On the equation $u_t - \Delta u + u^3 = f$

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Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma$. This note is concerned with the boundary value problem for the equation

$$-\Delta b + b^3 = 1 \quad (x \in \Omega)$$

under the boundary condition,

$$b|_{\Gamma} = b_0(x)$$

and also with the initial-boundary value problem for the equation

$$\frac{\partial u}{\partial t} - \Delta u + u^3 = f \quad (x \in \Omega, t \geq 0)$$

under the initial condition

$$u|_{t=0} = u_0(x)$$

and the boundary condition

$$u|_{\Gamma} = b_0(x)$$

We study the equation (3) when $u(x)$ is sufficiently close to a solution $b(x)$ of the equation (1) with the boundary condition (2), and prove that if $b(x)$ satisfies certain conditions, a solution $u$ converges to $b$ as $t \to \infty$.

The problem for the Navier-Stokes equations has been treated by Heywood in [1], [2].

Preliminaries

We denote by $L^p(\Omega)$, $1 \leq p < \infty$, the Banach space of all real functions on $\Omega$, with norm

$$|u|_p = \left( \frac{\int_{\Omega} |u(x)|^p \, dx}{p} \right)^{1/p}$$

For $p=2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \int_{\Omega} u(x)v(x) \, dx$$

and we set

$$|u| = (u, u)^{1/2}$$

$H^1(\Omega)$ is the space of functions of $L^2(\Omega)$ whose first derivatives (in the sense of distributions) are in $L^2(\Omega)$. $H^1(\Omega)$ is a Hilbert space with the scalar product

$$((u, v)) = (u, v) + \sum_{i=1}^3 (D_i u, D_i v), \quad D_i = \frac{\partial}{\partial x_i}.$$
$H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support contained in $\Omega$. We write

$$(\nabla u, \nabla v) = \sum_{i=1}^3 (D_i u_i, D_i v_i), \quad (\nabla u, \nabla u) = |\nabla u|^2.$$

**Lemma (Sobolev)** If $u \in H^1_0(\Omega)$, then

$$|u|^p \leq C |\nabla u|, \quad 1 \leq p \leq 6$$

Where $C_Q$ is a constant depending only on $\Omega$.

**Generalized solution**

We assume the functions $f, b$ are time independent and $b_0$ has an extension $b(x)$ into $\Omega$ satisfying

$$b \in L^4(\Omega)$$

(6) $\{ -\Delta b + b^3 - f \in L^4(\Omega) \}$

$v_0 = u_0 - b \in H^1_0(\Omega)$

**Definition** We call $u(x, t) = v(x, t) + b(x)$ a generalized solution of (3), (4), (5) in $\Omega \times (0, \infty)$ if $b$ satisfies (6) and if for all $T > 0$:

(i) $v \in L^t(0, T; H^1_0(\Omega) \cap L^4(\Omega)), v_t \in L^t(0, T; L^4(\Omega))$

(ii) $v(x, t) - v(x) \to 0$ as $t \to \infty$

(iii) $\int_0^T \{(v_t, \phi) + (\nabla v_t, \nabla \phi) + (v^3 + 3 bv^2 + 3 b^2 v, \phi) + (b^3 - \Delta b - f, \phi)\} dt = 0$ for all $\phi \in C^\infty(\Omega \times (0, T))$.

**Theorem** Let $f, u, b$ be given. Suppose a solution $b$ of equations (1), (2) satisfies the condition (6), and

(i) $1 - \frac{3}{2} C_Q |b| = \mu > 0$

(ii) $|v|_o + |\Delta u| + f - u^2 \leq \frac{\mu}{36 C_Q^4 |b|^2}$

Then the initial-boundary problem (3), (4), (5) has a generalized solution $u$ in $\Omega \times (0, \infty)$, and

$$|u(t) - b| \leq |v|_o \exp(-\mu C^{-2} Q).$$

**A Priori estimates**

We shall employ Galerkin’s method to prove the existence of generalized solutions.

Let $\{w_j(x)\}$ be a complete system of functions in $H^1_0(\Omega)$.

We suppose that $u = \sum_{j=1}^m g_{jm}(t) w_j(x)$ Let

$$v^m(x, t) = \sum_{j=1}^m g_{jm}(t) w_j(x), \quad m = 1, 2, \ldots.$$
be the solution of the system \((j=1,\ldots,m)\) of ordinary differential equations,

\[
(v^m_t, w_j) + (\nabla v^m, \nabla w_j) + (v^m, w_j) + (v^m, v^m) + 3b(v^m)^2 + 3b(x) = 0
\]

which satisfy the initial conditions \(g^m(0) = \|v\|_0\) and \(g^m(0) = 0\)

for \(j=1,\ldots, m\). There exists \(v^m\) in \([0, T]\), \(T > 0\).

By multiplying each equation \((7)\) by \(g^m\), summing \(\sum\), noting the Sobolev's lemma, inequality \((8)\) is obtained.

\[
\frac{1}{2} \frac{d}{dt} |v^m|^2 + \mu(|\nabla v^m|^2 + |v^m|^4) \leq 0.
\]

This shows that \(t_m = T\). According to the Sobolev's lemma, we have

\[
|v^m(t)| \leq |v_0| \exp\left(-\frac{\mu t}{C^2}\right)
\]

An applications of the Schwarz inequality to \((8)\) yields

\[
\mu(|\nabla v^m|^2 \leq |v_0| |v_t|
\]

By differentiating each equation \((7)\) with respect to \(t\), multiplying by \(\frac{d}{dt} g^m\) and summing \(\sum\), and using \((10)\), we obtain

\[
\frac{1}{2} \frac{d}{dt} |v^m|^2 + (1 - 6\mu^{1/2}C^3 |b|_1 |v^{1/2}|v_t^{1/2}) |v^m|^2 \leq 0.
\]

From the assumption of the theorem, it follows that \(|v^m_t|\) and hence \(|\nabla v^m|\) are bounded:

\[
|v^m_t| \leq |\Delta u_o + f - u_0|\]

\[
|\nabla v^m| \leq \mu^{1/2} |v_0|^{1/2} |\Delta u_o + f - u_0|^{1/2}.
\]

(the proof of the theorem)

By the estimates \((8), (11), (12)\) and the Rellich theorem, a subsequence \(\{v^k\}\) can be selected from \(\{v^m\}\) such that

\[
v^k \rightarrow v \text{ weakly in } L^2(0, T; H^1(\Omega))
\]

\[
v^k_t \rightarrow v_t \text{ weakly in } L^2(0, T; L^2(\Omega))
\]

\[
v^k \rightarrow v \text{ strongly and a.e. in } L^2(0, T; L^2(\Omega)),
\]

\[
(v^k)^3 \rightarrow v^3 \text{ weakly in } L^{4/3}(0, T; L^{4/3}(\Omega))
\]

According to well known results, it follows that \(v^3 = v^3\) and \(v\) is a generalized
solutions of the equations (3), (4), (5).

By (9), each $|v^m(t)|$ decays exponentially, uniformly in $m$.

Thus this estimate must hold for $|v(t)|$ also.

References

[1] J. Heywood, On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions,

    Stanford University, December 1967.