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On eventually covering families generated by the bracket function

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A family of sequences $\{[n\alpha_i + \beta_i] : 1 \leqq i \leqq k,\ n = 1, 2, \cdots\cdots\}$ is said to be an eventually covering family (ECF) if every sufficiently large integer occurs in exactly one of those sequences. If some α_i of an ECF is irrational, then all α_i in the ECF are irrational, and ECF's of that type were characterized by R. L. Graham [1]. But the situation is rather complicated if all α_i are rational numbers.

In this paper, we give several methods to construct ECF's of that type, and investigate some examples, and propose a conjecture. Finally, we list up all ECF with $k = 3$.

1. We start with several definitions and explain the notation used: **Q**, **R**, **N**, **Z** mean as usual. We denote by $[x]$ the greatest integer $\leqq x$, and by $\{x\}$ the fractional part of x. (a, b) means the greatest common divisor of a and b. For $v \in \mathbf{Z}$ and $w \in \mathbf{N}$, $[v, v + w] = \{v, v + 1, \cdots\cdots, v + w\}$. For $T \in \mathbf{N}$ and $x \in \mathbf{Z}$, $T < x > = x + T$, and this operation is applicable for a set of integers.

Definition 1. For $\alpha, \beta \in \mathbf{R}$ and $\alpha > 0$, $S(\alpha, \beta)$ denotes the set of integers $\{[\alpha n + \beta] : n \in \mathbf{N}\}$.

Definition 2. A finite family $\{S(\alpha_i, \beta_i) : 1 \leqq i \leqq k\}$ is said to be an *eventually covering family* (ECF) if every sufficiently large integer occurs in exactly one $S(\alpha_i, \beta_i)$.

Definition 3. We call a finite set of ordered pairs of integers $\{(r_i, m_i) : 1 \leqq i \leqq s\}$ is an *exactly covering set* (ECS) if each $n \in \mathbf{Z}$ satisfies exactly one of those relations $n \equiv r_i \pmod{m_i}$.

Definition 4. From a sequence $S(\alpha, \beta)$ and an ECS $\{(r_i, m_i) : 1 \leqq i \leqq s\}$ $(=E)$, we make the family of sequences $\{S(\alpha m_i, \alpha r_i + \beta) : 1 \leqq i \leqq s\}$. We denote that by $S[E]$.

If some α_i of an ECF is irrational, then all α_i in the ECF are irrational, and the following Theorem is proved in [1] : Any ECF in which some α_i is irrational is of the form $\{S_1[E_1], S_2[E_2]\}$ where $\{S_1, S_2\}$ is an ECF and E_1 and E_2 are ECS's. Thus in this case, the problem is reduced to that of ECS's.

However in case all α_i in an ECF are rational numbers, the situation is rather complicated. A. S. Fraenkel determined in [2] all the ECF's with $k = 2$. But it seems that we are far from the complete solution (cf. [3]). The object of this paper is to study ECF's of those type. About that case, we consider the problem to cover **Z** by sequences of type $[\alpha n + \beta]$ where n run through **Z** instead of **N**. This problem is essentially same with the original one and saves us from inessential complexities.

We note that if $\alpha = q/a$ where q and $a \in \mathbf{N}$, then the effect of β to $S(\alpha, \beta)$ depends only on the value of $[a\beta]$. Thus we define :

Definition 5. For $a, q \in \mathbf{N}$, $b \in \mathbf{Z}$, we denote $S(q, a, b) = \{[(qn + b)/a] : n \in \mathbf{Z}\}$.

Now it is easy to see the fact that a family of sequences $\{S(Q, a_i, b_i) : 1 \leqq i \leqq k\}$ is an ECF if and only if the following two conditions are satisfied :

$a_1 + a_2 + \cdots\cdots + a_k = Q$ and $S(Q, a_i, b_i) \cap S(Q, a_j, b_j) = \phi$ for all $i \neq j$.

Since a criterion for disjointness of those two sequences was given in [4], we may say that we have the fundamental tools to treat that problem. But there are many obstacles to be overcome, and for example we cannot say yet anything definitive about the famous conjecture of A. S. Fraenkel (cf. [3] P. 19).

The main object of this paper is to give several methods to construct ECF's. Precisely, we restate in §2 the criterion given in [4] for disjointness of two sequences $S(q_1, a, b)$ and $S(q_2, v, w)$. We explain in §3 a method to construct ECF's by the aid of pairs of ECS's. Investigating those ECF's, we prove a kind of uniqueness theorem and propose a conjecture.

In §4, another method to construct ECF's is given. That method springs from a method to construct an (infinite) ECS which is treated in [5]. The whole theory in which the "P-process" plays a central role will appear elsewhere. Finally in §5, we list up all ECF's with $k = 3$.

2. We restate here the criterion for disjointness of two sequences. For details we refer to [4].

We take two sequences $S(q_1, a, b)$ and $S(q_2, v, w)$. Assume that $(q_1, a) = (q_2, v) = 1$ and we define the numbers as follows : $(a, v) = t$, $(q_1, q_2) = q$, $a = tu$, $v = tf$.

(A) The two sequences $S(q_1, a, b)$ and $S(q_2, v, w)$ are disjoint with suitable two integers b and w if and only if

(1) $\quad lu + kf = q - 2uf(t-1) \quad$ holds with $\quad (l, k) \in \mathbf{N} \times \mathbf{N}$.

In case this condition is satisfied, we take the pair (L, K) of the solution of (1) such as $1 \leqq K \leqq u$. Furthermore if $L > f$, we define the numbers L_2 and K_2 by $L_2 = L - f$ and $K_2 = u - K$.

(B) Assume that (q_1, a) and (q_2, v) satisfy the condition of (A). Then $S(q_1, a, b)$ and $S(q_2, v, w)$ are disjoint if and only if

$uw - fb \in H \pmod q$ where $H = H_1 \cup H_2$,

$H_1 = \{uX + fY + uf(t-1) : 0 \leqq X \leqq L - 1,\ 1 \leqq Y \leqq K\}$,

$H_2 = \{uX + fY + uf(t-1) : 0 \leqq X \leqq L_2 - 1,\ K + 1 \leqq Y \leqq u\}$.

(C) We take T such as $uT \equiv f \pmod q$, then the set of w such that $S(q_1, a, -1) \cap S(q_2, v, w) = \phi$ is $G_1 \cup G_2 \pmod q$ where

$G_1 = \cup\ (iT) < [v - f, v - f + L - 1] > (0 \leqq i \leqq K - 1)$,

$G_2 = \cup\ (rT) < [-L - f(t-1), -(v+1)] > (0 \leqq r \leqq K_2 - 1)$.

3. We start this section with several definitions.

Definition 6. If each element of $S(q, a, b)$ $(=S)$ occurs exactly one $S(q_i, a_i, b_i)$ $(=S_i)$ and $S = \cup S_i (1 \leqq i \leqq k)$, we say that S is the *direct sum* of S_i and denote $S = S_1 \oplus S_2 \oplus \cdots\cdots \oplus S_k$.

To investigate an ECF, it is convenient if we take those q_i to be the same number.

Definition 7. For an ECF $\bigoplus S(Q, a_i, b_i)$ $(1 \leqq i \leqq k) = \mathbf{Z}$, we call the number Q the *size* of the ECF if $(Q, a_1, \cdots\cdots, a_k) = 1$. And we call a_i's *moduli* and b_i's *residues* of that ECF.

Remark. If there is no fear of misunderstanding, we denote the sequence S(Q, a, b) simply (a ; b) and $\bigoplus (a_i ; b_i) = \mathbf{Z}$.

Because of the relations $S(Q, a, b + Q) = S(Q, a, b)$ and $S(Q, a, b + aT) = T < S(Q, a, b) >$, the choice of residues of an ECF is free in two ways.

Definition 8. We say two ECF's $\{S(Q, a_i, b_i) : 1 \leqq i \leqq k\}$ and $\{S(Q, a_i, c_i) : 1 \leqq i \leqq k\}$ are equivalent if there exist $T \in \mathbf{Z}$ such that $c_i \equiv Ta_i + b_i \pmod Q$ holds for $1 \leqq i \leqq k$.

In the following, we identify the equivalent ECF's. For an ECF whose size = Q, the sum of its moduli is Q. In general the converse of that does not hold, but in case $k = 2$, the following result is known.

Proposition 1 (Fraenkel [2]). *If* $Q > a$, *then* $\mathbf{Z} = S(Q, a, b) \bigoplus S(Q, Q - a, -b - 1)$. *Conversely the all ECF with* $k = 2$ *are of this type.*

Definition 9. We say the two sequences given in Proposition 1 are *complementary* with each other.

Proposition 2. *If* $Q > ka$, *then* $S(Q, ka, b) = \bigoplus S(Q, a, [(Qi+b)/k])(0 \leqq i \leqq k-1)$. *If* $(Q, ka) = 1$, *then the choice of its residues is unique up to the equivalence given in Definition* 8.

Proof. We take the ECS $\{(i, k) : 0 \leqq i \leqq k - 1\}$ $(=E)$ and apply that to $S(Q, ka, b)$ $(=S)$. Then $S[E] = \{S(Qk, ka, Qi + b) : 0 \leqq i \leqq k - 1\}$. It is easy to see that $S(Qk, ka, Qi + b) = S(Q, a, [(Qi + b)/k])$. Thus we have the decomposition given in Proposition.

To prove the uniqueness, we first remark the fact : $S(Q, ka, b) = \bigoplus S(Q, a, b_i)$ $(=S_i)$ $(1 \leqq i \leqq k)$ if and only if $S_i \cap S_j = \phi$ for $i \neq j$ and each S_i is disjoint with $S(Q, Q - ka, -b - 1)$ which is the complementary sequence of S(Q, ka, b).

By its equivalence, we may assume $b = 0$ and denote $S^* = S(Q, Q - ka, -1)$. By (B) of §2, $S_i \cap S_j = \phi$ if and only if

(D) $b_i - b_j \in [a, Q - a] \pmod Q$.

Now we put $Q - ka = f$, then $(Q, f) = (f, a) = 1$. Thus by (C) of §2,

(i) If $f = 1$, then $K = 1$, $L = Q - a$. Thus S^* and S_i are disjoint if and only if $b_i \in [0, Q - a - 1] \pmod Q$. Since $Q - a - 1 = (k - 1)a$, (D) implies the fact that the set $\{b_i\}$ must equals with $\{0, a, \cdots\cdots, (k - 1)a\}$.

(ii) If $f > 1$, $(f, k) = (Q - ka, k) = (Q, k) = 1$, and $K = \{k/f\}f$, $L = 1 + [k/f]a$. Now we divide to two cases ;

(a) $(f > k)$. Then $L = 1$, $K = k$. Hence the cardinality of G of (C) in §2 = k. Thus the set $\{b_i\} = G$. (We denote $G = G_1 \cup G_2$.)

(b) $(k > f)$. Then $K_2 = (1 - \{k/f\})f$ and $L_2 = 1 + ([k/f] - 1)a$. Thus G is composed of K blocks whose length $= 1 + [k/f]a$ and K_2 blocks whose length $= 1 + ([k/f] - 1)a$. On the other hand, the set $\{b_i\}$ must satisfy (D). Hence the cardinality of b_i's contained in each block is at most $[k/f] + 1$ and $[k/f]$ respectively. The relation $([k/f] + 1)K + [k/f]K_2 = k$ implies that the extremal case is the actual case and the set $\{b_i\}$ is unique (mod Q).

In parallel to irrational case, we define :

Definition 10. An ECF is said to be a *standard* ECF (SECF) if that is $\mathbf{Z}[E]$ with suitable ECS E, or the form $\{S_1[E_1], S_2[E_2]\}$ where $\{S_1, S_2\}$ is an ECF and E_1 and E_2 are ECS's.

However in contrast to irrational case, there are ECF's which is not SECF. And we have no certain idea how to treat that type of ECF's. But examples seem to suggest that there are curious laws about the distribution of residues of an ECF.

Example. For Q = 13, there exist the following ECF's.

(i) $(6 ; -1) \oplus (3 ; 3) \oplus (2 ; 1) \oplus (1 ; 3) \oplus (1 ; 11)$,

(ii) $(6 ; -1) \oplus (3 ; 2) \oplus (2 ; 2) \oplus (1 ; 3) \oplus (1 ; 11)$,

(iii) $(8 ; -1) \oplus (2 ; 1) \oplus (1 ; 2) \oplus (1 ; 5) \oplus (1 ; 10)$,

(iv) $(3 ; -1) \oplus (4 ; 1) \oplus (2 ; 2) \oplus (2 ; 5) \oplus (2 ; 10)$.

Here (i) and (ii) shows that the choice of residues are not unique (up to equivalence). About (iii) and (iv), their moduli are proportional except the first one, and their residues are same. It seems plausible that example suggests the existence of some laws about the distributions of residues. As one of those, we propose :

Conjecture. For an ECF $\mathbf{Z} = \oplus S(Q, a_i, b_i)$ $(1 \leqq i \leqq k)$, we assume $(Q, a_i) = 1$ for all i. Then the sun of all its residues $\equiv (-k/2)$ (mod Q).

This conjecture holds for the case k = 2. And for SECF's which are generated by two NECS's (For that definition, we refer to [6].), this conjecture can be proved using Proposition 2. But for general cases, we can not see the reason of this phenomenon.

4. In [4], A. S. Fraenkel gave an ECF whose moduli are $1, 2, \cdots\cdots, 2^{w-1}$. In this section, we give ECF's which are, in a sense, thought to be its generalization. But those ECF's have a large background based on a method treated in [5] to construct an (infinite) ECS's. In particular the "P-process" plays a central role, and the ECF's given in Proposition 3 corresponds to the tree a, a, $\cdots\cdots$, a, P by the notation used in [5]. We treat the whole theory of that elsewhere.

Proposition 3. *Let* $w \geqq 2$, *and* $Q = a^w - 1$. *Then the following* $(a-1)w$ *sequences make an* ECF :

$(a^{w-1} ; ta^{w-1})$ $(0 \leqq t \leqq a-2)$, $(a^i ; sa^{w-1} + a^i(a-1) - 1)$ $(0 \leqq i \leqq w-2,\ 1 \leqq s \leqq a-1)$.

If $w \geqq 3$, *then the residues in the ECF are unique* (*It is not so if* $w = 2$ *and* $a \geqq 3$.).

Proof. By (B) of § 2, for $i \leqq j$, $(a^i ; b) \cap (a^j ; c) = \phi$ if and only if $c - a^{j-i}b \in [a^j, a^w - a^j + a^{j-i} - 2]$ (mod Q). Under some calculations, we can ascertain that the family of sequences satisfy that condition. Hence we shall prove its uniqueness. We denote $I(i, j) = [a^j, a^w - a^j + a^{j-i} - 2]$ (mod Q), and put $I^*(i, j) = [0, Q-1] - I(i, j)$ (mod Q).

We first take the sequences whose moduli are a^{w-1} and a^{w-2}. Then $(a^{w-2} ; b_s) \cap (a^{w-1} ; c_t) = \phi$ if and only if $c_t - ab_s \in I(w-2, w-1)$. Hence $-ab_s \in \cap(-c_t) < I(w-2, w-1) > (1 \leqq t \leqq a-1)$. Note that the cardinality of $I^*(w-2, w-1)$ is $2a^{w-1} - a$, and the number c_t's must differ at least a^{w-1} (mod Q).

(i) ($w = 2$). Then $(-c_t)$ $\langle I^*(0, 1)\rangle$ are relatively disjoint. Note the relation $Q = a^2 - 1 = a(a-1) + (a-1)$. Hence the cardinality of the set $\cap\ (-c_t) < I(w-2, w-1) >$ is $a - 1$, and that is exactly the set $\{-ab_s : 1 \leqq s \leqq a-1\}$. (It is easy to see that conversely those b's can be taken as residues of the ECF.)

(ii) ($w \geqq 3$). Because of the two relations $2a^{w-1} - a > a^{w-1}$ and $2a^{w-1} - a + (a-2)a^{w-1} = Q - (a-1)$, we see that the only chance to the cardinality of the set $\cap\ (-c_t) < I(w-2, w-1) >$ is $\geqq (a-1)$ lies when the numbers $c_1, c_2, \cdots, c_{a-1}$ make (by an appropriate order) the arithmetic progression whose difference $= a^{w-1}$. Thus we may assume (considering the equivalence) that $c_t = (t-1)a^{w-1}$ $(1 \leqq t \leqq a-1)$. And it is an easy deduction to see that then the set $\{b_s\} = \{-a^{w-2} - s\,a^{w-1} : 0 \leqq s \leqq a-2\}$ Now we proceed by induction on the order of moduli. Assume that an ECF $\{(a^j ; b_{j,s}) : 0 \leqq j \leqq w-1, 1 \leqq s \leqq a-1\}$ are given, and $b_{j,s}$ for $i+1 \leqq j \leqq w-1$ are taken as Proposition. We shall prove the uniqueness (mod Q) of the set $\{b_{i,s} : 1 \leqq s \leqq a-1\}$. We note $b_{i,s}$ simply by b_s.

Our plan of a proof is as follows. We take b_s so that $-1 \leqq b_s \leqq Q-2$ and express it by a-basis. Namly $b_s = -1 + \sum h_s(g)a^g$ $(1 \leqq g \leqq w-1, 0 \leqq h_s(g) \leqq a-1)$. And determine the coefficients $h_s(g)$ by applying the criterion of disjointness.

First because of disjointness with $(a^{w-1} : ta^{w-1})$ $(0 \leqq t \leqq a-2)$, we have $-a^{w-1-i}b_s \in [a^{w-1}, a^{w-1} + a^{w-i-1} - 2]$. Hence b_s is of the form $-1 + (a-1)a^i + \mu a^{i+1}$ where $1 \leqq \mu \leqq a^{w-i-1} - 1$. That means $h_s(g) = 0$ for $1 \leqq g \leqq i-1$, and $h_s(i) = a-1$. Next because of disjointness with $(a^{w-2} ; b_{w-2,s})$ $(1 \leqq s \leqq a-1)$, b_s must be of the form $-1 + \mu a^{i+2}$ (mod Q) where $1 \leqq \mu \leqq 2a^{w-2}(a-1) + a^{w-2-i} - 1$. Since $-1 \leqq b_s \leqq Q-2$ and $b_s \equiv -1 + (a-1)a^i$ (mod a^{i+1}), we have $a^{w-2}(a-1) \leqq \mu \leqq a^{w-2}(a-1) + a^{w-i-2} - 1$. Hence $h_s(i+1) = 0$. Using a similar reasoning, we have successively $h_s(i+2) = \cdots\cdots = h_s(w-3) = 0$. At the last step, by disjointness with $(a^{i+1} ; b_{i+1,s})$, we see that $b_s \equiv (-a^{w-2} + a^{i+1} - 1) + \mu a^{w-1}$ (mod Q) where $1 \leqq \mu \leqq 2a^{w-1} - 2a^{i+1} + a - 1$. Since $h_s(i) = a-1$ and $h_s(j) = 0$ for $1 \leqq j \leqq i-1$ or $i+1 \leqq j \leqq w-3$, we obtain the inequality $a^{w-1} - a^{i+1} \leqq \mu \leqq a^{w-1} - a^{i+1} + a - 1$. That means b_s is of the form $-1 + (a-1)a^i + sa^{w-1}$ $(0 \leqq s \leqq a-1)$. But the first one does not satisfy the condition for disjointness with $(a^{w-1} ; ta^{w-1})$ $(0 \leqq t \leqq a-2)$.

5. Finally we try another approach to the problem. Namely applying the criterion of § 2, we list up all possible ECF's for a fixed k. We treat here only a simple case, i.e. $k = 3$. For an investigation of ECF's with small k, we refer also to [7]. And we note that the conclusion of Proposition 4 supports the conjecture of A. S. Fraenkel.

Proposition 4. *Any ECF with* $k = 3$ *is equivalent to one of the following ones.*

(i) $(1 ; 0) \oplus (1 ; 1) \oplus (1 ; 2)$ $(Q = 3)$, (ii) $(2A ; -1) \oplus (B ; 0) \oplus (B ; Q/2)$ $(Q = 2(A + B))$, (iii) $(A ; -1) \oplus (B ; 0) \oplus (B ; (Q-1)/2)$ $(Q = A + 2B$, Q : odd) (iv) $(4 ; 0) \oplus (2 ; 5) \oplus (1 ; 4)$ $(Q = 7)$.

Proof. Take an ECF with $k = 3$. Let Q be the size of that ECF, thus the sum of three moduli a_1, a_2, a_3. If they are not distinct, then the ECF is SECF and it is an easy deduction to see that the ECF is equivalent to (i) - (iii) of Proposition. Hence we consider the case that all of a_i are distinct, and assume $a_1 > a_2 > a_3$. Put $(Q, a_1) = d_1$, (Q, a_2)

$= d_2$, $Q = d_1q_1 = d_2q_2$, $(q_1, q_2) = q$. By $(Q, a_1, a_2, a_3) = 1$, $(d_1, d_2) = 1$. Thus $Q = d_1d_2q$. Suppose $(a_1, a_2) = 1$, then by (B) of §2, the relation $(La_1)/d_1 + (Ka_2)/d_2 = q$ holds. Hence $(Ld_2)a_1 + (Kd_1)a_2 = Q$, but this relation contradicts with $a_1 + a_2 + a_3 = Q$. Thus we put $(a_1, a_2) = t > 1$, and $a_1 = td_1u_1$ and $a_2 = td_2u_2$. Then by (B) we have $Lu_1 + Ku_2 = q - 2(t-1)u_1u_2$. Denoting $u_1d_1 = v_1$ and $u_2d_2 = v_2$, we have $(Ld_2)v_1 + (Kd_1)v_2 = Q - 2(t-1)v_1v_2$. Substituting $Q = a_1 + a_2 + a_3$, and by $a_3 < a_2$ we obtain the inequality

(2) $2(t-1)v_1v_2 + (Ld_2)v_1 + (Kd_1)v_2 - tv_1 - 2tv_2 < 0$.

Since $t \geqq 2$, we have $v_1v_2 - v_1 - 2v_2 < 0$. Thus only the following three cases are possible : $(v_1, v_2) = (2, 1)$ or $(3, 1)$ or $(3, 2)$.

(i) $(v_1 = 2, v_2 = 1)$. Then $a_1 = 2t$, $a_2 = t$. Then by (2), $2(Ld_2) + Kd_1 < 4$. That means $d_1 = d_2 = K = L = 1$. And then three moduli is of the form $2t$, t, $t-1$. Now we consider the pair a_1 and a_3, and apply (A) of §2. Here $(2t, t-1) = 1$ or 2.

(a) If $(2t, t-1) = 1$, then $Q = K(2t) + L(t-1)$. Since $Q = 4t-1$, this relation holds only for $t = 2$, that is (iv).

(b) If $(2t, t-1) = 2$, then $Q - t(t-1) > 0$ implies $1 < t < 4$. Hence $t = 3$, $a_1 = 6$, $a_2 = 3$, $a_3 = 2$. But by (B) of §2, we see that no choice of residues for these moduli satisfy the disjointness condition.

(ii) $(v_1 = 3, v_2 = 1)$. Then by (2), $6(t-1) + 3Ld_2 + Kd_1 - 5t < 0$. But this inequality is impossible.

(iii) $(v_1 = 3, v_2 = 2)$. Then $12(t-1) + 3Ld_2 + Kd_1 - 6t < 0$ is also impossible.

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