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On a non-linear hyperbolic equation

By

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§ 1. Introduction

Let $\omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \omega$. In this note we consider the initial-boundary value problem of the form

(1.1) $\partial^2 u/\partial t^2 - \Delta u + G(u) = 0, \quad x \in \omega, \quad t > 0,$

(1.2) $u(x, t) = 0, \quad x \in \partial \omega, \quad t \geq 0,$

(1.3) \begin{align*}
    u(x, 0) &= u_0(x), \quad x \in \omega, \\
    \partial u/\partial t(x, 0) &= u_1(x), \quad x \in \omega.
\end{align*}

When the non-linear term $G$ is Hölder continuous, J. C. Saut gave existence of global solutions of the above problem in [3].

The purpose of the present note is to extend a part of the Saut's results. We make on a continuous function $G$ the following condition

(1.4) $|G(u)| \leq k_0 |u|^{\alpha} + k_1, \quad 0 < \alpha \leq 1,$

where $k_0, k_1$ are positive constants.

Remark 1.1. If $G$ is Hölder continuous, then $G$ satisfies (1.4).

We shall deal with real valued functions and use the notation in the book of J. L. Lions [2] and prove the following

Theorem. Suppose that $u_0 \in H^1(\partial \omega), \quad u_1 \in L^2(\omega)$. Then there exists a function $u$ such that

(1.5) $u \in L^\infty(0, T; H^1(\omega)), \quad T > 0,$

(1.6) $\partial u / \partial t \in L^\infty(0, T; L^2(\omega)), \quad T > 0,$

and which satisfies (1.1) in a generalized sense and (1.3).

Remark 1.2. The boundary condition (1.2) is implied by (1.5).
Remark 1.3. It follows from (1.5) and (1.6) that $u$ is continuous from $[0, T]$ to $L^2(\Omega)$, possibly after a modification on a set of measure zero and from (1.5), (1.6) and (1.1) that $\varphi u / \varphi t$ is continuous from $[0, T]$ to $H^{-1}(\Omega)$.

§ 2. Existence of an approximate solution

Suppose that $0 < a < 1$. We shall use the Galerkin's method. Let $\{W_i\}_{i=1,2,\ldots}$ be a complete system in $H^1_0(\Omega)$. We look for an approximate solution $u_m(x, t)$ of the form

$$u_m(t) = \sum_{i=1}^{\infty} g_{m, i}(t) w_i, \quad g_{m, i} \in C^2([0, T]).$$

(2.1)

The unknown functions $g_{m, i}$ are determined by the system of ordinary differential equations

$$u_m''(t, w_j) = \sum_{i=1}^{m} g_{m, i}(t) w_i, \quad (G(u_m(t)), w_i) = 0,$$

(2.2)

with initial conditions

$$u_m(0) = u_{0m}, \quad \dot{u}_m(0) = \sum_{i=1}^{m} a_{m, i} w_i u_i \quad \text{in } H^1_0(\Omega) \quad \text{as } m \to \infty,$$

(2.3)

$$u_m(0) = u_{0m}, \quad \dot{u}_m(0) = \sum_{i=1}^{m} \beta_{m, i} w_i u_i \quad \text{in } L^2(\Omega) \quad \text{as } m \to \infty.$$

(2.4)

Here $u' = \partial u / \partial t$, $u'' = \partial u / \partial t^2$, $(u, v) = \int_{\Omega} uv dx$,

(2.5)

$$a(u, v) = \sum_{i=1}^{m} \int_{\Omega} (\partial u / \partial x)(\partial v / \partial x) dx.$$

By general theory on the system of ordinary differential equations, there exists a solution of (2.2), (2.3) and (2.4) in an interval $[0, t_m]$, $t_m > 0$.

A priori estimate in §3 assure that $t_m = T$.

§ 3. A priori estimate

Multiplying (2.2) by $g_m'(t)$ and summing over $j$ from 1 to $m$, we get

$$\left(\frac{1}{2}\right) (d / dt) \left(|u_m'(t)|^2 + \|u_m(t)\|^2\right) + (G(u_m(t)), u_m'(t)) = 0,$$

(3.1)

Where $|u|^2 = (u, u)$, $\|u\|^2 = a(u, u)$.

Using (1.4), Hölder's inequality and Sobolev's inequality, we have
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\[
\left\{ \begin{array}{l}
\frac{1}{2} (d/dt) (|u_\ast(t)|^2 + \|u_\ast(t)\|^2) \leq k_2 |u_\ast| \|u_\ast\|^2 + k_3 |u_\ast|

\leq k_4 (|u_\ast_\ast|^{1+\sigma} + \|u_\ast\|^{1+\sigma}) + k_5 \quad \text{(by Young's inequality)}

\leq k_6 \left\{ \frac{1}{2} \left( |u_\ast_\ast|^2 + \|u_\ast\|^2 \right)^{\frac{1+\sigma}{2}} + k_7 \right. \]
\end{array} \right.
\]

Where \( k_2, \ldots, k_7 \) are positive constants.

We define \( e_m(t) \) by

\[
(3.3) \quad e_m(t) = \frac{1}{2} (|u_\ast_\ast(t)|^2 + \|u_\ast(t)\|^2),
\]

then by integrating with respect to \( t \) and using (2.3) and (2.4), we have an integral inequality

\[
(3.4) \quad e_m(t) \leq c_0 + c_1 t + c_2 \int_0^t e_m^{\frac{1+\sigma}{2}}(s) \, ds,
\]

Where \( c_0, c_1, c_2 \) are positive constants independent of \( m \).

From the results of integral inequalities (see [1]), we obtain

\[
(3.5) \quad e_m(t) \leq c_0 + c_1 t + M(t).
\]

Here \( M(t) \) is the maximal solution of ordinary differential equation of the form

\[
(3.6) \quad y'(t) = c_2 (c_0 + c_1 t + y)^{\frac{1+\sigma}{2}},
\]

with initial condition

\[
(3.7) \quad y(0) = 0.
\]

Since \( 0 < \frac{1+\sigma}{2} < 1 \), the above maximal solution \( M(t) \) exists in the large.

Hence

\[
(3.8) \quad \frac{1}{2} (|u_\ast_\ast(t)|^2 + \|u_\ast(t)\|^2) \leq c(T),
\]

and \( c(T) \) is a constant independent of \( m \).

This a priori estimate shows \( t_m = T \).

§ 4. Existence of global solutions

From a priori estimate (3.8) and Rellich's theorem, we can find a function \( u \) and a subsequence \( \{u_m\} \) of \( \{u_n\} \) such that

\[
(4.1) \quad u_m \to u \quad \text{in} \; L^\infty(0, T; H_0^1(\Omega)) \; \text{weakly star},
\]

\[
(4.2) \quad u_\ast \to u_\ast \quad \text{in} \; L^\infty(0, T; H^1(\Omega)) \; \text{weakly star},
\]
(4.3) \( u_n \to u \) in \( L^2(0, T; L^2(\Omega)) \) strongly and a.e. in \( \Omega \times [0, T] \).

By (4.3), (1.4) and a well-known lemma (see Lemma 1.3 [2]), we have

(4.4) \( G(u_n) \to G(u) \) in \( L^2(0, T; L^2(\Omega)) \) weakly,

Which implies the function \( u \) satisfies (1.1) in a generalized sense.

We next prove that \( u \) satisfies (1.3). From (4.1), (4.2) and Lemma 1.2 [2], in particular

(4.5) \( u_n(0) \to u(0) \) in \( L^2(\Omega) \) weakly

and since \( u_n(0) = u_{0,n} \to u_0 \) in \( H^1(\Omega) \), we have

(4.6) \( u(0) = u_0 \).

Using (2.2), (4.1), (4.2) and (4.4), we get

(4.7) \( (u'', w_i) \to (u'', w_i) \) in \( L^2(0, T) \) weakly.

It follows from (4.2), (4.7) that

(4.8) \( (u'(0), w_i) \to (u'(0), w_i) |_{t=0} = (u'(0), w_i) \) \( \forall i \).

Since \( (u'(0), w_i) = (u'_n, w_i) \to (u_i, w_i) \) \( \forall j \) we have

(4.9) \( u'(0) = u_1 \).

This completes the proof of theorem when \( 0 < \alpha < 1 \).

Remark 4.1. When \( \alpha = 1 \), we can more easily prove the existence of global solutions.

References

