On eventually covering families generated by
the bracket function II

Ryozo Morikawa

(Received April 20, 1983)

1. Introduction. This paper is a continuation of [4], and our main concern is the same. For two positive integers \( q, a \) and an integer \( b \), let \( S(q, a, b) \) denote the set of integers \( \lfloor (qn + b)/a \rfloor \) where \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \) and \( n \) runs through the set \( \mathbb{Z} \) of integers. A finite family \( \{ S(q_i, a_i, b_i) : 1 \leq i \leq k \} \) is said to be an eventually covering family (ECF) if they are mutually disjoint and the union of these set is \( \mathbb{Z} \). (This is an ECF of so to speak a rational type.)

We continue the study of the structure of ECF's of this type, which was initiated in [4]. Readers are assumed to be familiar with that paper, and we use the same notations and definitions without explanations.

We introduced in [4] the concept of a standard ECF (SECF). However in contrast to the case of ECF's of so to speak an irrational type, which is treated by R. L. Graham (cf. [2]), an ECF of our type is not necessarily an SECF. The ECF's of this type seem to have fairly complicated phenomena, and we have not yet certain idea how to treat these.

But it seems that the following proposition may be aimed as a cornerstone of a possible future theory:

"All ECF lies in the neighbourhood of a suitable SECF." Of course, this statement is worthless without defining the concept of the neighbourhood. And as we are yet in the dark how to define it, it may be indiscrete to propose such a vague proposition. But we hope that it may play a role of a guidepost.

Here we note the fact that SECF's are separated to the following two types:

(Type I) \( \mathbb{Z}[E] \) where \( E \) is an exactly covering set (ECS). (This ECF is essentially the same with \( E \), and does not appear in the case treated by R. L. Graham.)

(Type II) \( S_1[E_1] \cup S_2[E_2] \) where \( \{ S_1, S_2 \} \) is an ECF and \( E_1, E_2 \) are ECS's.

Now we explain the composition of this paper. In § 2, we add defini-
tions and notations used in this paper besides those of [4]. In § 3, we give a new characterization of an ECF, and prove the conjecture proposed in § 3 of [4]. In § 4, we construct ECF’s which lie near to SECF’s of Type I, and in § 5 those which lie near to SECF’s of Type II. Finally in § 6, we give a sufficient condition for an ECF to be an SECF (of Type II).

2. Here we explain the notations used in the following:

(i) We denote by $A(a, d, m)$ the arithmetic progression $a, a+d, \ldots, a+(m-1)d$.

(ii) For $m \in \mathbb{N}$, let $\sigma_m$ denote the map from $\mathbb{Z}$ into $\mathbb{C}$ defined by $\sigma_m(r) = \exp\left(\frac{2\pi ir}{m}\right)$ for $r \in \mathbb{Z}$.

We put $C(m) = \sigma_m(\mathbb{Z})$. For $P \in C(m)$, we say that the coordinate of $P$ is $r$ if $\sigma_m(r) = P$ and $0 \leq r \leq m-1$.

(iii) For two points $P, Q \in C(m)$ whose coordinates are $r$ and $s$ respectively, we define their distance $d(P, Q)$ by

$$d(P, Q) = \min(|r-s|, m-|r-s|).$$

3. We start with the following proposition.

**Proposition 1.** Let $q, a \in \mathbb{N}$ and assume that $(q, a) = 1$. Then $S(q, a, b) = \{z \in \mathbb{Z} : az \equiv (b-j) \pmod{q}, 0 \leq j \leq a-1\}$.

**Proof.** We take a number $\hat{a}$ such as $a\hat{a} = fq + 1$ with $f, \hat{a} \in \mathbb{Z}$. Then we have

$$(qn+b)/a = q(n-fb)/a + \hat{a}b.$$ 

Hence we have $S(q, a, b) = S(q, a, 0) + \hat{a}b$. Note that $\mathbb{Z}$ is the disjoint union of $\{an+fi : n \in \mathbb{Z}\}$ where $1 \leq i \leq a$. Now it is easy to see that the following relation holds:

$$[q\hat{a}/a] = [a\hat{a}/a] = \hat{a}i - 1 \equiv \hat{a}(i-a) \pmod{q}.$$ 

By putting $j = a-i$, we obtain easily the conclusion.

**Definition 1.** Let $q, a \in \mathbb{N}$ and $b \in \mathbb{Z}$. Assume first $(q, a) = 1$. We take the least positive integer $\hat{a}$ such that $a\hat{a} \equiv 1 \pmod{q}$. Then we define

$$V(q, a, b) = \{\hat{a}(b-j) : 0 \leq j \leq a-1\}.$$ 

And for the case $(q, a) = d > 1$, we put $q = dq'$ and $a = da'$. In this case we define $V(q, a, b)$ by

$$V(q, a, b) = \bigcup_{i=0}^{d-1} (V(q', a', [b/d]) + iq').$$
Then as an easy corollary of Proposition 1, we obtain the following criterion (#) for an ECF:

(#) The family \( \{S(Q, a_i, b_i) : 1 \leq i \leq k \} \) is an ECF if and only if the union of \( cV(Q, a_i, b_i) \) \( (1 \leq i \leq k) \) is just a complete set of residues modulo \( Q \), where \( c \) is a suitable integer such that \( (c, Q) = 1 \). We call \( c \) a multiplier.

**Remark.** Of course we can always take 1 as a multiplier. But in some cases, it is convenient if we take another suitable \( c \).

Now we prove the conjecture proposed in [4]. Namely we have

**Proposition 2.** Take an ECF \( \{S(Q, a_i, b_i) : 1 \leq i \leq k \} \). Assume that \( (Q, a_i) = 1 \) for all \( i \). Then the following relation holds.

\[
b_1 + b_2 + \cdots + b_k \equiv (-k/2) \pmod{Q}.
\]

(In case \( k \) is odd, then \( Q \) must be odd and this relation means as usual that \( 2(b_1 + b_2 + \cdots + b_k) + k \) is a multiple of \( Q \)).

**Proof.** First by the density considerations, we have

\[(1) \quad a_1 + a_2 + \cdots + a_k = Q.
\]

Secondly by applying (#), we have

\[
\sum_{i=1}^{k} a_i(a_i - 1)/2 \equiv Q(Q - 1)/2 \pmod{Q}.
\]

Now we divide to two cases.

(Q is odd) : Then (2) implies that \( \sum (b_i - (a_i - 1)/2) \equiv 0 \pmod{Q} \).

By (1), we obtain easily the conclusion.

(Q is even) : Then by \( (Q, a_i) = 1 \) for all \( i \), all \( a_i \)'s are odd. Thus (1) implies that \( k \) must be even. Now put \( Q = 2Q' \). Then we have

\[
\sum (b_i - (a_i - 1)/2) \equiv Q' \pmod{Q}
\]

Now again by (1), the conclusion of Proposition easily follows.

4. In this section, we construct a system of ECF's which lie near to SECF's of Type I.

We take an ECS \( \{(r_i, m_i) : 1 \leq i \leq s \} (= E) \). We may assume without loss of generality that the inequalities

\[
-(m_i - 1) \leq r_i \leq 0 \quad (1 \leq i \leq s)
\]

hold. We define \( \|E\| \) as follows and call it the size of \( E \).

\[
\|E\| = \text{l.c.m.} \{m_i : 1 \leq i \leq s \}.
\]
Now we assume that $E$ satisfies the following condition (*)

(*) We take $g \in \mathbb{N}$ such that $(g, \|E\|) = 1$. Assume that there are $g$ pairs $(r, \|E\|)$ in $E$ with $-g + 1 \leq r \leq 0$.

**Proposition 3.** Notations and assumptions being as above. We define the numbers $Q, a_i$ and $b_i$ as follows:

\[ Q = \|E\| - g, \quad a_i m_i = \|E\|, \quad g b_i \equiv a_i r_i \pmod{Q}, \]

where $(r_i, m_i)$ runs through the pairs of $E$ excluding the $g$ pairs mentioned in (*). Then the union of these $S(Q, a_i, b_i)$ makes an ECF of size $Q$.

**Proof.** We first note the relation $(g, \|E\|) = 1$ implies $(g, Q) = 1$. Thus $b_i$'s are well defined. (The ambiguity of $b_i$ modulo $Q$ does not effect to $S(Q, a_i, b_i)$.) And we have also $(a_i, Q) = 1$. Thus we apply (#) taking $g$ as a multiplier. We see easily $a_i g = r_i$ (mod $Q$). Thus the set $g V(Q, a_i, b_i)$ is the arithmetic progression $A(r_i, -m_i, a_i)$ (mod $Q$). Since $E$ is an ECS, we reach to the conclusion that the union of $A(r_i, -m_i, a_i)$ for all $i$ is just the set $[-\|E\| + 1, -g]$. Namely it is a complete set of residues modulo $Q$.

5. If we consider the problem to construct an ECF with a given set of moduli, there are two steps. The first step is to construct anyhow one ECF, and to list up all the possible choices of its residues is the second step. (Treating the second problem, we identify the equivalent ECF's.)

About the ECF's given in Proposition 3 of [4], the choices of their residues are unique. But in general, there are many possibilities of the residues of an ECF. We investigate in this section a system of ECF's and list up all the possible choices of their residues. (For the other examples, see [6].)

Let $a, w \in \mathbb{N}$ and $a, w \geq 2$. Put $Q = a^w + 1$. As a moduli set, we consider $a^i$ $(1 \leq i \leq w - 1)$ each of which is taken $a - 1$ times and 1 taken $a + 1$ times.

First we remark that there is an SECF whose moduli set is as above. We take a tree $a / P$, $(a), \ldots, (a) / /$ (We use the notation used in [2]). And operate the $P$-process $w - 2$ times. Now let $E_i$ be an NECS which correspond to this tree. Then we see easily that $S_1[E_1] \oplus S_2$ is the wanted one, where $S_1 = S(Q, a^w, 0)$ and $S_2 = S(Q, 1, -1)$.

To list up all the possible choices of its residues, we use the disjointness criterion obtained in [5], and apply (#) of § 3 taking $a^{w - 1}$ as a multiplier.

We use in the below the following notations:
(1) We denote \( S(Q, a, b) \) simply by \( (a; b) \) and \( a_0 \) by \( \sigma \).
(2) We denote the \( a - 1 \) residues of \( a^i \) by \( b_{i,s} (1 \leq s \leq a - 1) \).
(3) As noted above, we apply \((\#)\) and consider the set \( a^{w-1}V(Q, a^i, b_{i,s}) \). We denote this set by \( v(a^i; b_{i,s}) \).

We note here the following two facts:

(i) \( v(a^{w-1}; b_{w-1,s}) = [b_{w-1,s} - a^{w-1} + 1, b_{w-1,s}] \) (mod \( Q \)).
(ii) The \( a + 1 \) residues of the modulus \( 1 \) are uniquely determined if we determine the residues for all the other moduli \( a^i (1 \leq i \leq w - 1) \).

Now we settle first the case \( w = 2 \). As noted above, we must determine \( b_{i,s} (1 \leq s \leq a - 1) \). For that, it is necessary and sufficient if we take so that
\[
d(b_{i,s}, b_{i,t}) \geq a \text{ for } 1 \leq s, t \leq a - 1 \text{ and } s \neq t,
\]
where the distance is measured on \( C(a^2 + 1) \). Thus we arrange the coordinates of these residues as follows:
\[
0 \leq b_i < b_2 < \cdots < b_{a-1} \leq a^2,
\]
and we put
\[
g_i = b_{i+1} - b_i - a \quad (1 \leq i \leq a - 2) \text{ and } g_{a-1} = Q + b_i - b_{a-1} - a.
\]
Then we have
\[
g_1 + g_2 + \cdots + g_{a-1} = a + 1 \text{ and } g_i \geq 0 \quad (1 \leq i \leq a - 1).
\]
And it is easy to see that the cardinality of non equivalent combinations of residues is the cardinality of those \((g_1, g_2, \ldots, g_{a-1})\) classified by considering equivalent if they coincide by cyclic permutations. Its cardinality is
\[
\begin{cases}
(2a - 1)! / (a + 1)! (a - 1)! & \text{if } a \text{ is even} , \\
(2a - 1)! / (a + 1)! (a - 1)! + (2A - 1)! / (A + 1)! (A - 1)! & \text{if } a = 2A + 1.
\end{cases}
\]

Hereafter we assume \( w \geq 3 \). First we apply the disjointness criterion for \((a^{w-2}, b_{w-2,s})\) and \((a^{w-1}, b_{w-1,t})\). Then we see that they are disjoint if and only if
\[
b_{w-1,t} - ab_{w-2,s} \equiv [a^{w-1}, a^w - a^{w-1} + a] \text{ (mod } Q).\]
Thus we have
\[
(3) \quad -ab_{w-2,s} \equiv \prod_{i=1}^{a-1} [a^{w-1}, a^w - a^{w-1} + a] - b_{w-1,t} \text{ (mod } Q).
\]
Here we note the following three facts:

(i) The cardinality of \([a^{w-1}, a^w - a^{w-1} + a]\) is \( a^{w-1}(a - 2) + a + 1 \).
(ii) For any \( t_1 \neq t_2 \), \( d(\sigma (b_{w-1,t_1}), \sigma (b_{w-1,t_2})) \geq a^{w-1} \).
(iii) The cardinality of \((-ab_{w-2,s})\) (mod \( Q \)) is \( a - 1 \).

As noting that we are considering the combinations of residues up to
equivalence, we conclude easily from above facts that the possible choices of $b_{w-1,s}$ ($1 \leq s \leq a - 1$) are

$$b_{w-1,1} = 0, \ b_{w-1,s} = (s - 1)a^{w-1} + g_1 + g_2 + \cdots + g_{s-1} \ (2 \leq s \leq a - 1),$$

where $g_s$'s are non negative integers and $g_1 + g_2 + \cdots + g_{a-2} \leq 2$.

Now we determine $b_{i,s}$ ($1 \leq s \leq a - 1$) beginning from the case $i = w - 2$ and inductively by the magnitude of the moduli. In order that we investigate the structure of

$$\sigma \left( \bigcup_{s=1}^{w-1} v(a^{w-1}; b_{w-1,s}) \right).$$

We put $\mu = g_1 + g_2 + \cdots + g_{a-2}$ and treat three cases separately.

$(\mu = 2)$: In this case, there are $h_1$ and $h_2$ ($h_1 \leq h_2$) such that $g_{h_1}, g_{h_2} \geq 1$. (We treat the case $g_{h_1} = 2$ and the other $g$'s are 0, by interpreting the case as $h_1 = h_2$.) Now by easy calculations, we see that the set (4) is the $\sigma$-image of

$$\{a^{w-1} + j (j = 1, 2) \text{ and } [-2a^{w-1} + 1 + h : 1 \leq h \leq a^{w-1} - 1] \}
\text{ and } \{a^{w-2} \} (1 \leq j \leq a - 1) \text{ and } \{-2a^{w-1} + 1 + ha : 1 \leq h \leq a^{w-2} - 1\}.$$

It is easy to see that no other arithmetic progression (mod Q) of length $a^{w-2}$ whose difference $a$ is contained in (5). Thus we must choose the $a-1$ arithmetic progressions given in (6) as $v(a^{w-2}; b_{w-2,s})$ ($1 \leq s \leq a - 1$).

And now if we consider the case $i = w - 3$, we see that the relation of $b_{w-3,s}$ and $\{a^{w-3} + 1 + ha : 1 \leq h \leq a^{w-3} - 1\}$ is similar to that of the case $i = w - 2$. Thus we may proceed inductively, and obtain the conclusion that

$$\{b_{i,s} : 1 \leq s \leq a - 1\} = \{-a^{i-1} - a^{i+1} + ha^{w-1} : 1 \leq h \leq a - 1\} \text{ for } 1 \leq i \leq w - 2.$$ 

Thus the cardinality of possible choices of the residues is the cardinality of $\{ (h_1, h_2) : 1 \leq h_1 \leq h_2 \leq a - 2 \}$. Namely we have $(a - 1) (a - 2) / 2$.

$(\mu = 1)$: We assume $g_{h_1} = 1$ and the other $g$'s are 0. Then the set (4) is the $\sigma$-image of

$$\{a^{w-1} \text{ and } [-2a^{w-1} + 1 + h : 0 \leq h \leq a^{w-1} - 1]\}.$$

As noted above, we can omit the first element of (7) from the consideration of $v(a^{w-1}; b_{i,s})$ for $1 \leq i \leq w - 2$. And the second set is composed of

$$\{a^{w-2} \text{ (0} \leq j \leq a - 1\},$$
and no other arithmetic progression (mod $Q$) of length $a^{w-2}$ whose difference $= a$ is contained in (7). Thus we must choose $a - 1$ arithmetic progressions from those of (8) as $v(a^{w-2}; b_{w-2}, s)$ $(1 \leq s \leq a - 1)$. After that selection, we return to the same situation with $i = w - 3$. Thus we proceed inductively.

Since there are $a$ choices at each step, and $a - 1$ choices for $h$, the cardinality of the choices is $(a - 1)a^{w-2}$. (It is easy to see that they are not equivalent.)

$(\mu = 0)$: In this case the set (4) is the $\sigma$-image of

$$(9) \quad \{-2a^{w-1} + h : 0 \leq h \leq a^{w-1}\}.$$ 

It is easy to see that the set (9) is decomposed to arithmetic progressions

$$(10) \quad A(-2a^{w-1}, a, a^{w-2} + 1) \quad \text{and} \quad A(-2a^{w-1} + j, a, a^{w-2}) \quad (1 \leq j \leq a - 1),$$

and no other arithmetic progression with the difference $a$ and whose length is not shorter than $a^{w-2}$ is contained in (9).

Now the situation differs whether we adopt the first arithmetic progression of (10) for $v(a^{w-2}; b_{w-2}, s)$ or not. First we assume that we take that for $v(a^{w-2}; b_{w-2}, s_0)$. Then the initial term of $v(a^{w-2}; b_{w-2}, s_0)$ must be $-2a^{w-1}$ or $-2a^{w-1} + a$. We note that as above the remaining element (i.e. $-a^{w-1}$ or $-2a^{w-1}$ respectively) can not belong to any $v(a^j; b_i, s)$ for $1 \leq i \leq w - 3$. We must take the other $a - 2$ sets $v(a^{w-2}; b_{w-2}, s)$ from the latter set of (10). If we determine the choice, the same situation occurs for $i = w - 3$ as that of the case $\mu = 1$.

Secondly, if we assume that $\{v(a^{w-2}; b_{w-2}, s) : 1 \leq s \leq a - 1\}$ is just the latter set of (10), then the situation for $i = w - 3$ is the same with that of $i = w - 2$. Thus we proceed inductively.

As a conclusion, we obtain $2(a - 1) (a^{w-2} + a^{w-1} + \cdots + 1) + 1$ choices of non equivalent residues for the case $\mu = 0$.

Summing up the above discussions, we obtain

**Proposition 4.** Let take $a, w \in \mathbb{N}$ and assume $a, w \geq 2$. Then the cardinality of non equivalent ECF's of the form

$$\oplus (a^i; b_i, s) \quad (0 \leq i \leq w - 1, \ 1 \leq s \leq a - 1) \oplus (1; c_j) \quad (j = 1, 2)$$

is given by $a^{w-1} + a(a - 3)/2$ if $w \geq 3$. In case $w = 2$,

$$\begin{cases} (2a - 1)! / (a + 1)! (a - 1) & \text{if } a \text{ is even,} \\ (2a - 1)! / (a + 1)! (a - 1)! + (2A - 1)! / (A + 1)! (A - 1)! & \text{if } a = 2A + 1. \end{cases}$$

**Remark.** As remarked in the first part of this section, there is an
SECF among the ECF's given in Proposition 4. By suitable transformations of the tree mentioned there, we can interpret above ECF's as SECF's excluding those which arise from the case \( \mu = 2 \). And these excluding cases, whose cardinality is \( (a - 1)(a - 2)/2 \), seem not to be SECF's.

6. In this section, we give a sufficient condition for an ECF to be an SECF. Namely we have

**Proposition 5.** Take an ECF \( \{S(Q, a_i, b_i) : 1 \leq i \leq k\} \). Assume that \( (Q, a_i) = 1 \) for all \( i \) and \( a_i \) satisfies the inequality \( 3a_i \geq 2Q \). Then this ECF is an SECF which is equivalent to \( S(Q, a_i, -1) \oplus S[E] \) where \( E \) is an ECS and \( S = S(Q, Q - a_i, 0) \).

**Proof.** Without loss of generality, we may assume \( b_i = -1 \). Now take \((a_i; -1)\) and some \((a_i; b_i) \) \( (2 \leq i \leq k) \). Put \((a_i, a_i) = t_i, a_i = t_iA \) and \( a_i = t_iA_i \). Then by the disjointness criterion, there exist \((x, y) \in N \times N\) such that

\[
xA_i + yA_i = Q - 2(t_i - 1)A_i.
\]

From (11), we have \( Q/a_i > 2(1 - 1/t_i)A_i \). By the assumption, we have \( 3/2 > 2(1 - 1/t_i)A_i \). This inequality holds only for \( t_i = 1 \) or \( A_i = 1 \). In case \( A_i = 1 \), substituting again to (11), we see that \( t_i = 1 \) is also necessary.

Hence we assume \( t_i = 1 \) for all \( 2 \leq i \leq k \). Then (11) becomes

\[
xa_i + ya_i = Q \quad \text{with} \quad (x, y) \in N \times N.
\]

This relation holds only for \( x = 1 \). Thus we may put \( m_i a_i = Q - a_i \) for \( 2 \leq i \leq k \).

Now we apply (#) by taking \( a_i \) as a multiplier. Then we have

\[
a_iV(Q, a_i, -1) = [-a_i, -1] \quad \text{(mod } Q). \]

We remark that for \( i \geq 2 \), the relation \( -m_i a_i \equiv a_i \) (mod \( Q \)) implies that the set \( a_iV(Q, a_i, b_i) \) is an arithmetic progression (mod \( Q \)) of length \( a_i \) whose difference \( = m_i \). As it must be contained in \([0, Q - a_i - 1]\) (mod \( Q \)), it is easy to see that the only possibility is that the coordinates set of \( a_iV(Q, a_i, b_i) \) makes the arithmetic progression

\[
A(r_i, m_i, a_i) \quad \text{with suitable } 0 \leq r_i \leq m_i - 1.
\]

This mean that the set \( \{(r_i, m_i) : 2 \leq i \leq k\} \) is an ECS. Now it is an easy deduction to obtain the conclusion of Proposition.

**Remark 1.** The following ECF shows the necessity of the inequality of Proposition 5.

\[
Q = 13, (8; -1) \oplus (2; 1) \oplus (1; 2) \oplus (1; 5) \oplus (1; 10).
\]
Remark 2. If we loosen the inequality to $2 > Q/a_i$, we obtain an SECF or $A_i = 1$ for some $i$. (About the meaning of this phenomenon, we refer to [1].)

References

2. R. L. Graham, Covering the positive integers by disjoint sets of the form $\{[n\alpha + \beta]: n = 1, 2 \ldots \}$, ibid., 15(1973), 354–358.