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The radius of starlikeness for some analytic functions

By

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Abstract

Let $F(z)$ be regular in $|z|<1$ and $F(0)=0$, $F'(0)=1$. Let

\[ f(z) = \frac{1}{c+1} z^{1-c} \{z^c F(z)\}^{\prime} \]

and $c$ be a real and non-negative constant. We have determined the radius of starlikeness of order $\beta$ for $f(z)$ when $F(z)$ is starlike of order $\alpha$, $0 \leq \beta, \alpha < 1$.

1. Introduction

In this paper we shall treat a generalization of the recent results of H. S. Al-Amri [1]. The method used in the proof of the theorem is that of V.A. Zmorović [5]. The class $S^*(\alpha)$ is called the starlike functions of order $\alpha$. Analytically, $F(z) \in S^*(\alpha)$ if and only if $\text{Re} \{z F'(z)/F(z)\} > \alpha$ in $|z|<1$.

We shall denote by $P$ the class of regular functions $p(z)$ in $|z|<1$, $p(0)=1$ such that $\text{Re} \{p(z)\} > 0$. Let

\[
(1) \quad f(z) = \frac{1}{c+1} z^{1-c} (z^c F(z))^{\prime}, \quad \text{and } c \text{ be a real and non-negative constant.}
\]

Let $F(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t) \, dt$ be regular in $|z|<1$ and $F(0)=0$, $F'(0)=1$. If $F(z) \in S^*(\alpha)$, then there exists $p(z) \in P$ such that

\[
(2) \quad z F'(z)/F(z) = \alpha + (1-\alpha) p(z).
\]

From (1), (2) we get

\[
\int_0^z t^{c-1} f(t) \, dt = c + \alpha + (1-\alpha) p(z).
\]

Thus

\[
z f'(z)/f(z) = \beta
\]

(3) \quad \begin{array}{c}
\text{where } h = (c+\alpha)/(1-\alpha), 0 \leq \alpha < 1.
\end{array}

We shall determine $r_{\alpha, \beta, c}$ the radius of starlikeness of order $\beta$ for $f(z)$. Clearly, $r_{\alpha, \beta, c}$ is the smallest positive root of $Q_{\alpha, \beta, c} = 0$, where

\[
Q_{\alpha, \beta, c}(r) = \min_{p \in P} \min_{|z|=r < 1} \text{Re} \{-(c+\beta) + (1-\alpha)(h+p(z)) \}
\]
Thus this problem are reduced to seeking the quantity

\[ Q(r) = \min_{p \in F} \min_{|z| = r} \text{Re} \{ \Psi(p(z), zp'(z)) \}, \]

where \( \Psi(w; W) \) is an analytic function of the variables \( w \) and \( W \) in the \( W \)-plane and in the half plane \( \text{Re}(w) > 0 \).

By M. S. Robertson's variational method [4], the minimum in (5) is realized for functions of the form

\[ p(z) = \lambda_1 \frac{1 + ze^{-i\theta_1}}{1 - ze^{-i\theta_1}} + \lambda_2 \frac{1 + ze^{-i\theta_2}}{1 - ze^{-i\theta_2}}, \]

where \( \theta_1, \theta_2 \in [0, 2\pi] \), \( \lambda_1, \lambda_2 \geq 0 \) and \( \lambda_1 + \lambda_2 \geq 0 \) and \( \lambda_1 + \lambda_2 = 1 \).

We obtain \( r_*, \beta_*, \epsilon \) by an application of a theorem due to V. A. Zmorović.

Theorem A (V. A. Zmorović) Let \( \Psi(w; W) = M(w) + N(w) W \), where \( M(w) \) and \( N(w) \) are defined and finite in the half plane \( \text{Re}(w) > 0 \). Put

\[ w = \lambda_1 \frac{1 + z_1^n}{1 - z_1^n} + \lambda_2 \frac{1 + z_2^n}{1 - z_2^n}, \]

\[ W = \lambda_1 \frac{2mz_1^n}{(1 - z_1^n)^2} + \lambda_2 \frac{2mz_2^n}{(1 - z_2^n)^2}, \]

where \( z_1 \) and \( z_2 \) are arbitrary points on \( |z| = r < 1 \), \( m \) is a positive integer, \( \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1 \).

Then the function \( \Psi(w; W) \) has the form

\[ \Psi(w; W) = M(w) + \frac{\rho - \rho^*}{2} m(w^2 - 1)N(w) + \frac{1}{2} m(\rho^2 - \rho^*_2)N(w) e^{2i\psi}, \]

where

\[ (1 + z_k^n)/(1 - z_k^n) = a + \rho \exp(i\psi_k), \quad (k = 1, 2), \]

\[ w = a + \rho \exp(i\psi_0), \quad (0 < \rho < \rho_*), \]

\[ a = (1 + r^2)/(1 - r^2), \quad \rho = 2r^n/(1 - r^2), \]

Also

\[ \min \text{Re} \{ \Psi(w; W) \} = \Psi_{\rho_0}(w) \]

\[ = \text{Re} \{ M(w) + \frac{1}{2} m(w^2 - 1)N(w) \} - \frac{1}{2} m |N(w)| (\rho^2 - \rho^*_2). \]

The minimum in (7) is reached when

\[ \exp \{ i(2\psi - \arg N(w)) \} = 1. \]

Comparing problem (4) with the terms of Theorem A, we have \( m = 1 \), and

\[ z p(z)/(h + p(z)). \]
M(w) = -(c + β) + (1 - a)(w + h),
N(w) = 1/(w + h),
h = (c + a)/(1 - a), 0 ≤ a < 1.

From (4), (7) and these relations our problem is reduced to minimizing Ψs(w), where

\[ Ψs(w) = \text{Re}\left\{ -(c + β) + (1 - a)(w + h) + \frac{1}{2} \frac{w^2 - 1}{w + h} \right\} = -\frac{1}{2} \rho^2 - \rho \bar{h} \]

2. Radius of starlikeness

Theorem. Let \( F(z) \in \mathcal{S}^*(a) \), 0 ≤ a < 1,
\( f(z) = \frac{1}{c + 1} z^{1 - c} \left[ z^c F(z) \right]' \) in \( |z| < 1 \). Let \( r_{s, \beta} \) be the radius of the largest disk in which \( \text{Re}\{zf'(z)/f(z)\} > \beta \).

Put

\[ a^2 + 2(β - 2a - c)a + (2h + β + c)^2 - 4(2 - a)h^2 = 0, \]
\[ 4a^2 - 2(β - c + 2)a + (1 + β)(1 - c) \left( 1 - r^2 \right) + 2(3 - β + c)a - cβ - 2r + (1 + c)(1 - β) = 0, \]
where \( a = (1 + r^2)/(1 - r^2) \), \( h = (c + a)/(1 - a) \); also
\[ B_3 = 4(1 - c)a^4 + 2(1 + 12c - c^2)a^3 + (2c^3 + 14c^2 - 23c - 29) \alpha^2 - (6c^3 + 21c^2 + 29c - 14)\alpha + 4c^2 + 14c, \]
\[ B_4 = (2a^2 + c^2 \alpha + 3c\alpha + 3a - c^2 - 2)^2 + 2(1 - a)(2a + c)^2(2a^2 + c^2 \alpha + 3c\alpha + 3a - c^2 - 2) + (2a + c - 4) \left( 2a^2 + c^2 \alpha + 3c\alpha + 3a - c^2 - 2 \right)^2 - 4(2 - a) \left( c + a \right)^2 \left( 2a + c \right). \]

Then \( r_{s, β} \) is the smallest positive root \( r \) in (10) with \( 0 ≤ β ≤ \beta_0(a, c) \) and the smallest positive root \( r \) in (11) with \( \beta_0(a, c) ≤ β < 1 \), where \( \beta_0(a, c) \) is the smallest positive root of (12). These results are sharp.

Proof. Our problem is reduced to minimizing \( Ψ_s(w) \). This minimum is reached when the point \( w = (w - a)/\rho \) is fixed, and the chord passing through it and through the points \( a + \rho \exp i\phi \) is perpendicular to the vector \( \exp i\phi/2 \), where \( w + h = R \exp i\phi \).
By setting \( w = a + \xi + i\eta \), \( \rho^2 = \xi^2 + \eta^2 ≤ ρ^2 \).
Then (9) becomes
\[
\psi_\rho (w) = \psi_\rho (\xi ; \eta)
\]
(13) 
\[
= -(\alpha - \beta + h) + \left(\frac{3}{2} - \alpha (a + \xi + h) + \frac{1}{2} (a + \xi + h) R^2 - \frac{1}{2} (\xi^2 - r^2) R^2 \right),
\]
where 
\[
R^2 = (a + \xi + h)^2 + \eta^2.
\]
We now wish to minimize $\psi_\rho (w)$ as a function of $\eta$. A differentiation shows that
\[
\frac{\partial \psi_\rho}{\partial \eta} = \frac{1}{2} \eta R^2 S (\xi , \eta),
\]
where
\[
S (\xi , \eta) \geq S (-\rho , \rho) = 2((a - \rho)^2 + h(a - \rho) + 1)(a + \rho - \rho) > 0.
\]
Hence
\[
S (\xi , \eta) \geq S (-\rho , \rho) = 2((a - \rho)^2 + h(a - \rho) + 1)(a + \rho - \rho) > 0.
\]
Thus we see from (14) that $\Psi_\rho (\xi , \eta)$ is minimized on every constant of the circle $\xi = \xi_0 + \rho$ at the point $\eta = \eta_0$.

Therefore the minimum of $\psi_\rho (\xi ; \eta)$ in the disk $\xi^2 + \eta^2 \leq \rho^2$ occurs somewhere on the diameter $\eta = 0$. Setting $\eta = 0$ in (13), we have
\[
(15) \quad S (\xi , \eta) = S (\xi , 0) = \ell (R) = (2 - \alpha) R + (h^2 + a h) R^2 - (a + 2 h) (c + \beta).
\]

The absolute minimum of $\ell (R)$ is realized at
\[
(17) \quad R_0 = \left(\frac{(h^2 + a h)}{2 - \alpha}\right)^{1/2}
\]
where
\[
R_0 < a + h + \rho.
\]
However, if $R_0 < a + h - \rho$, then the minimum of $\ell (R)$ is attained at
\[
(18) \quad R_1 = a + h - \rho.
\]
The radius $r_{a, c}$ is therefore determined from either
\[
(19) \quad \ell (R_0) = 0,
\]
where $R_0$ is given by (17), or from
\[
(20) \quad \ell (R_1) = 0,
\]
where $R_1$ is given by (18).

These two equations coincide for some $\beta_0 = \beta_0 (a , c)$.

(19) and (20) can be reduced to (10) and (11), respectively.

From (10) and (11) we get,
\[
(21) \quad r_{a, c} = \frac{D}{2 a - \beta + c - 1 - D^{1/2}},
\]
where
\[
D = 4(2 - \alpha) h^2 - (2 h + \beta + c)^2 + (\beta - 2 a - c)^2.
\]
and
\[
(22) \quad r_2 = r_{n, \beta, c} = (1 + c)(1 - \beta) / \left( \left( \beta (c + a) + 2(1 - \alpha) - (c + 1) \alpha \right) + ((\beta (c + a) + 2(1 - \alpha) - (c + 1) \alpha)^2 \right.
\]
\[\left. - (1 + c)(1 - \beta)(4a^2 - 2\alpha + 2\alpha - 4\alpha - 1 - c + \beta - c^2) \right)^{1/2},
\]
respectively. To determine the \( \beta_0 = \beta_0(\alpha) \) that makes the transition from (21) to (22) set \( R_0 = R_1 \).

It follows then
\[
(23) \quad \beta_0 = (2 - \alpha) R_1^2 - \beta.
\]
From (20), (16) and (23), we obtain
\[
(24) \quad (3 - 2\alpha) a + 2\alpha - \beta + c = 2(2 - \alpha) R_1.
\]
From (24), (10), we obtain
\[
(25) \quad a = \frac{(1 - \alpha) \beta^2 + (3\alpha - 2\alpha + 5c) \beta - (2\alpha^2 + 3\alpha + c^2 + 3\alpha - c^2 - 2)}{(1 - \alpha)(2\alpha - \beta + c)}.
\]
Equation (12) is deduced from (10) and (25).

Note that \( r_{n, \beta, c} = r_1 \) cannot be used when \( \beta > \alpha \), since \( D < 0 \).

Therefore, \( r_{n, \beta, c} \) may be used for \( \beta > \alpha \). In fact \( r_2 \) may be used for \( \beta \geq \alpha \) (see [3]).

Now we determine the form of the extremal functions \( f_0(z) \) for Theorem.

Taking into account (8) and the fact that the minimum in case (19) is realized at a point on the diameter \( \eta = 0 \), we conclude that \( p(z) \) (see (6)) should in this case be taken in the form
\[
p(z) = \frac{1}{2} \left( \frac{1 + z e^{-\theta}}{1 - z e^{-\theta}} + \frac{1}{2} \frac{1 + z e^{-\theta}}{1 - z e^{-\theta}} \right),
\]
where \( \cos \theta \) is found from the equation
\[
h + (1 - r_1^2)(1 - 2r_1 \cos \theta + r_1^2) = R_0
\]
in which the quantities \( r_1 \) and \( R_0 \) are determined by formulas (21) and (17).

Then the extremal function \( f_0(z) \) can be represented by the expression
\[
f_0(z) = \frac{1}{c + 1} z(c + 1 - 2(c + 1)z) (1 - 2z \cos \theta + z^2)^{1/2},
\]
In case (20), \( p(z) \) is given by
\[
p(z) = \frac{1}{1 - z}. \quad \text{Hence the extremal function}
\]
\[
f_0(z) = \frac{1}{c + 1} z(c + 1 - (c + 2a - 1)z) (1 - z)^{1/2},
\]
The proof of Theorem is now completed.
This theorem was found by S. D. Bernardi [2] when $\alpha = \beta = 0$ and $c$ is positive integer.

References

5. V. A. Zmorovic, On bounds of convexity for starlike functions of order $\alpha$ in the circle $|z| < 1$ and in the circular region $0 < |z| < 1$, Mat. Sb. 68 (1965), 518-526; English transl., Amer. Math. Soc. Transl. (2) 80 (1969), 203-213.