On a non-linear hyperbolic equation

By

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§1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. In this note we consider the initial-boundary value problem of the form

\begin{align*}
(1.1) & \quad \partial^2 u / \partial t^2 - \Delta u + G(u) = 0, \quad x \in \Omega, \quad t > 0, \\
(1.2) & \quad u(x, t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
(1.3) & \quad \left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(x, 0) = u_0(x), \quad x \in \Omega, \\
\frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad x \in \partial \Omega.
\end{array} \right.
\end{align*}

When the non-linear term $G$ is Hölder continuous, J. C. Saut gave existence of global solutions of the above problem in [3].

The purpose of the present note is to extend a part of the Saut's results. We make on a continuous function $G$ the following condition

\begin{equation}
|G(u)| \leq k_0|u|^\alpha + k_1, \quad 0 < \alpha \leq 1,
\end{equation}

where $k_0, k_1$ are positive constants.

Remark 1.1. If $G$ is Hölder continuous, then $G$ satisfies (1.4).

We shall deal with real valued functions and use the notation in the book of J. L. Lions [2] and prove the following

Theorem. Suppose that $u_0 \in H^0(\Omega), \quad u_1 \in L^2(\Omega)$. Then there exists a function $u$ such that

\begin{align*}
(1.5) & \quad u \in L^\infty(0, T; H^0(\Omega)), \quad T > 0, \\
(1.6) & \quad \partial u / \partial t \in L^\infty(0, T; L^2(\Omega)), \quad T > 0,
\end{align*}

and which satisfies (1.1) in a generalized sense and (1.3).

Remark 1.2. The boundary condition (1.2) is implied by (1.5).
**Remark 1.3.** It follows from (1.5) and (1.6) that $u$ is continuous from $[0, T]$ to $L^2(\Omega)$, possibly after a modification on a set of measure zero and from (1.5), (1.6) and (1.1) that $\partial u / \partial t$ is continuous from $[0, T]$ to $H^{-1}(\Omega)$.

§ 2. Existence of an approximate solution

Suppose that $0 < \alpha < 1$. We shall use the Galerkin’s method. Let $\{w_i\}_{i=1,2,\ldots}$ be a complete system in $H^1_0(\Omega)$. We look for an approximate solution $u_m(x, t)$ of the form

$$u_m(t) = \sum_{i=1}^{m} g_m(t) w_i, \quad g_m \in C^2([0, T]).$$

The unknown functions $g_m$ are determined by the system of ordinary differential equations

$$\begin{cases}
(u''(t), w_j) + a(u_m(t), w_j) + (G(u_m(t)), w_j) = 0, \\
1 \leq j \leq m,
\end{cases}$$

with initial conditions

$$\begin{align*}
(2.3) \quad & u_m(0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{m} \alpha_i w_i \rightarrow u \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad m \rightarrow \infty, \\
(2.4) \quad & u'_m(0) = u_{0m}, \quad u_{0m} = \sum_{i=1}^{m} \beta_i w_i \rightarrow u \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad m \rightarrow \infty.
\end{align*}$$

Here $u' = \partial u / \partial t$, $u'' = \partial^2 u / \partial t^2$, $(u, v) = \int_{\Omega} uv dx$, $\alpha, \beta \in \mathbb{R}$, and

$$a(u, v) = \sum_{i=1}^{m} \int_{\Omega} (\partial u / \partial x)(\partial v / \partial x) dx.$$  

By general theory on the system of ordinary differential equations, there exists a solution of (2.2), (2.3) and (2.4) in an interval $[0, t_m]$, $t_m > 0$.

A priori estimate in §3 assure that $t_m = T$.

§ 3. A priori estimate

Multiplying (2.2) by $g'_m(t)$ and summing over $j$ from 1 to $m$, we get

$$\begin{align*}
(3.1) \quad & \frac{1}{2} \left( \frac{d}{dt} \right) \left| u''(t) \right|^2 + \left| u_m(t) \right|^2 + \left( G(u_m(t)), u_m(t) \right) + (G(u_m(t)), u'_m(t)) = 0,
\end{align*}$$

where $|u|^2 = (u, u)$, $\|u\|^2 = a(u, u)$.

Using (1.4), Hölder's inequality and Sobolev's inequality, we have
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\[
\left\{ \begin{align*}
\frac{1}{2} \frac{d}{dt} (|u_\tau'(t)|^2 + \|u_\tau(t)\|^2) & \leq k_2 |u_\tau'| \|u_\tau\| + k_3 |u_\tau| \\
& \leq k_4 (|u_\tau'|^{1+\sigma} + \|u_\tau\|^{1+\sigma}) + k_5 \text{ (by Young's inequality)} \\
& \leq k_6 \left( \frac{1}{2} (|u_\tau'|^2 + \|u_\tau\|^2) \right)^{\frac{1+\sigma}{2}} + k_7
\end{align*} \right.
\]

Where \( k_2, \ldots, k_7 \) are positive constants.

We define \( e_\tau(t) \) by

\[
e_\tau(t) = \left( \frac{1}{2} (|u_\tau'|^2 + \|u_\tau(t)\|^2) \right),
\]

then by integrating with respect to \( t \) and using (2.3) and (2.4), we have an integral inequality

\[
e_\tau(t) \leq c_0 + c_1 t + c_2 \int_0^t e_\tau^{\frac{1+\sigma}{2}}(s) \, ds,
\]

Where \( c_0, c_1, c_2 \) are positive constants independent of \( \tau \).

From the results of integral inequalities (see [1]), we obtain

\[
e_\tau(t) \leq c_0 + c_1 t + M(t).
\]

Here \( M(t) \) is the maximal solution of ordinary differential equation of the form

\[
y' = c_2 (c_0 + c_1 t + y)^{\frac{1+\sigma}{2}},
\]

with initial condition

\[
y(0) = 0.
\]

Since \( 0 < \frac{1+\sigma}{2} < 1 \), the above maximal solution \( M(t) \) exists in the large. Hence

\[
\left( \frac{1}{2} (|u_\tau'|^2 + \|u_\tau(t)\|^2) \right) \leq c(T),
\]

and \( c(T) \) is a constant independent of \( \tau \).

This a priori estimate shows \( t_\tau = T \).

\section{Existence of global solutions}

From a priori estimate (3.8) and Rellich's theorem, we can find a function \( u \) and a subsequence \( \{u_\tau\} \) of \( \{u_\tau(t)\} \) such that

\[
\text{(4.1) } u_\tau \rightharpoonup u \text{ in } L^\infty(0, T; \mathcal{H}_0^1(\Omega)) \text{ weakly star,}
\]

\[
\text{(4.2) } u_\tau' \rightharpoonup u' \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly star,}
\]
(4.3) \( u_\nu \to u \) in \( L^2(0, T; L^2(\Omega)) \) strongly and a.e. in \( \Omega \times [0, T] \).

By (4.3), (1.4) and a well-known lemma (see Lemma 1.3 [2]), we have

(4.4) \( G(u_\nu) \to G(u) \) in \( L^{2\alpha}(0, T; L^{2\alpha}(\Omega)) \) weakly,

Which implies the function \( u \) satisfies (1.1) in a generalized sense.

We next prove that \( u \) satisfies (1.3). From (4.1), (4.2) and Lemma 1.2 [2], in particular

(4.5) \( u_\nu(0) \to u(0) \) in \( L^2(\Omega) \) weakly

and since \( u_\nu(0) = u_{\nu,0} \to u_0 \) in \( H^1_0(\Omega) \), we have

(4.6) \( u(0) = u_0 \).

Using (2.2), (4.1), (4.2) and (4.4), we get

(4.7) \( (u'', w_i) \to (u'', w_i) \) in \( L^{\frac{\alpha}{\alpha - 1}}(0, T) \) weakly.

It follows from (4.2) (4.7) that

(4.8) \( (u''(0), w_i) \to (u', w_i) \mid_{t=0} = (u'(0), w_i) \) \( \forall \mathcal{J} \).

Since \( (u''(0), w_i) = (u_{\nu,}, w_i) \to (u_0, w_i) \) \( \forall \mathcal{J} \) we have

(4.9) \( u'(0) = u_1 \).

This completes the proof of theorem when \( 0 < \alpha < 1 \).

Remark 4.1. When \( \alpha = 1 \), we can more easily prove the existence of global solutions.

References

