<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>記事</td>
<td>長崎大学教養部紀要 [自然科学 ⑬] 1978, 18, p.31-36</td>
</tr>
</tbody>
</table>

この記事は長崎大学の学術出版物「長崎大学教養部紀要」に掲載されています。
The radius of starlikeness for some analytic functions

By

Toshiaki YOSHIIKAI

(Received September 25, 1972)

Abstract

Let $F(z)$ be regular in $|z| < 1$ and $F(0) = 0$, $F'(0) = 1$. Let

$$f(z) = \frac{1}{c+1} z^{1-c} \{z^c F(z)\}'$$

and $c$ be a real and non-negative constant. We have determined the radius of starlikeness of order $\beta$ for $f(z)$ when $F(z)$ is starlike of order $\alpha$, $0 \leq \alpha, \beta < 1$.

1. Introduction

In this paper we shall treat a generalization of the recent results of H. S. Al-Amri [1]. The method used in the proof of the theorem is that of V.A. Zmorovich [5]. The class $S^*(\alpha)$ is called the starlike functions of order $\alpha$. Analytically, $F(z) \in S^*(\alpha)$ if and only if $\text{Re} \{z F'(z) / F(z)\} > \alpha$ in $|z| < 1$.

We shall denote by $P$ the class of regular functions $p(z)$ in $|z| < 1$, $p(0) = 1$ such that $\text{Re} \{p(z)\} > 0$. Let

$$f(z) = \frac{1}{c+1} z^{1-c} \{z^c F(z)\}'$$

and $c$ be a real and non-negative constant.

Let $F(z) = (c+1) z^c \int_{t=0}^{t=z} \{c-1 \{t^c-1 f(t)\} dt$ be regular in $|z| < 1$ and $F(0) = 0$, $F'(0) = 1$.

If $F(z) \in S^*(\alpha)$, then there exists $p(z) \in P$ such that

$$z F'(z) / F(z) = \alpha + (1 - \alpha) p(z).$$

From (1), (2) we get

$$z^c f(z) / \int_{t=0}^{t=z} \{c-1 \{t^c-1 f(t)\} dt = c + \alpha + (1 - \alpha) p(z).$$

Thus

$$zf'(z) / f(z) - \beta$$

$$= - (c + \beta) + (1 - \alpha) (h + p(z)) + zp'(z) / (h + p(z)),$$

where $h = (c + \alpha) / (1 - \alpha), 0 \leq \alpha < 1$.

We shall determine $r_{\alpha, \beta, c}$ the radius of starlikeness of order $\beta$ for $f(z)$.

Clearly, $r_{\alpha, \beta, c}$ is the smallest positive root of $Q_{\alpha, \beta, c} = 0$, where

$$Q_{\alpha, \beta, c} (r) = \min_{p \in P} \min_{|z| < 1} \text{Re} \{-(c + \beta) + (1 - \alpha)(h + p(z))\}$$
Thus this problem are reduced to seeking the quantity
\[
Q(r) = \min_{p \in P} \min_{u = r^{1/2}} \text{Re}(\Psi(p(z), zp'(z))\{ \Psi(w; W) \text{ is an analytic function of the variables } w \text{ and } W \text{in the } W\text{-plane and in the half plane } \text{Re}(w) > 0.}
\]

By M. S. Robertson's variational method [4], the minimum in (5) is realized for functions of the form
\[
\Psi(z) = e^{i\theta_1} + e^{i\theta_2} - i, \quad \lambda_1, \lambda_2 \geq 0 \text{ and } \lambda_1 + \lambda_2 = 1.
\]

We obtain \( r_*, \beta, \epsilon \) by an application of a theorem due to V. A. Zmorovic.

Theorem A (V. A. Zmorovic). Let \( \Psi(w; W) = M(w) + N(w) W \), where \( M(w) \) and \( N(w) \) are defined and finite in the half plane \( \text{Re}(w) < 0 \).

Put
\[
w = \lambda_1 \frac{1 + z_1^n}{1 - z_1^n} + \lambda_2 \frac{1 + z_2^n}{1 - z_2^n},
\]

\[
W = \lambda_1 \frac{2m z_1^n}{(1 - z_1^n)^2} + \lambda_2 \frac{2m z_2^n}{(1 - z_2^n)^2},
\]

where \( z_1 \) and \( z_2 \) are arbitrary points on \( |z| = r < 1 \), \( m \) is a positive integer, \( \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1 \).

Then the function \( \Psi(w; W) \) has the form
\[
\Psi(w; W) = M(w) + \frac{1}{2} m (w^2 - 1) N(w) + \frac{1}{2} m (\rho^2 - \rho_0^2) N(w) e^{i\varphi}.
\]

where
\[
(1 + z_k^n)/(1 - z_k^n) = a + \rho \exp(i\psi_k) \quad (k = 1, 2),
\]

\[
w = a + \rho \exp(i\psi_0) \quad (0 \leq \rho \leq \rho_0),
\]

\[
|z_1| = |z_2| = r, \quad a = (1 + r z_1^n)/(1 - r z_1^n),
\]

\[
\rho = 2 r^n/(1 - r z_1^n), \quad c^\psi = i \exp \{i(\psi_1 + \psi_2)/2\}.
\]

Also
\[
\min \text{Re}(\Psi(w; W)) = \Psi_\rho(w)
\]

\[
= \text{Re} \{ M(w) + \frac{1}{2} m (w^2 - 1) N(w) \} - \frac{1}{2} m |N(w)| (\rho^2 - \rho_0^2).
\]

The minimum in (7) is reached when
\[
\exp \{i(2\psi + \arg N(w))\} = 1.
\]

Comparing problem (4) with the terms of Theorem A, we have \( m = 1 \), and
On the radius of starlikeness and convexity for some univalent functions

\[ M(w) = -(c + \beta) + (1 - \alpha)(w + h), \]
\[ N(w) = \frac{1}{(w + h)}, \]
\[ h = \frac{(c + \alpha)}{(1 - \alpha)}, \quad 0 \leq \alpha < 1. \]

From (4), (7) and these relations our problem is reduced to minimizing \( \Psi_s(w) \), where

\[ \Psi_s(w) = \operatorname{Re} \left\{ - (c + \beta) + (1 - \alpha)(w + h) + \frac{1}{2} \frac{w^2 - 1}{w + h} \frac{\rho^2 - \rho^2_s}{w + h} \right\} \]

2. Radius of starlikeness

Theorem. Let \( F(z) \in S^*(\alpha) \), \( 0 \leq \alpha < 1 \),
\[ f(z) = \frac{1}{c + 1} - z^{1 - c}[z^c F(z)]' \text{ in } |z| < 1. \] Let \( r_{\alpha, \beta, c} \) be the radius of the largest disk in which \( \operatorname{Re}(zf'(z)/f(z)) \geq \beta \).

Put
\[ a^2 + 2(\beta - 2c - \alpha) + 2(2h + \beta + c)^2 - 4(2 - \alpha)h^2 = 0, \]
\[ 4a^2 - 2(\beta - c + 2) + (1 + \beta)(1 - c) \geq r^2 \]
\[ + 2 \left\{ (3 - \beta + c) - c - \beta - 2 \right\} r + (1 + c)(1 - \beta) = 0, \]
where \( a = (1 + r^2)/(1 - r^2), \ h = (c + \alpha)/(1 - \alpha); \)
\[ B_1 \beta^3 + B_2 \beta^2 + B_3 \beta + B_4 = 0 \]
where
\[ B_1 = 4(1 - \alpha)(2c + (2 - c)\alpha - \alpha^2), \]
\[ B_2 = -4a^4 + 4(c + 1)\alpha^2 + (9 - 28c + 8c^2)\alpha^2 \]
\[ + 2(21c - 16c^2)\alpha + 33c^2, \]
\[ B_3 = 4(1 - \alpha)a^4 + 2(1 + 12c - c^2)\alpha^2 + (2c^3 + 14c^2 + 23c - 29)\alpha^2 \]
\[ - (6c^3 + 21c^2 + 29c - 14)\alpha + 4c^2 + 14c, \]
\[ B_4 = (2a^2 + \alpha^2 + 3ca + 3\alpha^2 - c^2 + 2) \]
\[ + 2(1 - \alpha)(2a + c)(2a^2 + c^2 + 3ca + 3\alpha^2 - \alpha^2 - 2) \]
\[ + (2a + c)^2(2a - ca + 3c)^2 - 4(2 - \alpha)(c + 1)^2(2a + c). \]

Then \( r_{\alpha, \beta, c} \) is the smallest positive root \( r \) in (10) with \( 0 \leq \beta \leq \beta_0(\alpha, c) \) and the smallest positive root \( r \) in (11) with \( \beta_0(\alpha, c) \leq \beta < 1 \), where \( \beta_0(\alpha, c) \) is the smallest positive root of (12). These results are sharp.

Proof. Our problem is reduced to minimizing \( \Psi_s(w) \). This minimum is reached when the point \( w \) \( |w - a| < \rho \) is fixed, and the chord passing through it and the points \( a + \rho \exp(\phi k = \pi) \) is perpendicular to the vector \( \exp(\psi)/2 \), where \( w + h = R \exp(\phi). \)

By setting \( w = a + \xi + i\eta, \ \rho^2 = \xi^2 + \eta^2 \leq \rho^2. \)
Then (9) becomes

\[ \psi_\rho (w) = \psi_\rho (\xi ; \eta) \]

\[ = -(c + \beta + h) + \left( \frac{3}{2} - a \right) (a + \xi + h) + \frac{1}{2} (h^2 - 1) (a + \xi + h) R^2 - \frac{1}{2} (\rho^2 - \xi^2 - \eta^2) R^{-1}, \]

where \( R^2 = (a + \xi + h)^2 + \eta^2. \)

We now wish to minimize \( \psi_\rho (w) \) as a function of \( \eta \). A differentiation shows that

\[ \frac{\partial \psi_\rho}{\partial \eta} = \frac{1}{2} \eta R^4 S(\xi, \eta), \]

where

\[ S(\xi, \eta) = (\xi^2 + 2(a + h) \xi + \rho^2 + \eta^2 + 2(a + h)^2) R^2 - 2(h^2 - 1)(a + \xi + h) - (\xi^2 + 4(a + h) \xi + \rho^2 + 2(a + h)^2 - 2(h^2 - 1))(a + \xi + h). \]

But the last expression in (15) is an increasing function of \( \xi \) in the interval \([-\rho, \rho]\). Hence

\[ S(\xi, \eta) \geq S(-\rho, \rho) = 2((a - \rho)^2 + 2h(a - \rho) + 1)(a + h - \rho) > 0. \]

Thus we see from (14) that \( \psi_\rho (\xi, \eta) \) is minimized on every chord \( \xi = \) constant of the circle \( \xi^2 + \eta^2 = \rho^2 \) at the point \( \eta = 0 \).

Therefore the minimum of \( \psi_\rho (\xi, \eta) \) in the disk \( \xi^2 + \eta^2 \leq \rho^2 \) occurs somewhere on the diameter \( \eta = 0 \). Setting \( \eta = 0 \) in (13), we have

\[ \psi_\rho (\xi, 0) = \ell(R) = (2 - a) R + (h^2 + ah) R^{-1} - (a + 2h) - (c + \beta). \]

The absolute minimum of \( \ell(R) \) is realized at

\[ R_0 = \left( \frac{(h^2 + ah)/(2 - a)}{(2 - a)^{1/2}} \right)^{1/2} \]

where \( R_0 < a + h + \rho \).

However, if \( R_0 \notin [a + h - \rho, a + h + \rho] \), then the minimum of \( \ell(R) \) is attained at

\[ R_1 = a + h - \rho. \]

The radius \( r_{a, c} \) is therefore determined from either

\[ \ell(R_0) = 0, \]

where \( R_0 \) is given by (17), or from

\[ \ell(R_1) = 0, \]

where \( R_1 \) is given by (18).

These two equations coincide for some \( \beta_0 = \beta_0 (a, c) \).

(19) and (20) can be reduced to (10) and (11), respectively.

From (10) and (11) we get,

\[ r_1 = r_{a, c} \]

\[ = \left( \frac{(2a - \beta + c - 1 - D^{1/2})/(2 a - \beta + c + 1 - D^{1/2})}{(2a - \beta + c + 1 - D^{1/2})} \right)^{1/2}, \]

where \( D = 4(2 - a) h^2 - (2h + \beta + c)^2 + (\beta - 2a - c)^2 \).
On the radius of starlikeness and convexity for some univalent functions

and

\[(22) \quad r_z = r_{\alpha, \beta, c} \]

\[= \frac{(1 + c)(1 - \beta)}{((\beta (c + a) + 2(1 - a) - (c + 1)a)} \]

\[+ \frac{(\beta (c + a) + 2(1 - a) - (c + 1)a)^2}{(1 + c)((1 - \beta)(4a^2 - 2a\beta + 2ca - 4a + 1 - c + \beta - c\beta))^{1/4}}, \]

respectively. To determine the \( \beta_0 = \beta(\alpha) \) that makes the transition from (21) to (22) set \( R_0 = R_1 \).

It follows then

\[(23) \quad h^2 + ah = (2 - a)R_1^2. \]

From (20), (16) and (23), we obtain

\[(24) \quad (3 - 2\alpha)a + 2\alpha - \beta + c = 2(2 - a)\rho. \]

From (24), (10), we obtain

\[(25) \quad a = \frac{(1 - \alpha)\beta^2 + (3\alpha - 2c + 5c)\beta - (2a^2 + 3ca + c^2a + 3a - c - 2)}{(1 - \alpha)(2\alpha - \beta + c)} \]

Equation (12) is deduced from (10) and (25).

Note that \( r_{\alpha, \beta, c} = r_1 \) cannot be used when \( \beta > a \), since \( D < 0 \).

Therefore, \( r_{\alpha, \beta, c} \) may be used for \( \beta > a \). In fact \( r_2 \) may be used for \( \beta \geq a \) (see [3]).

Now we determine the form of the extremal functions \( f_0(z) \) for Theorem.

Taking into account (8) and the fact that the minimum in case (19) is realized at a point on the diameter \( \eta = 0 \), we conclude that \( p(z) \) (see (6)) should in this case be taken in the form

\[ p(z) = \frac{1}{2} \frac{1 + ze^{-\theta}}{1 - ze^{-\theta}} + \frac{1}{2} \frac{1 + ze^{\theta}}{1 - ze^{\theta}}, \]

where \( \cos \theta \) is found from the equation

\[ h + (1 - r_1^2)(1 - 2r_1\cos \theta + r_1^2) = R_0 \]

in which the quantities \( r_1 \) and \( R_0 \) are determined by formulas (21) and (17).

Then the extremal function \( f_0(z) \) can be represented by the expression

\[ f_0(z) = \frac{1}{c + 1} z(c + 1 - 2(c + a)\cos \theta + (c + 2a - 1)z^2)(1 - 2z\cos \theta + z^2)^{-1/2}. \]

In case (20), \( p(z) \) is given by \( p(z) = \frac{1}{1 - z} \). Hence the extremal function

\[ f_0(z) = \frac{1}{c + 1} z(c + 1 - (c + 2a - 1)z)(1 - z)^{-1/2}. \]

The proof of Theorem is now completed.
This theorem was found by S. D. Bernardi \[2\] when \( a = \beta = 0 \) and \( c \) is positive integer.

References


5. V. A. Zmorovic', On bounds of convexity for starlike functions of order \( \alpha \) in the circle \( |z| < 1 \) and in the circular region \( 0 < |z| < 1 \), Mat. Sb. 68 (1965), 518-526; English transl., Amer. Math. Soc. Transl. (2) 80 (1969), 203-213.