\phi'(z)} = \Re \left\{ \frac{(1-\lambda)}{1+(1-2\beta)\lambda} \cdot \frac{1+(1-2\beta)\omega(z)}{1+(1-2\beta)\lambda - \omega(z)} \right\} \geqslant -\frac{1-\lambda}{(1+(1-2\beta)\lambda)^2} \cdot \frac{1-\lambda-2(\beta+(1-2\beta)\lambda)|z|-\frac{1-\lambda}{1+(1-2\beta)\lambda} z^2}{1+(1-2\beta)\lambda - \omega(z)^2}.$$

Using the right-hand part of inequality in (4), we obtain that

$$\Re \frac{f'(z)}{\phi'(z)} > 0 \text{ for } |z| < R^* < 1.$$

This shows that $f(z)$ is univalent and close-to-convex for $z < R^*$.

Theorem 2. With the same hypothesis as is Theorem 1, $f(z)$ maps the disk $|z| < R$ onto a convex domain, where $R$ is the smallest positive root of the equation

$$q(\gamma, \alpha, \beta, \lambda) = 0,$$

where

$$q(\gamma, \alpha, \beta, \lambda) = (1-\alpha) \left\{ (1-\lambda) + 2\left\{ (1-\lambda)\alpha - (1+(1-2\beta)\lambda) \right\} \gamma \right.$$

$$\left. - \left\{ (1-\lambda)(1-2\alpha) + 4\alpha(\beta+(1-2\beta)\lambda) + 4(1-\beta) + (1-2\beta)(1+(1-2\beta)\lambda) \right\} \gamma^2 + 2(1-2\alpha)(1-2\beta)(1+(1-2\beta)\lambda) \gamma^4 \right\}.$$

Proof. Now from (1),

(5) $f'(z) = \left\{ (1-\lambda)\phi'(z)p(z) \right\} / (1-\lambda p(z))$, for $|z| < R^* < 1$.

From equation (5), we have

$$\frac{zf^*(z)}{f'(z)} = \frac{zp^*(z)}{\phi'(z)} + \frac{zp'(z)}{p(z)} - \frac{1}{1-\lambda p(z)}.$$

In (1), it is shown that

(6) $\Re \left\{ \frac{z\phi^*(z)}{\phi'(z)} \right\} \geqslant -\frac{2(1-\alpha)\gamma}{1+\gamma}$ for $|z| = \gamma < 1$.

From Lemma 4, we obtain

(7) $|z| p'(z) / p(z) | \leqslant -\frac{2(1-\beta)\gamma}{(1-\gamma)(1+(1-2\beta)\gamma)}$.

Using (6), (7) and Lemma 3,
(8) \( \text{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \geq 1 - \frac{2(1-\alpha)\gamma}{1+\gamma} - \frac{2(1-\beta)\gamma}{(1+(1-2\beta)\lambda)\gamma} \cdot \frac{1}{1-\lambda-(1+(1-2\beta)\lambda)\gamma} \),

for \( |z| < R^* < 1 \).

Simplifying the right-hand side of (8), we obtain

\[
\frac{q(\gamma, \alpha, \beta, \lambda)}{(1+\gamma)(1+(1-2\beta)\gamma)(1-\lambda-(1+(1-2\beta)\lambda)\gamma)}.
\]

Since \( q(0, \alpha, \beta, \lambda) > 0 \),

\( q(R^*, \alpha, \beta, \lambda) < 0 \),

and \( q(R, \alpha, \beta, \lambda) = 0 \),

it follows that \( f(z) \) maps the disk \( |z| < R^* < 1 \) onto a convex domain.

This completes the proof.

Remark. If \( \alpha = \beta = 0, \lambda = 0 \), we see that \( f(z) \in \mathbb{C}(0, 0) \). The Koebe function \( z \frac{z}{1-z} \) is in \( \mathbb{C}(0, 0) \) relative for \( z \) and the least positive root of

\[ q(\gamma, 0, 0, 0) = \gamma^4 - 2\gamma^3 - 6\gamma^2 - 2\gamma + 1 \]

is \( 2 - \sqrt{3} \), the radius of convexity for the class \( S \). (\( S \) is the class of normalized univalent functions analytic in \( |z| < 1 \).)

Theorem 3. Let \( f(z) = z + a_2z^2 + \ldots \) be analytic for \( |z| < 1 \) and \( \phi(z) \in \mathbb{K}(\alpha) \).

If \( \text{Re} \left\{ \frac{f'(z)}{(\lambda f'(z) + (1-\lambda)\phi'(z))} \right\} > \beta, 0 < \beta < 1, 0 < \lambda < 1 \) for \( |z| < 1 \), then we have for \( |z| < R^* \)

(9) \( f'(z) \leq \frac{(1-\lambda)(1+(1-2\beta)\gamma)}{(1-\lambda)-(1+(1-2\beta)\lambda)\gamma} \cdot \frac{1}{(1-\gamma)^{2(1-\alpha)}} \),

(10) \( f'(z) \geq \frac{(1-\lambda)(1-(1-2\beta)\gamma)}{(1-\lambda)+(1+(1-2\beta)\lambda)\gamma} \cdot \frac{1}{(1+\gamma)^{2(1-\alpha)}} \).

Equality holds in (9) for the function

\[
f_1(z) = \int_0^z \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}} dt
\]

and equality holds in (10) for the function

\[
f_2(z) = \int_0^z \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}} dt.
\]

Proof. Using Lemma 2, we put

(11) \( \frac{f'(z)}{\lambda f'(z) + (1-\lambda)\phi'(z)} = \frac{1+(1-2\beta)\lambda}{1-G(z)} \cdot \frac{1}{G(z)} \),

where \( G(0) = 0, |G(z)| < 1 \) for \( |z| < 1 \).

Therefore by Schwarz's lemma, (11) yields
On certain classes of analytic functions in the unit disk

\[
\frac{1-(1-2\beta)r}{1+r} \leq \frac{f'(z)}{\lambda f'(z) + (1-\lambda)\phi'(z)} \leq \frac{1+(1-2\beta)r}{1-r} \quad (|z| = r < 1),
\]
or
\[
\frac{1+r}{1-(1-2\beta)r} \leq \frac{\lambda f'(z) + (1-\lambda)\phi'(z)}{f'(z)} \leq \frac{1-r}{1+(1-2\beta)r}.
\]

Using (12) and Lemma 5, the result follows.

Now, let

\[
f(z) = \int_{0}^{r} \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda) - (1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}} \, dt
\]
and

\[
\phi(z) = \int_{0}^{r} \frac{1}{(1-t)^{2(1-\alpha)}} \, dt.
\]

For \(\phi(z)\), we can able to show that \(\phi(z) \in \mathcal{K}(\alpha)\), and for \(f(z)\), \(\phi(z)\), it is shown that

\[
\text{Re} \frac{f'(z)}{\lambda f_1(z) + (1-\lambda)\phi'(z)} = \text{Re} \frac{1+(1-2\beta)z}{1-z} > \beta.
\]

Therefore \(f_1(z)\) satisfies the conditions of Theorem 3. The proof that \(f_2(z)\) satisfies the conditions of Theorem 3. is similar, with

\[
\phi(z) = \int_{0}^{r} \frac{1}{(1+t)^{2(1-\alpha)}} \, dt.
\]

Theorem 4. With the same hypothesis as in Theorem 3, we have for \(|z| < R^*\)

\[
|f(z)| \leq \int_{0}^{r} \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda) - (1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}} \, dt.
\]

(14) \quad \|f(z)\| \geq \int_{0}^{r} \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda) + (1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}} \, dt.

Equality holds in (13) for \(f_1(z)\) in Theorem 3. and in (14) for \(f_2(z)\) in Theorem 3. The proof. To prove (13), let \(z = re^{i\theta}\). Then

\[
|f(re^{i\theta})| \leq \int_{0}^{r} |f'(te^{i\theta})| \, dt
\]

\[
\leq \int_{0}^{r} \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda) - (1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}} \, dt.
\]

To prove (14), let \(z_0, |z_0| = r\), be chosen in such a way that \(|f(z_0)| \leq |f(z)|\), for all \(z, |z| = r\).

If \(L(z_0)\) is the pre-image of the segment \(o, f(z_0)\), then

\[
|f(z)| \geq |f(z_0)| \geq \int_{L(z_0)} |f'(z)| \, dz.
\]
This completes the proof.

References


