On eventually covering families generated by  
the bracket function  

Ryozo Morikawa  

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A family of sequences \( \{[n\alpha + \beta] : 1 \leq i \leq k, n = 1, 2, \ldots \} \) is said to be an eventually covering family (ECF) if every sufficiently large integer occurs in exactly one of those sequences. If some \( \alpha \) of an ECF is irrational, then all \( \alpha \) in the ECF are irrational, and ECF's of that type were characterized by R. L. Graham [1]. But the situation is rather complicated if all \( \alpha \) are rational numbers.

In this paper, we give several methods to construct ECF's of that type, and investigate some examples, and propose a conjecture. Finally, we list up all ECF with \( k = 3 \).

1. We start with several definitions and explain the notation used: \( \mathbb{Q}, \mathbb{R}, \mathbb{N}, \mathbb{Z} \) mean as usual. We denote by \( \lfloor x \rfloor \) the greatest integer \( \leq x \), and by \( \{x\} \) the fractional part of \( x \). \( (a, b) \) means the greatest common divisor of \( a \) and \( b \). For \( v \in \mathbb{Z} \) and \( w \in \mathbb{N} \), \( \{v, v + w\} = \{v, v + 1, \ldots, v + w\} \). For \( T \in \mathbb{N} \) and \( x \in \mathbb{Z} \), \( T < x > = x + T \), and this operation is applicable for a set of integers.

Definition 1. For \( \alpha, \beta \in \mathbb{R} \) and \( \alpha > 0 \), \( S(\alpha, \beta) \) denotes the set of integers \( \{[\alpha n + \beta] : n \in \mathbb{N} \} \).

Definition 2. A finite family \( \{S(\alpha_i, \beta) : 1 \leq i \leq k\} \) is said to be an eventually covering family (ECF) if every sufficiently large integer occurs in exactly one \( S(\alpha_i, \beta) \).

Definition 3. We call a finite set of ordered pairs of integers \( \{(r_i, m_i) : 1 \leq i \leq s\} \) is an exactly covering set (ECS) if each \( n \in \mathbb{Z} \) satisfies exactly one of those relations \( n \equiv r_i \pmod{m_i} \).

Definition 4. From a sequence \( S(\alpha, \beta) \) and an ECS \( \{(r_i, m_i) : 1 \leq i \leq s\} \) (EC), we make the family of sequences \( \{S(\alpha m_i, \alpha r_i + \beta) : 1 \leq i \leq s\} \). We denote that by \( S[EC] \).

If some \( \alpha_i \) of an ECF is irrational, then all \( \alpha_i \) in the ECF are irrational, and the following Theorem is proved in [1]: Any ECF in which some \( \alpha_i \) is irrational is of the form \( \{S_1(\beta), S_2(\beta)\} \) where \( (S_1, S_2) \) is an ECF and \( E_1 \) and \( E_2 \) are ECS's. Thus in this case, the problem is reduced to that of ECS's.

However in case all \( \alpha \) in an ECF are rational numbers, the situation is rather complicated. A. S. Fraenkel determined in [2] all the ECF's with \( k = 2 \). But it seems that we are far from the complete solution (cf. [3]). The object of this paper is to study ECF's of those type. About that case, we consider the problem to cover \( \mathbb{Z} \) by sequences of type \( \{[\alpha n + \beta] \} \) where \( n \) run through \( \mathbb{Z} \) instead of \( \mathbb{N} \). This problem is essentially same with the original one and saves us from inessential complexities.

We note that if \( \alpha = q/a \) where \( q \) and \( a \in \mathbb{N} \), then the effect of \( \beta \) to \( S(\alpha, \beta) \) depends only on the value of \( [a\beta] \). Thus we define:
Definition 5. For \( a, q \in \mathbb{N}, b \in \mathbb{Z} \), we denote \( S(q, a, b) = \{(qn + b)/a : n \in \mathbb{Z}\} \).

Now it is easy to see the fact that a family of sequences \( \{S(Q, a_i, b_i) : 1 \leq i \leq k\} \) is an ECF if and only if the following two conditions are satisfied:

\[ a_1 + a_2 + \cdots + a_k = Q \quad \text{and} \quad S(Q, a_i, b_i) \cap S(Q, a_j, b_j) = \emptyset \quad \text{for all } i \neq j. \]

Since a criterion for disjointness of those two sequences was given in [4], we may say that we have the fundamental tools to treat that problem. But there are many obstacles to be overcome, and for example we cannot say yet anything definitive about the famous conjecture of A. S. Fraenkel (cf. [3] P. 19).

The main object of this paper is to give several methods to construct ECF's. Precisely, we restate in §2 the criterion given in [4] for disjointness of two sequences \( S(q_1, a, b) \) and \( S(q_2, v, w) \). We explain in §3 a method to construct ECF's by the aid of pairs of ECS's. Investigating those ECF's, we prove a kind of uniqueness theorem and propose a conjecture.

In §4, another method to construct ECF's is given. That method springs from a method to construct an (infinite) ECS which is treated in [5]. The whole theory in which the "P-process" plays a central role will appear elsewhere. Finally in §5, we list up all ECF's with \( k = 3 \).

2. We restate here the criterion for disjointness of two sequences. For details we refer to [4].

We take two sequences \( S(q_1, a, b) \) and \( S(q_2, v, w) \). Assume that \((q_1, a) = (q_2, v) = 1\) and we define the numbers as follows: \((a, v) = t, (q_1, q_2) = q, a = tu, v = tf\).

(A) The two sequences \( S(q_1, a, b) \) and \( S(q_2, v, w) \) are disjoint with suitable two integers \( b \) and \( w \) if and only if

\[ lu + kf = q - 2uf(t - 1) \quad \text{holds with} \quad (l, k) \in \mathbb{N} \times \mathbb{N}. \]

In case this condition is satisfied, we take the pair \((L, K)\) of the solution of (1) such as \( 1 \leq K \leq u \). Furthermore if \( L > f \), we define the numbers \( L_2 \) and \( K_2 \) by \( L_2 = L - f \) and \( K_2 = u - K \).

(B) Assume that \((q_1, a)\) and \((q_2, v)\) satisfy the condition of (A). Then \( S(q_1, a, b) \) and \( S(q_2, v, w) \) are disjoint if and only if

\[ uw - fb \in H \pmod{q} \quad \text{where} \quad H = H_1 \cup H_2, \]

\[ H_1 = \{uX + fY + uf(t - 1) : 0 \leq X \leq L - 1, 1 \leq Y \leq K\}, \]

\[ H_2 = \{uX + fY + uf(t - 1) : 0 \leq X \leq L_2 - 1, K + 1 \leq Y \leq u\}. \]

(C) We take \( T \) such as \( uT \equiv f \pmod{q} \), then the set of \( w \) such that \( S(q_1, a, -1) \cap S(q_2, v, w) = \emptyset \) is \( G_1 \cup G_2 \pmod{q} \) where

\[ G_1 = U (iT) < [v - f, v - f + L - 1] > (0 \leq i \leq K - 1), \]

\[ G_2 = U (rT) < [-L - f(t - 1), -(v + 1)] > (0 \leq r \leq K_2 - 1). \]

3. We start this section with several definitions.

Definition 6. If each element of \( S(q, a, b) \) (= \( S\)) occurs exactly one \( S(q_i, a_i, b_i) \) (= \( S_i \)) and \( S = \bigcup S_i (1 \leq i \leq k) \), we say that \( S \) is the direct sum of \( S_i \) and denote \( S = S_1 \oplus S_2 \oplus \cdots \oplus S_k \).

To investigate an ECF, it is convenient if we take those \( q_i \) to be the same number.
Definition 7. For an ECF $\oplus S(Q, a_i, b_i) (1 \leq i \leq k) = \mathbb{Z}$, we call the number $Q$ the size of the ECF if $(Q, a_1, \ldots, a_k) = 1$. And we call $a_i$'s moduli and $b_i$'s residues of that ECF.

Remark. If there is no fear of misunderstanding, we denote the sequence $S(Q, a, b)$ simply $(a; b)$ and $\oplus (a_i; b_i) = \mathbb{Z}$.

Because of the relations $S(Q, a, b + Q) = S(Q, a, b)$ and $S(Q, a, b + aT) = T < S(Q, a, b)$, the choice of residues of an ECF is free in two ways.

Definition 8. We say two ECF's $(S(Q, a_i, b_i) : 1 \leq i \leq k)$ and $(S(Q, a_i, c_i) : 1 \leq i \leq k)$ are equivalent if there exist $T \in \mathbb{Z}$ such that $c_i = Ta_i + b_i \pmod{Q}$ holds for $1 \leq i \leq k$.

In the following, we identify the equivalent ECF's. For an ECF whose size $= Q$, the sum of its moduli is $Q$. In general the converse of that does not hold, but in case $k = 2$, the following result is known.

Proposition 1 (Fraenkel [2]). If $Q > a$, then $\mathbb{Z} = S(Q, a, b) \oplus S(Q, Q - a, -b - 1)$.

Conversely the all ECF with $k = 2$ are of this type.

Definition 9. We say the two sequences given in Proposition 1 are complementary with each other.

Proposition 2. If $Q > ka$, then $S(Q, ka, b) = \oplus S(Q, a, [(Qi+b)/k]) (0 \leq i \leq k-1)$.

If $(Q, ka) = 1$, then the choice of its residues is unique up to the equivalence given in Definition 8.

Proof. We take the ECS $\{E(i, k) : 0 \leq i \leq k - 1\}$ and apply that to $S(Q, ka, b)$ $\oplus S(Q, ka, Qi + b) (0 \leq i \leq k - 1)$. It is easy to see that $S(Qk, ka, Qi + b) = S(Q, a, [(Qi + b)/k])$. Thus we have the decomposition given in Proposition.

To prove the uniqueness, we first remark the fact: $S(Q, ka, b) \oplus S(Q, a, b) (= S_1) (1 \leq i \leq k)$ if and only if $S_i \cap S_j = \emptyset$ for $i \neq j$ and each $S_i$ is disjoint with $S(Q, Q - ka, -b - 1)$ which is the complementary sequence of $S(Q, ka, b)$.

By its equivalence, we may assume $b = 0$ and denote $S^* = S(Q, Q - ka, -1)$.

By (B) of § 2, $S_1 \cap S_2 = \emptyset$ if and only if

$\begin{align*}
&\quad \text{(D) } b_i - b_j \in [a, Q - a] \pmod{Q}.
\end{align*}$

Now we put $Q - ka = f$, then $(Q, f) = (f, a) = 1$. Thus by (C) of § 2,

(i) If $f = 1$, then $K = 1, L = Q - a$. Thus $S_i$ and $S_j$ are disjoint if and only if $b_i \in [0, Q - a - 1] \pmod{Q}$. Since $Q - a - 1 = (k - 1)a$, (D) implies the fact that the set $\{b_i\}$ must equals with $\{0, a, \ldots, (k - 1)a\}$.

(ii) If $f > 1$, $(f, k) = (Q - ka, k) = (Q, k) = 1$, and $K = [k/f]f, L = 1 + [k/f]a$. Now we divide to two cases:

(a) $(f > k)$. Then $L = 1, K = k$. Hence the cardinality of $G$ of (C) in § 2 $= k$.

Thus the set $\{b_i\} = G$. (We denote $G = G_1 \cup G_2$.)

(b) $(k > f)$. Then $K_2 = (1 - [k/f])f$ and $L_2 = 1 + ([k/f] - 1)a$. Thus $G$ is composed of $K$ blocks whose length $= 1 + [k/f]a$ and $K_2$ blocks whose length $= 1 + ([k/f] - 1)a$. On the other hand, the set $\{b_i\}$ must satisfy (D). Hence the cardinality of $b_i$'s contained in each block is at most $[k/f] + 1$ and $[k/f]$ respectively. The relation $([k/f] + 1)K + [k/f]K_2 = k$ implies that the extremal case is the actual case and the
set \{b_i\} is unique (mod \(Q\)).

In parallel to irrational case, we define:

**Definition 10.** An ECF is said to be a standard ECF (SECF) if that is \(Z[E]\) with suitable ECS \(E\), or the form \(\{S_1 [E_1], S_2 [E_2]\}\) where \(\{S_1, S_2\}\) is an ECF and \(E_1\) and \(E_2\) are ECS's.

However in contrast to irrational case, there are ECF's which is not SECF. And we have no certain idea how to treat that type of ECF's. But examples seem to suggest that there are curious laws about the distribution of residues of an ECF.

**Example.** For \(Q = 13\), there exist the following ECF's.

(i) \((6 ; -1) \oplus (3 ; 3) \oplus (2 ; 1) \oplus (1 ; 3) \oplus (1 ; 11),\)

(ii) \((6 ; -1) \oplus (3 ; 3) \oplus (2 ; 2) \oplus (1 ; 3) \oplus (1 ; 11),\)

(iii) \((8 ; -1) \oplus (2 ; 1) \oplus (1 ; 2) \oplus (1 ; 5) \oplus (1 ; 10),\)

(iv) \((3 ; -1) \oplus (4 ; 1) \oplus (2 ; 2) \oplus (2 ; 5) \oplus (2 ; 10).\)

Here (i) and (ii) shows that the choice of residues are not unique (up to equivalence). About (iii) and (iv), their moduli are proportional except the first one, and their residues are same. It seems plausible that example suggests the existence of some laws about the distributions of residues. As one of those, we propose:

**Conjecture.** For an ECF \(Z = \oplus S(Q, a_1, b_0(1 \leq i \leq k))\), we assume \((Q, a_i = 1\) for all \(i\). Then the sum of all its residues \(= (-k/2) \mod Q\).

This conjecture holds for the case \(k = 2\). And for SECF's which are generated by two NECS's (For that definition, we refer to [6]), this conjecture can be proved using Proposition 2. But for general cases, we can not see the reason of this phenomenon.

4. In [4], A. S. Fraenkel gave an ECF whose moduli are 1, 2, \ldots, 2\(^w\)\(^{-1}\). In this section, we give ECF's which are, in a sense, thought to be its generalization. But those ECF's have a large background based on a method treated in [5] to construct an (infinite) ECS's. In particular the "P-process" plays a central role, and the ECF's given in Proposition 3 corresponds to the tree \(a, a, \ldots, a, P\) by the notation used in [5]. We treat the whole theory of that elsewhere.

**Proposition 3.** Let \(w \geq 2\), and \(Q = a^w - 1\). Then the following \((a - 1)^w\) sequences make an ECF:

\[(a^{w-1} ; ta^{w-1}) (0 \leq t \leq a - 2), (a^i ; sa^{w-1} + a^i(a - 1) - 1) (0 \leq i \leq w - 2, 1 \leq s \leq a - 1).\]

If \(w \geq 3\), then the residues in the ECF are unique (It is not so if \(w = 2\) and \(a \geq 3\)).

**Proof.** By (B) of § 2, for \(i \leq j\), \((a^i ; b) \cap (a^j ; c) = \emptyset\) if and only if \(c - a^{i-b} \in [a^i, a^w - a^i + a^i - 2] \mod Q\). Under some calculations, we can ascertain that the family of sequences satisfy that condition. Hence we shall prove its uniqueness. We denote \(I(i, j) = [a^i, a^w - a^i + a^i - 2] \mod Q\), and put \(I^*(i, j) = [0, Q - 1] - I(i, j) \mod Q\).

We first take the sequences whose moduli are \(a^{w-1}\) and \(a^{w-2}\). Then \((a^{w-1} ; b_0) \cap (a^{w-2} ; c_0) = \emptyset\) if and only if \(c_0 = ab_0 \in I(w - 2, w - 1)\). Hence \(-ab_0 \in I(-c_0) < I(w - 2, w - 1) > (1 \leq t \leq a - 1)\). Note that the cardinality of \(I^*(w - 2, w - 1)\) is \(2a^{w-1} - a\), and the number \(c_0\)'s must differ at least \(a^{w-1} \mod Q\).
(i) \((w = 2)\). Then \((-c_0 \langle I^*(0, 1) \rangle\) are relatively disjoint. Note the relation \(Q = a^2 - 1 = a(a - 1) + (a - 1)\). Hence the cardinality of the set \(\cap (-c_0 < I(w - 2, w - 1)) > a - 1\), and that is exactly the set \(\{ -a b_s : 1 \leq s \leq a - 1 \}\). (It is easy to see that conversely those b's can be taken as residues of the ECF.)

(ii) \((w \geq 3)\). Because of the two relations \(2a^{w-1} - a > a^{w-1}\) and \(2a^{w-1} - a + (a - 2)a^{w-1} = Q - (a - 1)\), we see that the only chance to the cardinality of the set \(\cap (-c_0 < I(w - 2, w - 1)) > a - 1\), and that is exactly the set \(\{ -a b_s : 1 \leq s \leq a - 1 \}\). (It is easy to see that conversely those b's can be taken as residues of the ECF.)

Our plan of a proof is as follows. We take \(b_s\) so that \(-1 \leq b_s \leq Q - 2\) and express it by a-basis. Namely \(b_s = -1 + \sum h_s(g) a^g (1 \leq g \leq w - 1, 0 \leq h_s(g) \leq a - 1)\).

Proposition 4. Any ECF with \(k = 3\) is equivalent to one of the following ones.

(i) \((1 ; 0) \oplus (1 ; 1) \oplus (1 ; 2) (Q = 3)\), (ii) \((2A ; -1) \oplus (B ; 0) \oplus (B ; Q/2) (Q = 2(A + B))\), (iii) \((A ; -1) \oplus (B ; 0) \oplus (B ; (Q - 1)/2) (Q = A + 2B, Q : odd)\)

Proof. Take an ECF with \(k = 3\). Let \(Q\) be the size of that ECF, thus the sum of three moduli \(a_1, a_2, a_3\). If they are not distinct, then the ECF is SECF and it is an easy deduction to see that the ECF is equivalent to (i) - (iii) of Proposition. Hence we consider the case that all of \(a_i\) are distinct, and assume \(a_1 > a_2 > a_3\). Put \((Q, a_3) = d_3, (Q, a_2) = d_2, (Q, a_1) = d_1\).
\[ d_1, Q = d_1q_1 = d_1q_2, (q_1, q_2) = q. \] By \((Q, a_1, a_2, a_3) = 1, (d_1, d_2) = 1. \) Thus \( Q = d_1d_2Q. \) Suppose \((a_1, a_2) = 1, \) then by \((B) \) of \( \S 2, \) the relation \((La_1)/d_1 + (Ka_2)/d_2 = q\) holds. Hence \((Ld_2)a_1 + (Kd_1)a_2 = Q, \) but this relation contradicts with \( a_1 + a_2 + a_3 = Q. \) Thus we put \((a_1, a_2) = t > 1, \) and \( a_1 = td_1u_1 \) and \( a_2 = td_2u_2. \) Then by \((B) \) we have \( Lu_1 + Ku_2 = q - 2(t - 1)u_1u_2. \) Denoting \( u_1d_1 = v_1, \) and \( u_2d_2 = v_2, \) we have \( (Ld_2)v_1 + (Kd_1)v_2 = Q - 2(t - 1)v_1v_2. \) Substituting \( Q = a_1 + a_2 + a_3, \) and by \( a_3 < a_2 \) we obtain the inequality

\[(2) \quad 2(t - 1)v_1v_2 + (Ld_2)v_1 + (Kd_1)v_2 - tv_1 - 2tv_2 < 0.
\]

Since \( t \geq 2, \) we have \( v_1v_2 = v_1 - 2v_1 < 0. \) Thus only the following three cases are possible: \((v_1, v_2) = (2, 1) \) or \((3, 1) \) or \((3, 2). \)

(i) \((v_1 = 2, v_2 = 1). \) Then \( a_1 = 2t, a_2 = t. \) Then by \((2), 2(Ld_2) + Kd_1 < 4. \) That means \( d_1 = d_2 = K = L = 1. \) And then three moduli is of the form \( 2t, t, t - 1. \)

Now we consider the pair \( a_1 \) and \( a_2, \) and apply \((A) \) of \( \S 2. \) Here \((2t, t - 1) = 1 \) or \((2). \)

(a) If \((2t, t - 1) = 1, \) then \( Q = K(2t) + L(t - 1). \) Since \( Q = 4t - 1, \) this relation holds only for \( t = 2, \) that is \((iv). \)

(b) If \((2t, t - 1) = 2, \) then \( Q - t(t - 1) > 0 \) implies \( 1 < t < 4. \) Hence \( t = 3, \) \( a_1 = 6, a_2 = 3, a_3 = 2. \) But by \((B) \) of \( \S 2, \) we see that no choice of residues for these moduli satisfy the disjointness condition.

(ii) \((v_1 = 3, v_2 = 1). \) Then by \((2), 6(t - 1) + 3Ld_2 + Kd_1 - 5t < 0. \) But this inequality is impossible.

(iii) \((v_1 = 3, v_2 = 2). \) Then \( 12(t - 1) + 3Ld_2 + Kd_1 - 6t < 0 \) is also impossible.

References

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