<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>On the equation u_t-Δu + u^3=f</td>
</tr>
<tr>
<td>著者</td>
<td>Okamoto, Kazuo</td>
</tr>
<tr>
<td>発行年月日</td>
<td>1978</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/16509">http://hdl.handle.net/10069/16509</a></td>
</tr>
</tbody>
</table>
On the equation $u_t - \Delta u + u^3 = f$

Kazuo Okamoto

Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^3$ with sufficiently smooth boundary $\Gamma$. This note is concerned with the boundary value problem for the equation

$$(1) \quad -\Delta b + b^3 = 1 \quad (x \in \Omega)$$

under the boundary condition,

$$(2) \quad b|_{\Gamma} = b_o(x)$$

and also with the initial-boundary value problem for the equation

$$(3) \quad u_t - \Delta u + u^3 = f \quad (x \in \Omega, \ t \geq 0)$$

under the initial condition

$$(4) \quad u|_{t=0} = u_o(x)$$

and the boundary condition

$$(5) \quad u|_{\Gamma} = b_o(x)$$

We study the equation (3) when $u(x)$ is sufficiently close to a solution $b(x)$ of the equation (1) with the boundary condition (2), and prove that if $b(x)$ satisfies certain conditions, a solution $u$ converges to $b$ as $t \to \infty$.

The problem for the Navier-Stokes equations has been treated by Heywood in [1], [2].

Preliminaries

We denote by $L^p(\Omega)$, $1 \leq p < \infty$, the Banach space of all real functions on $\Omega$, with norm

$$\|u\|_p = \left( \int_{\Omega} |u(x)|^p \ dx \right)^{1/p}$$

For $p=2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \int_{\Omega} u(x)v(x) \ dx,$$

and we set

$$\|u\| = (u, u)^{1/2},$$

$H^1(\Omega)$ is the space of functions of $L^2(\Omega)$ whose first derivatives (in the sense of distributions) are in $L^2(\Omega)$. $H^1(\Omega)$ is a Hilbert space with the scalar product

$$((u, v)) = (u, v) + \sum_{i=1}^3 (D_i u, D_i v), \quad D_i = \frac{\partial}{\partial x_i}. $$
$H^1_0(\Omega)$ is the closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support contained in $\Omega$. We write

$$(\nabla u, \nabla v) = \sum_{i=1}^3 (D_i u, D_i v), \quad (\nabla u, \nabla u) = |\nabla u|^2.$$ 

**Lemma (Sobolev)** If $u \in H^1(\Omega)$, then

$$|u| \leq C |\nabla u|, \quad 1 \leq p \leq 6$$

Where $C_\Omega^p$ is a constant depending only on $\Omega$.

**Generalized solution**

We assume the functions $f, b$ are time independent and $b_0$ has an extension $b(x)$ into $\Omega$ satisfying

$$b \in L^4(\Omega)$$

(6) $$\begin{cases} -\Delta b + b^3 - f \in L^4(\Omega) \\ \nu_0 = u_0 - b \in H^1_0(\Omega) \end{cases}$$

**Definition** We call $u(x, t) = v(x, t) + b(x)$ a generalized solution of (3), (4), (5) in $\Omega \times (0, \infty)$ if $b$ satisfies (6) and if for all $T > 0$ : 

(i) $v \in L^q(0, T; H^1_0(\Omega) \cap L^q(\Omega), \nu_t \in L^q(0, T; L^q(\Omega))$

(ii) $v(x, t) - v(x) \to 0$ as $t \to \infty$

(iii) $\int_0^T \langle (v_t, \phi) + (\nabla v, \nabla \phi) + (v^3 + 3b v^2 + 3b^2 v, \phi) + (b^3 - \Delta b - f, \phi) \rangle \, dt = 0$ for all $\phi \in C_0^\infty(\Omega \times (0, T) )$.

**Theorem** Let $f, u_0, b$ be given. Suppose a solution $b$ of equations (1), (2) satisfies the condition (6), and

(i) $1 - \frac{3}{2} \frac{C}{\Omega} |b| = \mu > 0$

(ii) $|v|_o \cdot |\Delta u|_o + f - u_0^2| \leq \frac{\mu}{36 \frac{C^p}{\Omega} |b|^2}$

Then the initial-boundary problem (3), (4), (5) has a generalized solution $u$ in $\Omega \times (0, \infty)$, and

$$|u(t) - b| \leq |v|_o \exp (-\mu \frac{C^{-2}}{\Omega}).$$

**A Priori estimates**

We shall employ Galerkin’s method to prove the existence of generalized solutions.

Let $\{w_j(x)\}$ be a complete system of functions in $H^1_0(\Omega)$. We suppose that $u_0 = |v|_o w_1 + b$ Let

$$v^m(x, t) = \sum_{i=1}^m g^m_j(t) w_i(x)$$

$m = 1, 2, \ldots.$
be the solution of the system \((j=1, \ldots, m)\) of ordinary differential equations,
\[
 (7) \quad (v^m_t, w_j) + (\nabla v^m, \nabla w_j) + ((v^m)^3 + 3b(v^m)^2 + 3b^2 v^m, w_j) = 0
\]
which satisfy the initial conditions \(g^m_j(0) = |v^m_o|\) and \(g^m_j(0) = 0\)
for \(j=1, \ldots, m\). There exists \(v^m\) in \([0, t]\), \(t > 0\).
By multiplying each equation (7) by \(g^m_j\), summing \(\Sigma\), noting the Sobolev's lemma, inequality (8) is obtained.
\[
 (8) \quad \frac{1}{2} \frac{d}{dt} |v^m_t|^2 + \mu(|\nabla v^m|^2 + |v^m|^4) \leq 0.
\]
This shows that \(t_m = T\). According to the Sobolev's lemma, we have
\[
 (9) \quad |v^m_t(t)| \leq |v^m_o| \exp\left(-\frac{\mu t}{\mathcal{C}^2}\right)
\]
An application of the Schwarz inequality to (8) yields
\[
 (10) \quad \mu|\nabla v^m|^2 \leq |v^m_o| \cdot |v^m_t|
\]
By differentiating each equation (7) with respect to \(t\), multiplying by \(\frac{d}{dt} g^m_{jm}\)
(t), summing \(\Sigma\), and using (10), we obtain
\[
 \frac{1}{2} \frac{d}{dt} |v^m_t|^2 + (1 - 6\mu^{1/2} C_3 |b|_1 |v^m_o|^{1/2} |v^m_t|^{1/2} |v^m_t|^{1/2}) |v^m_t|^2 \leq 0.
\]
From the assumption of the theorem, it follows that \(|v^m_t|\) and hence \(|\nabla v^m_t|\)
are bounded :
\[
 (11) \quad |v^m_t| \leq |\Delta u^m_o + f - u^m_o|\]
\[
 (12) \quad |\nabla v^m_t| \leq \mu^{1/2} |v^m_o|^{1/2} |\Delta u^m_o + f - u^m_o|^{1/2}.
\]
(the proof of the theorem)

By the estimates (8), (11), (12) and the Rellich theorem, a subsequence \(\{v^k\}\) can be selected from \(\{v^m\}\) such that
\[
v^k \to v\ \text{weakly in} \ L^2(0, T; H^1_0(\Omega))
\]
\[
v^k_t \to v_t\ \text{weakly in} \ L^2(0, T; L^2(\Omega))
\]
\[
v^k \to v\ \text{strongly and a.e. in} \ L^2(0, T; L^2(\Omega)),
\]
\[
(v^k)^3 \to v'\ \text{weakly in} \ L^{4/3}(0, T; L^{4/3}(\Omega)).
\]
According to well known results, it follows that \(v' = v^3\) and \(v\) is a generalized
solutions of the equations (3), (4), (5).

By (9), each $|v^m(t)|$ decays exponentially, uniformly in $m$.
Thus this estimate must hold for $|v(t)|$ also.

References

[1] J. Heywood, On stationary solutions of the Navier-Stokes equations as limits of nonstationary solutions,

    Stanford University, December 1967.