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Some examples of covering sets

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1. We proposed in [1] a method to construct covering sets. In this paper, we give some more examples of covering sets.

Namely, we first give a skeleton covering Whose least modulus = 24, which is generated by $\langle 2, 3, 5, 7, 11, 13, 17, 19, 23 \rangle$. From that skeleton covering, we can obtain a covering set with the same generators (cf. [1], Theorem 5).

Secondly, we compare several covering sets whose least modulus = 8, and give some remarks on construstion of the best covering set.

Readers are assumed to be familiar with [1], and we use the same notation.

2. We give the table of a skeleton covering with the least modulus = 24. About the rule of notation, consult [1], especially § 4 and § 8;

2/2, 3/2, 3 : 3, 5, 7/2, 3 : 3, 5, 7 : 3, 5, 3 : 5, 7, 11, 5, (2.3.5) : 7, 11, 13, 17, 3, 19, (2.3.7)/2, 3 : 3, 5, ($2^{3}3$): 3, 5, ($2^{2}3^{2}$) : 5, 7, 11, 5, ($2^{2}3$.5) : 7, 2, 23, 2, 3, ($2^{2}3$.7), ($2^{2}7$) : P, 5 (patt. $3^2 \longrightarrow 3^3$), (3^3) : 5, 7, 11, ($3^{2}5$)-2 : P, 2, (2.3^3) : P, 7 (patt. $5 \longrightarrow 5^2$), 11, (5^2)-2 : 7, 11, 13, (5.7)-2.3 : 11, 13, 17, 2, 2, 3, (3.11), (5.11)-2.3 : P, 13, 3, (3.5^2)-2 : P, 11 (patt. $7 \longrightarrow 7^2$), 3, (7^2)-2.3 : 11, 13, 3, 2, 2, 2, (2.3.11), (7.11)-2.3 : 13, 7, 3, 2, 2, 2, (2.13), (3.13)-2, (7.13)-2.3 : 17, 2, 2, 3, 5, 2, 2, 3, 5, 7, (2.17), (3.17)-2, (7.17)-2.3 : P, ($3^{2}7$)-2 : 19, 2, 2, 3, 5, 13, 17, 2, 2, 3, 5, 7, (2.19), (3.19)-2, (7.19)-2.3/P, (2^5) : 3, 2, ($2^{4}3$) : 3, 5, ($2^{3}3^{2}$): 5, 2, ($2^{3}3$.5), 2, ($2^{3}5$) : P, 5 (patt. $3^2 \longrightarrow 3^3$) : 5 (patt. $3 \longrightarrow 3^2$), 7 (patt. $3 \longrightarrow 3^2$), 23, (4) ($2^{2}3^{2}5$) : P, 7 (patt. $5 \longrightarrow 5^2$), 2, (4) ($2^{2}5^{2}$) : 7 (patt. $2 \longrightarrow 2^2$), 2, ($2^{2}3$.5.7), (4) ($2^{2}5$.7) : 11, 13, 2, 2, 23, 3, (8) ($2^{2}3$.5.11), (10) ($2^{2}3$.11) : P, 13, 3, (4) ($2^{2}3$.5^2) : P, 2, 3, (4) ($2^{2}7^{2}$), (6) ($2^{2}3$.7^2) : P, ($2^{3}7$) : 23, 2, 13, 17, 19, 2, 2, 2, 3, 3, 5, 7, (2.23)-2, (3.23)-2^2, (7.23)-$2^{2}3$: P, ($2^{3}3$.7) : P, (2) ($2^{2}3^{2}7$) : P, 7 (patt. $5 \longrightarrow 5^2$), 11 : 7, 11, 13, (5) ($3^{2}5$.7)-2 : 11, 13, 17, 2, 2, 2, 19, (9) ($3^{2}11$)-2 : P, ($2^{2}3^{3}$) : 11, 13, 2, 2, 2, 3, (7) ($5^{2}11$)-2.3 : P, (5.7^2)-2, (5) (3.5.7^2)-2 : 11, 13, (2.5.7.11), 2, 2, (5.7.11), (9) (3.5.7.11)-2 : 2, 7, 2, 2, 2, 2, (7) (3.5.7.13)-2, (11) (5.7.13)-2 : P, 13 (patt. $11 \longrightarrow 11^2$), 17 (patt. $11 \longrightarrow 11^2$), 2, (11^2)-2, (3.11^2), (5.11^2)-2.3 : 13, 3, 2, 2, (11.13)-2, (7) (3.11.13)-2, (5.11.13)-2.3 : 17, 2, 2, 2, 7, 2, 2, 5, (11.17)-2, (11) (3.11.17)-2, (5.11.17)-2.3 : P, ($2^{2}11$) : P, ($2^{2}5$.11) : P, ($3^{2}5$.11)-2 : 3, 2, 2, 2, 3, (5.13), (7) (3.5.13)-2, ($5^{2}13$)-2.3 : P, ($3^{2}5^{2}$)-2 : P, ($3^{2}7^{2}$)-2 : P, 2, 2, 2, (6) (2.3.11^2), (7.11^2)-2.3 : 2, 2, 3, 2, (9) (7.11.13)-2.3 : P, ($3^{2}7$.11)-2 : P : P, ($2^{2}7$.11) : P, ($2^{2}3$.7.11) : P, 2, 2, 2, 3, (13^2)-2.3.7 : P, 2, 2, (7^{2}13)-2.3 : P, (3^{2}7.13)-2 : P, (2^{2}13) : P, (2^{2}7.13) : P, (2^{2}3.7.13) : P, 2, 2, 3, 5, 2, 2, 3, 5, (17^{2})-2.3.7 : P, (2^{2}17) : P, (2^{2}3.17) : P, (3^{2}17)-2 : 5, (5.17)-2.3 : P, (2^{2}7.17) : P, (2^{2}3.7.17) : P, (3^{2}7.17)-2 : 5, (5.7.17)-2.3 : P, 2, 2, (7^{2}17)-2.3 : P, 2, 2, 3, 5, 13, 2, 2, 3, 5, 7, (19^{2})-2.3.7 : P, (2^{2}19) : P, (2^{2}3.19) : P, (3^{2}19) -2 : 5, (5.19)-2.3 : (13.19)-2, (3.13.19), 2, (2.3.13.19), 2, (9) (7.13.19)-2.3 : 2, 2, 2, 3, 5, 2, (17.19)-2.3, (13) (7.17.19)-2.3 : P, (2^{2}7.19) : P, (2^{2}3.7.19) : P, (3^{2}7.19)-2 : 5, (5.7.19)-2.3 : P, 2, 2, (7^{2}19)-2.3/3, 2, (2^{4}3^{2}) : P, (2^{5}3) : P, 5 : 2, 2, (2^{3}3^{2}5) : P, 7, 2, (4) (2^{3}5^{2}) : P, (2^{4}5) : P, (2^{4}3.5) : 2, 2, 13, 17, 19, 2, 2, 2, 5, 5, (3^{2}23)-2, (14) (5.23)-2^{2}3^{2} : P, (2^{3}3.5^{2}) : P : P, 2, 2, 2, (5) (2^{2}5.11^{2}), (8) (2^{2}3.11^{2}), (10) (2^{2}3.5.11^{2}) : 2, 3, 2, 2, (5) 2, (2^{2}11.13), (8) (2^{2}3.11.13), (10) (2^{2}3.5.11.13), (12) (2^{2}5.11.13) : P, (2^{3}3.11) : P, (2^{3}3.5.11) : 2, 2, 3, 2, 2, 2, (11.23)-2^{2}, (3.5.11.23)-2^{2}, (17) (3.11.23)-2^{2}, (5.11.23)-2^{2} : P, (2) (2^{2}3^{2}5.11) : 3, 2, 2, 2, 3, (6) (2.5.13), (8) (2^{2}3.5.13), (10) (2^{2}5^{2}13), (12) (2^{2}3.5^{2}13) : P, (2) (2^{2}3^{2}5^{2}) : P, (2^{3}7^{2}) :P, (2) (2^{2}3^{2}7^{2}) : P, 2, 13 (patt. 23 $\longrightarrow$ 23^{2}), 17 (patt. 23 $\longrightarrow$ 23^{2}), 2, 2, 2, 3, 3, 5, 7, (23^{2})-2^{2}3.7 : P, (2^{3}23) : (3.13.23)-2.7, (7.13.23)-2^{2}, (8) (13.23), (10) (2.13.23), (12) (2^{2}.13.23) : 2, 2, 2, 2, (17.23)-2^{2}3, (2^{2}3.7.17.23), (12) (7.17.23)-2.3, (2^{2}7.17.23) : 2, 2, 2, 2, 3, 13, (19.23)-2^{2}3, (2^{2}3.7.19.23), (14) (7.19.23)-2.3, (2^{2}7.19.23) : P, (2^{3}3.23) : P, (2^{3}7.23) : P, (2^{3}3.7.23) : P, (2^{2}3^{2}23) -7 : P, (3^{2}7.23)-2 : (5.7.23)-2^{2}, (3.5.7.23)-2 : P, (7^{2}23)-2^{2}3 : 11 (patt.5 $\longrightarrow$ 5^{2}), 13 (patt. 5 $\longrightarrow$ 5^{2}), 2, 2, 2, 2, 19 (patt. 5 $\longrightarrow$ 5^{2}) : P, (5) (3^{2}5.7^{2})-2 : (2.3^{2}5.7.11), 2, (3) 2, 2, (6) (3^{2}5.7.11) : (3^{2}5.7.13), 2, (2.3^{2}5.7.13), 2, 2, 2, (7) (3^{2}13)-2 : P, 2, (2.3^{2}11^{2}), 2, (6) (3^{2}11^{2}), (9) (3^{2}5.11^{2})-2 : 13, 2, 2, 2, (11) (3^{2}5.11.13)-2 : 2, 2, 2, 2, (3^{2}11.17), 2, (2.3^{2}11.17), 5, (11) (3^{2}5.17)-2, (15) (3^{2}5.11.17)-2 : P : P : P, (2^{2}3^{2}11) : 2, 2, 2, 2, 2, 2, (5.11.19)-2.3^{2}, (13) (11.19)-2.3^{2} : P, 2, 2, 2, (7) (5^{2}11^{2})-2.3 : 2, 2, 2, 2, (9) (5^{2}11.13)-2.3 : P, (2^{2}3.5^{2}11) : P : P, (2^{2}5^{2}11) : P, (3^{2}5^{2}11)-2 : P, (5.7.11^{2})-2, 2, (9) (3.5.7.11^{2})-2 : (5.7.11.13)-2, 2, 2, (11) (3.5.7.11.13)-2 : P : P : P, (2^{2}3.5.7.13) : P, (5.7^{2}13)-2, (5) (3.5.7^{2}13)-2 : P, (2^{2}3.13) : P : P : P, (2^{2}5.7.13) : P, (2^{2}11^{2}) : P, 2, (11.13^{2}), 2, (2.11.13^{2}), (7) (3.11.13^{2})-2, (5.11.13^{2})-2.3 : P, (3^{2}11.13)-2 : P : P : P, 2, 2, 2, 2, 2, 3, (11.17^{2}) -2, (11) (3.11.17^{2})-2, (5.11.17^{2})-2.3 : P : P, (2^{2}3.11.17) : P, (2^{2}5.11.17) : (7.11.17) -2.3, (5.7.11.17)-2, (3.5.7.11.17) : P, (2^{2}11.17) : P, (2^{2}3.5.11.17) : P, (5^{2}11.17) -2.3 : P, (3^{2}5^{2}13)-2 : P, (2^{2}5.13) : P : P : P, (3^{2}5.13)-2 : P, (2^{2}3.7.11^{2}) : P, (2^{2}7.11^{2}) : P : P, (2^{2}3.7.11.13) : P, (2^{2}7.11.13) : P, (3^{2}7.11.13)-2 : P : P, (2^{2}3.7.13^{2}) : P, (2^{2}7.13^{2}) : P, (2^{2}13^{2}) : P, (3^{2}7.13^{2})-2 : P, (2^{2}7^{2}13) : P, (2^{2}3.7^{2}13) : P, (2^{2}17^{2}) : P, (2^{2}3.17^{2}) : P, (3^{2}17^{2})-2 : 5, (5.17^{2})-2.3 : P, (2^{2}7.17^{2}) : P, (2^{2}3.7.17^{2}) : P, (3^{2}7.17^{2})-2 : 5, (5.7.17^{2})-2.3 : P, (5^{2}17)-2 : P, (3.5^{2}7.17)-2 : P, (2^{2}7^{2}17) : P, (2^{2}3.7^{2}17) : P, (2^{2}19^{2}) : P, (2^{2}3.19^{2}) : P, (3^{2}19^{2})-2 : 5, (5.19^{2})-2.3 : (13.19^{2})-2, (3.13.19^{2}), 2, (2.3.13.19^{2}), 2, (9) (7.13.19^{2})-2.3 : P, (2^{2}7.19^{2}) : P, (2^{2}3.7.19^{2}) : P, (3^{2}7.19^{2})-2 : 5, (5.7.19^{2})-2.3 : P, 2, 2, (7^{2}19^{2})-2.3 : P, (5^{2}19)-2 : P : P, (2^{2}13.19) : P, (2^{2}7.17.19) : P : P : P : 5 : P, (2^{2}17.19) : P, (5^{2}7.19)-2 : P, (2^{2}7^{2}19) : P, (2^{2}3.7^{2}19)/P, 5 : P, (2^{5}3^{2}) : 5, 2, (2^{3}3^{3}5) : P, (2^{4}3^{2}5) : P : 7, (2^{3}5^{2}7)-3, (4) (2^{3}3.5.7), (6) (2^{3}5.7) : P, (2^{4}5^{2}) : P, (2^{3}3^{2}5.23) : P : (5.13.23)-2^{2}3, (3^{2}5.13.23), (9) (2.3^{2}5.13.23), (11) (2^{2}3^{2}5.13.23) : 2, 2, (3^{2}17.23)-2, (7) (5.17.23)-2^{2}3^{2},

$(2^23^217.23)$: 2, 2, 2, 2, $(3^219.23)-2$, (9) $(5.19.23)-2^23^2$, $(2^23^219.23)$: P, (2^33^223) : P, $(2^35.23)$: P, $(2^33.5.23)$: P, $(5^223)-2.3$: P, $(2^25^223)-3$, $(3^25^223)-2$: P, $(2^33.5.11^2)$: P, $(2^33.11^2)$: P, $(2^35.11^2)$: P, $(2^35.11.13)$: P, (2) $(2^23^211.13)$: P, $(2^33.11.13)$: P, $(2^311.13)$: P, $(2^33.5.11.13)$: P, $(2^33.5.11.23)$: P : P, $(3^211.23)$ -2 : P, $(2^311.23)$: P, $(2^33.11.23)$: P, $(2^35.11.23)$: P, (2) $(2^23^25^213)$: P, $(2^33.5.13)$: P, (2^35^213) : P, $(2^33.5^213)$: P, (2) $(2^23^25.13)$: P, (2^323^2) : P, $(2^33.23^2)$: P, $(2^37.23^2)$: P, $(2^33.7.23^2)$: P, $(2^23^223^2)-7$: P, $(3^27.23^2)-2$: $(5.7.23^2)-2^2$, $(3.5.7.23^2)$ -2 : P, $(7^223^2)-2^23$: P, $(2^33.7.17.23)$: P, $(2^317.23)$: P, $(2^33.17.23)$: P, $(2^37.17.23)$: P, $(2^33.7.19.23)$: P, $(2^319.23)$: P, $(2^33.19.23)$: P, $(2^37.19.23)$: P, $(3^27.19.23)-2$: (1) $(2^213.19.23)$, (3) $(13.19.23)-2.3$, (8) $(2^23.13.19.23)$, (10) $(7.13.19.23)$, (12) $(2.7.13.19.23)$: P, $(2^23^25^211)$: P : P: P : P, $(2^23^27.11)$: P : P : P, $(2^23^25.7.13)$: P : P : P : P, $(2^23^211^2)$: P : P, $(3^211.13^2)$, (3) $(2.3^211.13^2)$, (7) $(3^213^2)-2$, (11) $(3^25.11.13^2)-2$: P, $(2^23^25.11.13)$: P, (2^23^213) : P : P, $(2^23^25.11.17)$: P : P, $(2^23^211.17)$: P : P : P, (3) $(3^25^211.17)-2$: P, $(2^23.5.11.19)$: P : P, $(2^25.11.19)$: P, $(2^23^211.19)$: P, $(2^23.11.19)$: P, $(2^211.19)$: P, $(2^23.5^211^2)$: P, $(2^25^211^2)$: P : P, $(2^25^211.13)$: P, $(2^23.5^211.13)$: P : P : P : P : P : P, $(2^211.13^2)$: P : P : P, $(2^211.17^2)$: P, $(2^23.11.17^2)$: P, $(2^25.11.17^2)$: P, $(2^23.5.11.17^2)$: P, $(3^211.17^2)-2$: P, $(5^217^2)-2$: P, $(3.5^217^2)-2$: P, $(5^219^2)-2$: P : P, $(2^213.19^2)$: P, $(3.5^219^2)-2$: P, $(2^27^219^2)$: P, $(2^23.7^219^2)$: P/7, 7, 2, (4) (2^43^35) : P, 7, 2 : P : P, (2) $(2^33.7^2)$, (4) $(2^33.5^27^2)$, (6) $(2^35^27^2)$: P, $(2^35.17.23)$: P : P, $(2^35.19.23)$: P : P, $(2^33.5.19.23)$: P, $(2^33^219.23)$/2, 2, (2^33^27), $(2^43.5.7)$, (2^43^27), $(2^33^25.7)$, (2^33^37) : 2, $(2^45.7)$: P, (2^53^35) : 7, (2) $(2^33^35^27)$, (4) $(2^33^25^27)$: P/P, (2^53^27) : P, (2^43^37) : P : P, (2) $(2^33^35^27^2)$, (4) $(2^33^25^27^2)$ //

3. We consider the following skeleton covering which is generated by $\langle 2, 3, 5 \rangle$, whose least modulus = 8 :

2/2, 3/2, 3 : 3, 5, 3/P, (2^3) : 3, 2, (2^23) : P, 5, (3^2) : 5, 2 ,(2.5), $(3.5)-2$: P, 2, (2.3^2)/P, 5 : P, (2^33) : 5, 2, (3) $(3^25)-2$: P, $(5^2)-2.3$: 2, (2^25) : 2, (2^23^2)/5, 2, $(2^23.5)$, (4) (2^23^25) : P, (3) $(3^25^2)-2$: 2 : P, (2^35) : P, (2^33^2)/P, (2) (2^25^2), (4) $(2^23.5^2)$: P, $(2^33.5)$: P //

To obtain a covering set from this skeleton covering, we may take two lines, i. e. (A) introduce a number w which is relatively prime to 30, and obtain a covering set generated by $\langle 2, 3, 5, w \rangle$, or (B) seek a covering set with the same generators. The existence of those covering sets is assured in [1] by Theorem 4' and Theorem 5.

However, these covering sets obtained by that are not always the best ones. First of all, the question arises how we define the best covering. Several authors have sought a covering set whose moduli are taken from divisors of a number N, and tried to obtain that by a smaller N. For example, C. Krukenberg gave in [2] the following covering set whose least modulus = 8, and N $=2^53^25^27$. Using our notation, that is;

2/2, 3/2, 3 : 7, 2, 3/(2^3), 2 : 5, 5, 3 : (2.7), $(3.7)-2$, 2, 2, 3, 5 : 2, (2^23) :

(2.3^2), 2, $(3^2)/(2^4)$, 2 : 5, (2.5), (3.5), (2^25), 2 : 5, (2) (2.3.5), (4) $(2^23.5)$: 5, 5 : 2, (2^27) : 2, $(2^23.7)$: (3^27)-2, 2 : 5, (5.7), (3) (2.5.7), 2 : 2, (2^33) : 2, $(2^23^2)/3$, 3, (5^2)-2, (3.5^2), (2^25^2), 2 : (2^35)-3 : (3) $(2.3.5^2)$, (5) $(2^23.5^2)$: 5, (2) (3^25), (4) 2 : 5, (2) (2.3^25), (4) (2^23^25) : 2, (2^37) : 2, $(2^33.7)$: 2, (2^23^27) : (1) (5^27), (3) (2.5^27), 2 : 2, $(2^25.7)$: 2, (2^43) : 2, (2^33^2) /5, 5, 3 : 5, 5, 3 : (2^35^2)-3 : (2) (3^25^2), (4) 2 : (1) (2^33^25) : (2) (2.3^25^2), (4) $(2^23^25^2)$: 2, (2^47) : 2, $(2^43.7)$: 2, (2^33^27) : 2, (2^25^27) : 2, $(2^35.7)$: (2^5), (2^53) : 2, $(2^43^2)/5$, (3) (2^45)-2 : 5, (4) $(2^43.5)$: 5, 5 : 5, (4) $(2^53.5)$: 5 : 5, 5 : (1) $(2^33^25^2)$: (1) (2^57) : (1) $(2^53.7)$: 2, (2^43^27) : 2, (2^35^27) : 2, $(2^45.7)$: (1) $(2^53^2)/(3)$ (2^45^2)-2 : (4) $(2^43.5^2)$: 5 : 5, (4) (2^43^25) : (4) $(2^53.5^2)$: : 5, (4) (2^53^25) : 5 : (1) (2^53^27) : 2, (2^45^27) : (1) $(2^55.7)/$: (4) $(2^43^25^2)$: (4) $(2^53^25^2)$: : (1) (2^55^27) //

Now, we consider the following skeleton covering;

2/2, 3/2, 3 : 7, 2, $3/(2^3)$, P : 5, 5, 3 : (2.7), (3.7)-2, 2, 2, 3, 5 : P, (2^23) : (2.3^2), 2, $(3^2)/P$, (2.5), (3.5), (2^25), 2 : P, (2) (2.3.5), (4) $(2^23.5)$: 5 (patt. 3 (0) $\longrightarrow$3 (2) 3 (0)), 5 (patt. 3 (1) $\longrightarrow$3 (2) 3 (1)) : P, (2^27) : P, $(2^23.7)$: (3^27) -2, 2 : P, (5.7), (3) (2.5.7), 2 : P, $(2^23^2)/(2^35)$-3 : P, (2^23^27) : P, $(2^25.7)$ //
(N. B. The two patternizations which appear in the table are ambiguous. Hence we noted the congruences attached to the corresponding arcs.)

Here, the above covering set may be regarded as obtained from that skeleton covering. But if we compare the two diagrams and scrutinize the process, we find the fact that very subtle considerations are paied. We think that it is more natural to seek a covering set with smaller kind of generators.

Anyhow we must say that we have as yet no settled way to construct efficiently a covering set or a skeleton covering.

References

[1] R. Morikawa, On a method to construct covering sets, (to appear).
[2] C. Kruhenberg, Covering set of the integers, Ph. D. Dissertation, University of Illi nois, (1971).