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*この文脈においては、特定の解析関数クラスに関する研究を扱っています。
On certain classes of analytic functions
in the unit disk*

By

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Abstract

In this note we shall consider the problem of finding the radius of convexity in some
univalent region for functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and satisfy
$Re \left\{ f'(z) / \left( \lambda f'(z) + (1 - \lambda) \phi'(z) \right) \right\} > \beta$ for $|z| < 1$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$
where $\phi(z) = z + b_2 z^2 + \cdots$ is analytic, univalent and convex of order $\alpha$, $0 \leq \alpha < 1$.
Finally a distortion theorem for $f(z)$ is shown.

1. Introduction

Let $k(\alpha)$ denote the class of functions analytic in $|z| < 1$ and of the form
$\phi(z) = z + b_2 z^2 + \cdots$,
such that $Re \left\{ z \phi''(z) / \phi'(z) + 1 \right\} > \alpha$ for $|z| < 1$ and $0 \leq \alpha < 1$.
Then $\phi(z)$ is said to be convex of order $\alpha$.

we say that an analytic function $f(z) = z + a_2 z^2 + \cdots$ is in the class $C(\alpha, \beta)$ if there
exists a function $\phi(z) \in k(\alpha)$ such that
$Re \left\{ f'(z) / \phi'(z) \right\} > \beta$, $0 \leq \beta < 1$.

Kaplan [2] proved that $C(0, 0)$, the class of close-to-convex functions, is univalent in
$|z| < 1$.

In this paper we shall consider the radius of convexity of $f(z)$ under the conditions
which are pointed out in abstract. And finally we state on the distortion theorems for
$f(z)$.

2. Proof of the theorems

In proving the theorems we will make use of the following lemmas.

* Some of these results had been reported in the Kyushu branch of Mathematical
Lemma 1. If \( h(z) = 1 + d_1 z + \cdots \) is analytic for \( |z| < 1 \) and \( \text{Re}\ h(z) > \alpha \),

\[
0 \leq \alpha < 1, \text{then}
\]

\[
1 - \frac{(1 - 2\alpha)}{1 + |z|} \leq \text{Re}\ h(z) \leq \frac{1 + (1 - 2\alpha)}{1 - |z|}.
\]

Lemma 2. Let \( p(z) \) be analytic for \( |z| < 1 \), \( p(0) = 1 \).

Then \( \text{Re}\ p(z) > \beta \), \( 0 \leq \beta < 1 \), if and only if

\[
p(z) = \frac{1 + (1 - 2\beta) \omega(z)}{1 - \omega(z)},
\]

where \( \omega(z) \) is analytic, \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) for \( |z| < 1 \).

Lemma 3. Let \( p(z) = 1 + c_1 z + \cdots \) is analytic and \( \text{Re}\ p(z) > \beta \), then \( 0 \leq \beta < 1 \), for \( |z| < 1 \).

\[
(1 - \lambda) |p(z)|^{-1} \leq (1 - |z|) / (1 - \lambda - (1 + (1 - 2\beta)\lambda)|z|)
\]

for \( |z| < \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} < 1 \), where \( 0 \leq \lambda < 1 \).

Proof. Using Lemma 1,

\[
1 - \frac{(1 - \lambda)}{1 + (1 - 2\beta)\lambda} \leq \frac{1}{1 - |z|} \leq \frac{1}{1 - \lambda - (1 + (1 - 2\beta)\lambda)|z|}.
\]

for \( |z| < \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} < 1 \).

Lemma 4. With the same hypothesis as in Lemma 3, we have

\[
\frac{zp'(z)}{p(z)} \leq \frac{2\gamma}{(1 - \gamma)\left\{1 + \frac{\beta}{1 - \beta} - (1 - \gamma)\right\}}
\]

for \( |z| = \gamma < 1 \).

Lemma 5. Let \( \phi(z) \in k(\alpha) \). Then

\[
|\phi'(z)| \leq \frac{1}{(1 - \gamma)^2(1 - \alpha)}, \quad |\phi'(z)| \geq \frac{1}{(1 + \gamma)^2(1 - \alpha)}
\]

for \( |z| = \gamma < 1 \).

Theorem 1. Let \( f(z) = z + a_2 z^2 + \cdots \) be analytic for \( |z| < 1 \) and \( \phi(z) \in k(\alpha) \).

If \( \text{Re}\{f'(z) / (\lambda f'(z) + (1 - \lambda)\phi'(z))\} > \beta \),

\[
0 \leq \beta < 1, \quad 0 \leq \lambda < 1 \quad \text{for} \quad |z| < 1, \quad \text{then} \quad \text{Re}\ \frac{f'(z)}{\phi'(z)} > 0
\]

for \( |z| < R = \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} \).

Proof. Let

\[
(1) \quad p(z) = f'(z) / (\lambda f'(z) + (1 - \lambda)\phi'(z)) = 1 + c_1 z + \cdots,
\]

then \( p(z) \) is analytic and \( \text{Re}\ p(z) < \beta \) for \( |z| < 1 \).

Now from (1),

\[
(2) \quad \frac{f'(z)}{\phi'(z)} = \frac{(1 - \lambda)p(z)}{1 - \lambda p(z)}.
\]
The expression (2) is valid for those \( z \) for which \( 1 - \lambda p(z) \leq 0 \) for \( |z| < 1 \).

Since \( |p(z)| \leq \frac{1 + (1 - 2\beta)z}{1 - |z|} \), \( 1 - \lambda p(z) \leq 0 \) in particular if \( |z| < \frac{1 - \lambda}{1 + (1 - 2\beta)\lambda} \).

From Lemma 2, by Schwarz's lemma,

\[
(3) \quad |\omega(z)| \leq |z|.
\]

Using Lemma 2 and (3),

\[
\text{Re} \left\{ \frac{f''(z)}{p'(z)} \right\} = \text{Re} \left\{ \frac{(1 - \lambda)}{2(1 + (1 - 2\beta)\lambda)} \cdot \left( \frac{1 + (1 - 2\beta)\lambda}{1 + (1 - 2\beta)\lambda} - \omega(z) \right) \right\}
\]

\[
(4) \quad \geq \frac{1 - \lambda}{2(1 + (1 - 2\beta)\lambda)^2} \cdot \frac{1 - \lambda - 2(\beta + (1 - 2\beta)\lambda) |z| - (1 - 2\beta)(1 + (1 - 2\beta)\lambda) |z|^2}{1 - \lambda - 2(1 - 2\beta)(1 + (1 - 2\beta)\lambda) |z|^2}.
\]

Using the right-hand part of inequality in (4), we obtain that

\[
\text{Re} \left\{ \frac{f''(z)}{p'(z)} \right\} > 0 \text{ for } |z| < R^* < 1.
\]

This shows that \( f(z) \) is univalent and close-to-convex for \( z < R^* \).

Theorem 2. With the same hypothesis as in Theorem 1, \( f(z) \) maps the disk \( |z| < R \) onto a convex domain, where \( R \) is the smallest positive root of the equation

\[
q(\gamma, \alpha, \beta, \lambda) = 0,
\]

where

\[
q(\gamma, \alpha, \beta, \lambda) = (1 - \alpha) \left\{ (1 - \lambda) + 2 \left\{ (1 - \lambda)\alpha - (1 + (1 - 2\beta)\lambda) \right\} \gamma 
- \left\{ (1 - \lambda)(1 - 2\alpha) + 4\alpha(\beta + (1 - 2\beta)\lambda) + 4(1 - \beta)(1 - 2\beta)(1 + (1 - 2\beta)\lambda) \right\} \gamma^2 
+ 2(1 - 2\alpha)(1 - 2\beta)(1 + (1 - 2\beta)\lambda) \gamma^4 \right\}.
\]

Proof. Now from (1),

\[
(5) \quad f'(z) = \left( (1 - \lambda)\phi'(z)p(z) \right) \left/ \left( 1 - \lambda p(z) \right) \right.,
\]

for \( |z| < R^* < 1 \).

From equation (5), we have

\[
\frac{zf''(z)}{f'(z)} = \frac{z\phi''(z)}{\phi'(z)} + \frac{zp'(z)}{p(z)} \frac{1}{1 - \lambda p(z)}.
\]

In (1), it is shown that

\[
(6) \quad \text{Re} \left\{ \frac{z\phi''(z)}{\phi'(z)} \right\} \geq - \frac{2(1 - \alpha)\gamma}{1 + \gamma} \text{ for } |z| = \gamma < 1.
\]

From Lemma 4, we obtain

\[
(7) \quad \left| z \frac{p'(z)}{p(z)} \right| \leq \frac{2(1 - \beta)\gamma}{(1 - \gamma)(1 + (1 - 2\beta)\gamma)}.
\]

Using (6), (7) and Lemma 3,
(8) \[ \text{Re} \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \geq 1 - \frac{2(1-\alpha)\gamma}{1+\gamma} - \frac{2(1-\beta)\gamma}{(1+(1-2\beta)\lambda)\gamma} \cdot \frac{1}{1-\lambda-(1+(1-2\beta)\lambda)\gamma}, \]
for \(|z| < R^* < 1\).

Simplifying the right-hand side of (8), we obtain
\[ \frac{q(\gamma, \alpha, \beta, \lambda)}{(1+\gamma)(1+(1-2\beta)\gamma)(1-\lambda-(1+(1-2\beta)\lambda)\gamma)}. \]

Since \(q(\alpha, \beta, \lambda) > 0\),
\[ q(R^*, \alpha, \beta, \lambda) < 0, \]
and \(q(R, \alpha, \beta, \lambda) = 0\),
it follows that \(f(z)\) maps the disk \(|z| < R(<R^*<1)\) onto a convex domain.
This completes the proof.

**Remark.** If \(\alpha = \beta = 0, \lambda = 0\), we see that \(f(z) \in C(0, 0)\). The Koebe function \(z(1-z)^{-2}\) is in \(C(0, 0)\) relative for \(\frac{z}{1-z}\) and the least positive root of
\[ q(\gamma, 0, 0, 0) = \gamma^4 - 2\gamma^3 - 6\gamma^2 - 2\gamma + 1 \]
is \(2 - \sqrt{3}\), the radius of convexity for the class \(S\). (\(S\) is the class of normalized univalent functions analytic in \(|z| < 1\).)

**Theorem 3.** Let \(f(z) = z + a_2z^2 + \ldots\) be analytic for \(|z| < 1\) and \(\phi(z) \equiv k(\alpha)\).

If \(\text{Re} \left\{ \frac{f'(z)}{(\lambda f'(z) + (1-\lambda)\phi'(z))} \right\} > \beta, 0 \leq \beta < 1, 0 \leq \lambda < 1\) for \(|z| < 1\), then we have for \(|z| < R^*\)

\[ \text{Re} \left\{ \frac{f'(z)}{(\lambda f'(z) + (1-\lambda)\phi'(z))} \right\} > \beta \]

Equality holds in (9) for the function
\[ f_1(z) = \int_0^z \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^{2(1-\alpha)}} \, dt \]
and equality holds in (10) for the function
\[ f_2(z) = \int_0^z \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}} \, dt. \]

**Proof.** Using Lemma 2, we put
\[ \frac{f'(z)}{\lambda f'(z) + (1-\lambda)\phi'(z)} = \frac{1+(1-2\beta)G(z)}{1-G(z)}, \]
where \(G(o) = 0, |G(z)| < 1\) for \(|z| < 1\).

Therefore by Schwarz's lemma, (11) yields
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\[
\frac{1-(1-2\beta)\gamma}{1+\gamma} \leq \left| \frac{f'(z)}{\lambda f'(z)+(1-\lambda)\phi'(z)} \right| \leq \frac{1+(1-2\beta)\gamma}{1-\gamma} \quad (|z| = \gamma < 1),
\]
or
\[
\frac{1+\gamma}{1-(1-2\beta)\gamma} \geq \left| \frac{\lambda f'(z)+(1-\lambda)\phi'(z)}{f'(z)} \right| \geq \frac{1-\gamma}{1+(1-2\beta)\gamma}.
\]

Using (12) and Lemma 5, the result follows.

Now, let

\[
f(z) = \int_0^z \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^2(1-\alpha)} \ dt
\]

and

\[
\phi(z) = \int_0^z \frac{1}{(1-t)^2(1-\alpha)} \ dt.
\]

For \(\phi(z)\), we can able to show that \(\phi(z) \equiv k(\alpha)\), and for \(f(z)\), \(\phi(z)\), it is shown that

\[
\text{Re} \frac{f'(z)}{\lambda f_1(z)+(1-\lambda)\phi'(z)} = \text{Re} \frac{1+(1-2\beta)z}{1-z} > \beta.
\]

Therefore \(f_1(z)\) satisfies the conditions of Theorem 3. The proof that \(f_2(z)\) satisfies the conditions of Theorem 3. is similar, with

\[
\phi(z) = \int_0^z \frac{1}{(1-t)^2(1-\alpha)} \ dt.
\]

**Theorem 4.** With the same hypothesis as in Theorem 3, we have for \(|z| < R^*\)

\[
|f(z)| \leq \int_0^r \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^2(1-\alpha)} \ dt.
\]

\[
|f(z)| \geq \int_0^r \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^2(1-\alpha)} \ dt.
\]

Equality holds in (13) for \(f_1(z)\) in Theorem 3. and in (14) for \(f_2(z)\) in Theorem 3.

**Proof.** To prove (13), let \(z = re^{i\theta}\). Then

\[
|f(re^{i\theta})| \leq \int_0^r |f'(te^{i\theta})| \ dt
\]

\[
\leq \int_0^r \frac{(1-\lambda)(1+(1-2\beta)t)}{(1-\lambda)-(1+(1-2\beta)\lambda)t} \cdot \frac{1}{(1-t)^2(1-\alpha)} \ dt.
\]

To prove (14), let \(z_0, |z_0| = \gamma\), be chosen in such a way that \(|f(z_0)| \leq |f(z)|\), for all \(z, |z| = \gamma\).

If \(L(z_0)\) is the pre-image of the segment \(o, f(z_0)\), then

\[
|f(z)| \geq |f(z_0)| \geq \int_{L(z_0)} |f'(z)| \ |dz|.
\]
\[ \int_0^r \frac{(1-\lambda)(1-(1-2\beta)t)}{(1-\lambda)+(1-(1-2\beta)\lambda)t} \cdot \frac{1}{(1+t)^{2(1-\alpha)}} \, dt. \]

This completes the proof.

References


