On eventually covering families generated by
the bracket function III

Ryozo Morikawa

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1. Introduction. Let $Z, N$ and $C$ mean as usual. And we write $S(q, a, b)$ the set $\{ \lfloor (qn + b)/a \rfloor : n \in Z \}$ where $\lfloor x \rfloor$ means the greatest integer $\leq x$.

We take numbers $q, a_1, a_2, v_i, v_2 \in N$ such as

$$q - a_1 v_i + a_2 v_2 \quad \text{and} \quad (q, a_1) = (q, a_2) = (a_1, a_2) = 1.$$ 

We consider in this paper the problem to list up eventually covering families which are composed of $v_i + v_2$ sequences of the form

$$S(q, a_i, b^{(i)}) 1 \leq i \leq v_i \quad \text{and} \quad S(q, a_2, b^{(j)}) 1 \leq j \leq v_2.$$ 

We assume for readers to be familiar with our preceding papers [2]. But for the convenience of readers, we list here the definitions and results quoted from these papers without explanations:

(From I) Definitions of an eventually covering family (ECF) ; of moduli and residues of an ECF ; of equivalence between ECF’s.

(From II) We quote criterion (2) for an ECF.

Now in order to state our result, we introduce the following numbers.

We take $t \in Z$ such that

$$a_1 t \equiv a_2 \pmod{q}.$$ 

Moreover, we take $v_2$ integers such as

$$0 \leq c_0 \leq c_1 \leq \ldots \leq c_{v_2} - 1 \leq (v_1 - 1).$$ 

Theorem. Notations being as above. Assume that the family of $v_1 + v_2$ sequences

$$S(q, a_i, b^{(i)}) 1 \leq i \leq v_i \quad \text{and} \quad S(q, a_2, b^{(j)}) 1 \leq j \leq v_2$$

makes an ECF. Then the residue set $\{ b^{(j)} : 1 \leq j \leq v_2 \}$ is equivalent to the set $\{-ht - c_0 : 0 \leq h \leq v_2 - 1\}$. Conversely, this set can be a residue set of a suitable ECF of above form.

We remark here the following two facts.

(i) If we determine the residue set $\{ b^{(j)} : 1 \leq j \leq v_2 \}$, then the residue
set \( \{ b_i^{(i)} : 1 \leq i \leq v_i \} \) is determined uniquely.

(ii) Among the residue sets taken as above, equivalent ones are mixed. Thus the problem to classify these to equivalent classes is remained.

About these points, we shall discuss in §7. But in the following, our main effort is devoted to give a proof of Theorem.

2. We first explain the notations and definitions used.

(i) If \( f \in \mathbb{Z} \) and \( h \in \mathbb{N} \) or \( 0 \), we write \([f, f + h] = \{f, f + 1, \ldots, f + h\}\). This set is called a segment of \( \mathbb{Z} \) of length \( h + 1 \).

(ii) Let \( q \) be as in (1), which we consider as fixed. \( \sigma \) denotes the map from \( \mathbb{Z} \) to \( \mathbb{C} \) defined by \( \sigma(r) = \exp (2\pi i r/q) \) for \( r \in \mathbb{Z} \). We put \( C(q) = \sigma(\mathbb{Z}) \). The \( \sigma \)-image of a segment of \( \mathbb{Z} \) is called a segment of \( C(q) \), the length of which is defined as its cardinality \( (\leq q) \).

(iii) Let \( t \in \mathbb{Z} \) of (3), which we consider as fixed. For \( S \subset \mathbb{Z} \), we denote \( t(S) = \{ts : s \in S\} \). And for \( A \subset C(q) \), we define \( t(A) = \sigma(t(\sigma^{-1}(A))) \).

(iv) For \( P \in C(q) \), we say \( r \in \mathbb{Z} \) to be the coordinate of \( P \), if \( \sigma(r) = P \) and \( 0 \leq r \leq q - 1 \). For \( P, Q \in C(q) \) whose coordinates are \( r \) and \( s \) respectively, we define their distance in \( C(q) \) by \( \operatorname{Min}(|r-s|, q-|r-s|) \).

(v) We call a sequence of the form \( S(q, a_i, b_i) \) as an \( a_i \)-sequence, and call a segment of \( \mathbb{Z} \) or of \( C(q) \) whose length is \( a_i \) as an \( a_i \)-segment \((i = 1, 2)\).

(vi) We use frequently the suffix sets \( H, M \) and \( H \times M \) where \( H = [0, v_2 - 1] \) and \( M = [0, a_2 - 1] \).

3. We give in this section preliminary discussions.

We put \( A_i = \sigma([b_i^{(i)} - a_i + 1, b_i^{(i)}]) \) \( 1 \leq i \leq v_i \) and \( B_j = \sigma([b_j^{(j)} - a_2 + 1, b_j^{(j)}]) \) \( 1 \leq j \leq v_2 \). Then by criterion (#), we see that a family of form (2) makes an ECF if and only if

\[
(5) \bigcup_{i=1}^{v_i} t(A_i) \bigcup_{j=1}^{v_2} B_j = C(q).
\]

Hence we seek a condition for (5). We put \( \bigcup_{i=1}^{v_i} A_i = A \). We consider the residues up to their equivalence. Hence we may assume \( A_1 = \sigma([1, a_1]) \), and \( A_i \)'s are arranged by their order of suffix to the positive direction of \( C(q) \).

We denote \( g_i \) for \( 1 \leq i \leq v_i - 1 \) the cardinality of the gap points of \( C(q) \) between \( A_i \) and \( A_{i+1} \) (Namely we have \( g_i \equiv b_i^{(i+1)} - b_i^{(i)} - a_i \pmod{q} \)). And let \( g_{A_i} \) be that of \( A_{V_i} \) and \( A_i \). Then we have
We denote $\langle g_1, g_2, \ldots, g_{v_l} \rangle = G$ and 
$C(q) - t(A) = R(G)$. We assume (by exchanging if necessary) $a_1 \geq a_2$. Hence we have

$$ (6) \quad g_1 + g_2 + \ldots + g_{v_l} = v_2a_2 \text{ and } g_i \geq 0 \quad (1 \leq i \leq v_l). $$

We use frequently the relations $\langle v_i, a_2 \rangle = \langle v_2, a_1 \rangle = 1$, which follows easily from (1).

4. We start with

**Definition 1.** For $G = \langle g_1, g_2, \ldots, g_{v_l} \rangle$, we put

$$ n(h, m) = \begin{cases} 0 & \text{if } h + mv_2 < g_{v_l} \\ n & \text{if } g_{v_l} + g_{v_l-1} + \ldots + g_{v_l-n+1} \leq h + mv_2 < g_{v_l} + g_{v_l-1} + \ldots + g_{v_l-n} \end{cases} $$

where $(h, m) \in H \times M$.

Here we remark that $n(h, m)$ satisfies

$$ (9) \quad \begin{cases} h + mv_2 < h' + m'v_2 \implies n(h, m) \leq n(h', m'), \\ 0 \leq n(h, m) \leq v_i - 1 \text{ where } (h, m), (h', m') \in H \times M. \end{cases} $$

And conversely, if we are given a set $\{n(h, m): (h, m) \in H \times M\}$ which satisfies (9), we have $G = \langle g_1, g_2, \ldots, g_{v_l} \rangle$ which satisfies (8) by putting

$$ (10) \quad g_i = \text{the cardinality of } (h, m) \text{ such as } n(h, m) = v_i - i. $$

**Definition 2.** We introduce an order to $H \times M$ by

$$ (h, m) < (h', m') \text{ if and only if } h + mv_2 < h' + m'v_2. $$

**Lemma 1.** $R(G) = \{\sigma (-ht + mv_i - n(h, m)a_2): (h, m) \in H \times M\}$.

**Proof.** Note that $R(G) = t(C(q) - A)$. We take the points of $C(q) - A$ starting from $\sigma(0)$ and run on $C(q)$ to the negative direction. Then

(i) before to reach to the $(h + mv_2 + 1)$-th point from the starting point, we meet $A_i$'s exactly $n(h, m)$ times.

(ii) If we meet $A_i$, then the coordinates of points of $C(q) - A$ jump $-a_i$.

Hence by (3), the coordinates of $t(C(q) - A)$ jump $-a_2 \pmod{q}$.

(iii) Relation (1) implies $tv_2 \equiv -v_1 \pmod{q}$.

These three facts lead easily to the assertion of Lemma.

We put

$$ (11) \quad P(h, m) = -ht + mv_i - n(h, m)a_2. $$

Now remember that our problem is to obtain a criterion for $R(G)$ to be a
disjoint union of \( a_2 \)-segments. And we try to construct an \( a_2 \)-segment of the form \( \sigma(\{P(h, m) : m \in M\}) \) for each \( h \in H \). (Later it will be shown that the all segments which satisfy (5) are made by such means.)

**Lemma 2.** For each \( h \in H \), \( \sigma(\{P(h, m) : m \in M\}) \) is an \( a_2 \)-segment of \( C(q) \) if and only if \( n(h, m) = \lfloor (c + mv_i)/a_2 \rfloor \) for \( m \in M \) with a suitable \( c \in [0, v_i - 1] \).

**Proof.** By (7) and (9), \( \sigma(\{P(h, m) : m \in M\}) \) makes an \( a_2 \)-segment of \( C(q) \) if and only if \( \{P(h, m) : m \in M\} \) makes an \( a_2 \)-segment of \( Z \). We fix \( h \in H \), and put \( Q(m) = mv_i - n(h, m)a_2 \). Then it is enough if we seek a condition for \( \{Q(m) : m \in M\} \) to be an \( a_2 \)-segment of \( Z \).

First we assume that \( \{Q(m)\} \) is an \( a_2 \)-segment. We denote it by \( B \). Then we have

\[
-(a_2 - 1) \leq Q(m) - Q(0) \leq a_2 - 1.
\]

Hence we have

(12) \[ \lfloor mv_i/a_2 \rfloor \leq n(h, m) - n(h, 0) \leq \lfloor mv_i/a_2 \rfloor + 1. \]

Now we consider the set \( \{mv_i - \lfloor mv_i/a_2 \rfloor a_2 : m \in M\} \). Then we see that it is an \( a_2 \)-segment. We denote it \( B_0 \). Since \( B_0 \) is obtained by putting \( n(h, 0) = 0 \) and taking the left-hand sided equality of (12) for all \( m \in M \). Hence \( B_0 \) is an \( a_2 \)-segment made of the (possible) largest integers. Thus we may take \( c \) such as \( B_0 - c = B \).

Then it is easy to see that for \( B \), the relations

\[
n(h, m) = \lceil (mv_i + c)/a_2 \rceil \quad \text{for } m \in M
\]

hold. Here \( c \) must be \( \leq v_i - 1 \), because of the inequality \( n(h, m) \leq v_i - 1 \).

The converse part of Lemma is obvious.

**Proposition 1.** We take \( \{c_h : h \in H\} \) which satisfies (4). Then the set \( \{-ht - c_h : h \in H\} \) can be taken as a residue set \( \{b_{j}^{(1)} : 1 \leq j \leq v_2\} \) of (2).

**Proof.** We put \( n(h, m) = \lceil (c_h + mv_i)/a_2 \rceil \) for \( (h, m) \in H \times M \). Then these numbers satisfy (9). And by Lemma 2, the set \( \{P(h, m) : m \in M\} \) makes an \( a_2 \)-segment for each \( h \).

Now we may take the largest integer of each segment as a residue of an \( a_2 \)-sequence. By the property of \( c \) stated in the proof of Lemma 2, we see that the number is \(-ht - c_h + a_2 - 1\). We translate these numbers \(-a_2 - 1\) simultaneously, and obtain the assertion.
5. As a kind of converse of Proposition 1, we have

**Proposition 2.** All the residue set \( \{ b^{(i)}_x : 1 \leq j \leq v_2 \} \) of an ECF of form

(2) is equivalent to one of the sets given in Proposition 1.

We shall prove this proposition by investigating the structure of \( R(G) \). And we give in this section some preliminary discussions.

Obviously, we may assume \( v_2 \geq 2 \). And under suitable cyclic permutation of the suffix of \( g_i \) (that corresponds to a translation of \( b^{(i)} \_l \leq i \leq v_1 \)), and by the pigeon-hole rule, we may assume

\[
v_2 - 1 < g_{v_1} + g_{v_1} - 1 + \ldots + g_{v_1} - k \quad \text{with} \quad k \leq [v_1/a_2].
\]

Namely we have

(13) \( n(v_2 - 1, 0) \leq [v_1/a_2] \).

Moreover, we assume that

(14) \( n(0, 0) = 0 \).

(If not the case, we substract \( n(0, 0) \) simultaneously from all \( n(h, m) \). That corresponds to a translation of \( R(G) \) on \( C(q) \)).

6. We start with the following

**Definition 3.** For \( h \in H \), we take \( r \in H \) such that \( h = -ra_1 \ (\text{mod} \ v_2) \). We put \( \mu(h) = r \).

**Lemma 3.** The coordinate of \( \sigma(-ht) \) where \( h \in H \) is given by

(15) \( (\lfloor (ra_1 - 1)/v_2 \rfloor + 1)v_1 + ra_2 \) where \( r = \mu(h) \).

And they lie on \( C(q) \) with the same order (in the positive direction) of \( r \).

**Proof.** We consider the difference (mod \( q \)) of \( a_1 \) multiples of \( -ht \) and (15). Then we have by (1) and (3),

\[
-a_1ht - a_1a_2r - a_1v_1 (\lfloor (ra_1 - 1)/v_2 \rfloor + 1)
\equiv a_2v_2 (\lfloor (ra_1 - 1)/v_2 \rfloor + 1 - (ra_1 + h)/v_2) \ (\text{mod} \ q).
\]

By Definition 3, \( ra_1 + h \equiv 0 \ (\text{mod} \ v_2) \). Hence the value put in the parenthesis is 0. Moreover as easily seen, the value of (15) lies in \([0, q - 1]\), and it increases as \( r \) increases.

**Corollary.** The coordinate of \( \sigma(P(h, m)) \) is congruent modulo \( q \) to

(15) \( (\lfloor (ra_1 - 1)/v_2 \rfloor + m + 1)v_1 + (r - n(h, m))a_2 \) where \( r = \mu(h) \).
In the following, \( r \) plays a main role instead of \( h \).

**Definition 4.** For \((r, m) \in H \times M\), we put
\[
\begin{align*}
  v(r, m) &= \left\lfloor \frac{(ra_1 - 1)/v_2} \right\rfloor + m + 1, \\
  Pr(m) &= v(r, m)v_1 + (r - n(h, m))a_2, \\
  Qr(m) &= v(r, m)v_1 + ra_2.
\end{align*}
\]
We call \( v(r, m) \) the \( v \)-part of \( Pr(m) \) (or \( Qr(m) \)). We say the pair \((h, m)\) is attached to \( Pr(m) \) (or \( Qr(m) \)), where \( \mu(h) = r \). We call \( r \) the \( r \)-part of \( Pr(m) \) (or \( Qr(m) \)). And we consider \( r \) cyclicly. (The next suffix of \( r = v_2 - 1 \) is \( r = 0 \) etc.)

Hereafter, we assume that \( R(G) \) is composed of disjoint \( a_2 \)-segments.

**Lemma 4.** Each \( a_2 \)-segment contains exactly one \( \sigma(Pr(0)) \) \((r \in H)\). And the order of these \( a_2 \)-segments on \( C(q) \) is same with that of \( r \).

**Proof.** We put \( \mu(h) = r \) and \( \mu(h') = r + 1 \). Then the coordinates of \( \sigma(Pr(0)) \) and of \( \sigma(Pr+1(0)) \) are congruent modulo \( q \) to
\[
\begin{align*}
  v(r, 0)v_1 + (r - n(h, 0))a_2 \quad \text{and} \\
  v(r + 1, 0)v_1 + (r + 1 - n(h', 0))a_2,
\end{align*}
\]
respectively.

Note that by (13), we have \( n(h, 0)a_2 \leq v_1 \) for \( h \in H \). Hence

(Case 1) If \( v(r, 0) < v(r + 1, 0) \), the distance of \( \sigma(Pr(0)) \) and \( \sigma(Pr+1(0)) \) on \( C(q) \) is \( \geq a_2 \).

(Case 2) Assume that \( v(r, 0) = v(r + 1, 0) \), then by Definition 3, we have \( h' = h - a_1 \). Hence by (9), \( n(h, 0) \geq n(h', 0) \).

Thus in either cases, we conclude that \( \sigma(Pr+1(0)) \) lies in the positive direction of \( \sigma(Pr(0)) \) on \( C(q) \) with distance \( \geq a_2 \).

The same reasoning works for \( r = v_2 - 1 \) and \( r + 1 = 0 \).

**Definition 5.** We denote \( D_r \) the \( a_2 \)-segment which contains \( \sigma(Pr(0)) \).

Our aim is to prove that \( D_r = \sigma(\{Pr(m) : m \in M\}) \) for all \( r \in H \). Here we consider that \( Pr(m) \) is the \(-n(h, m)a_2\) translation of \( Qr(m) \). And we remark the facts that an applicable translation is to the negative direction of \( C(q) \), whose length is a multiple of \( a_2 \) and \( \leq (v_1 - 1)a_2 \).

**Definition 6.** Take \( f \in M \).

(i) We collect from \( \{Qr(m) : (r, m) \in H \times M\} \) all of which satisfies
v(r, m) ≡ f (mod a₂), and arrange them from left to right by the order of their magnitude. We call this sequence the central members of the f-th line.

(ii) Next, we take from \{Q_r(m)\} all of which satisfies \(Q_r(m) > q - a₂\) and \(v(r, m) ≡ f + a₁ (mod a₂)\). And after arranging the numbers \(Q_r(m) - q\) by the order of their magnitude, we join the sequence to the left-hand side of (i). We call the sequence the left members of the f-th line.

(iii) Finally, we take all of \(Q_r(m)\) such as \(Q_r(m) < (v_i - 1)a₂\) and \(v(r, m) ≡ f - a₁ (mod a₂)\). And after arranging the numbers \(Q_r(m) + q\) by their order of magnitude, join that sequence to the right-hand side of (i). We call them the right members.

We call the total sequence of (i), (ii) and (iii) as the f-th line. (By calling it, we allow an ambiguity modulo a₂. Namely we call the f-th line as the \((f + a₂)\)-th line etc.)

We attach the pair \((h, m)\) for \(Q_r(m) \geq q\) of (ii) or (iii), and call r the r-part of them.

**Lemma 5.** The members of the f-th line satisfy the following properties.

(i) They are arranged from left to right by the order of their magnitude. And the r-parts of these members increase one by one. (We consider r cyclicly.)

(ii) The differences of the consecutive members are either \(a₂\) or \((v_i + 1) a₂\).

(iii) For two consecutive members with difference \(a₂\), the order of the \((h, m)\) pairs attached to them is inverse to that of their magnitude.

**Proof.** We first take \(Q_r(m)\) which is the central members of the f-th line. Then

\[ v(r, m) = [ (r a₁ - 1)/v₂] + m + 1 ≡ f (mod a₂). \]

Thus for each \(r \in H\), there exists exactly one \(m \in M\). We take \(Q_{r+1}(m')\) from them. Note that the v-part varies by a multiple of a₂, and \(m \in M\). Hence we have \(v(r, m) \leq v(r + 1, m')\).

(Case 1) If \(v(r, m) < v(r + 1, m')\), then we have

\[ Q_{r+1}(m') - Q_r(m) \geq (v_i + 1)a₂. \]

(Case 2) If \(v(r, m) = v(r + 1, m')\), then

\[ Q_{r+1}(m') - Q_r(m) = a₂. \]

(a) If \([ (r a₁ - 1)/v₂] < [ (r + 1)a₁ - 1)/v₂]\), then \(m > m'\).
(b) If \( \frac{(ra_1 - 1)}{v_2} = \frac{(r + 1)a_1 - 1}{v_2} \), then \( m = m' \), and we have, by Definition 3, \( h > h' \) where \( \mu(h) = r \) and \( \mu(h') = r + 1 \) respectively.

Thus we have proved the assertions for the central members of the \( f \)-th line.

Now we take up a left member. Note that it is the \( -q \) translation of a central member of the \( (f + ai) \)-th line. Hence it is enough if we check the assertions of Lemma for the jointed part of the left and the central members.

The first central member of the \( f \)-th line is \( f v_1 \). On the other hand, we put \( Q_r(m) \) the last central member of the \( (f + ai) \)-th line. Then \( r = v_2 - 1 \) and \( m \equiv \left[ \frac{a_1}{v_2} \right] + f \) (mod \( a_2 \)). Now \( q - a_2 < Q_r(m) \) implies \( m \equiv \left[ \frac{a_1}{v_2} \right] + f \). Thus we have \( Q_r(m) - q = f v_1 + a_2 \). It is easy to check the property (iii).

Finally, note that the jointed part of the right and the central members is the \( q \) translation of that of the left and the central members of the \( (f - a_i) \)-th line. Thus we proved the assertions of Lemma.

**Definition 7.** We make a table by putting together all the \( f \)-th line for \( f \subseteq M \). We denote it \( T_1 \).

And we make a table \( T_2 \) by translating each member of \( T_1 \) by \( -n(h, m) \) \( a_2 \) where \( (h, m) \) is the attached pair to the member. Then Lemma 5 shows that the order of the members of \( T_2 \) is same with that of \( T_1 \). Hence we use corresponding terminology of \( T_2 \) such as the central members etc. And we attach to each member the same \( (h, m) \) pair.

Hereafter we adopt a slightly modified definition of coordinate in \( C(q) \). Namely, for the points of \( D_0 \), we take their coordinates in \([ -a_2 + 1, a_2 - 1 ] \). (Remember that by (11), \( P_0(0) = 0 \).) Then the coordinates of each \( D_r \) (\( r \subseteq H \)) makes a complete set of residues modulo \( a_2 \), and they are contained in \( T_2 \).

**Definition 8.** From \( T_2 \), we eliminate the members which are not the coordinates of points of \( D_r \) (\( r \subseteq H \)). We denote this table \( T_3 \). We interpret the same terminology of \( T_2 \) to \( T_3 \).

Then as noted above, there are \( v_2 \) members in each line of \( T_3 \), and the \( (r + 1) \)-th member from left belongs to \( D_r \).
Lemma 6. All members of $T_3$ is the central member of each line.

Proof. Assume that there is a left member in the $f$-th line of $T_3$. We put it $P_1 - q$. Then $P_1$ is a central member of the $(f + a_1)$-th line of $T_2$. Hence there is a member of $T_3$ in the $(f + a_1)$-th line, which is not central. We put it $P_2$.

First we assume that $P_2$ is a right member. Then we have $P_1 < P_2$ on the same $(f + a_1)$-th line of $T_2$. Now take $Q_1$ and $Q_2$ from $T_1$ which correspond to $P_1$ and $P_2$ respectively. Note that the members of $T_3$ are contained in $[-a_2 + 1, q - 1]$, and the members of $T_1$ are contained in $[-a_2 + 1, q + (v_1 - 1)a_2]$. Thus we have

(a) The fact that $P_1 - q$ is a left member of $T_3$ implies $\sigma([P_1, Q_1]) \notin \sigma(0)$ or $\sigma(P_1) \notin D_0$.

(b) The fact that $P_2$ is a right member of $T_3$ implies $\sigma([P_2, Q_2]) \notin \sigma(0)$ and $\sigma(P_2) \in D_r$ such as $1 \leq r \leq v_2 - 1$. On the other hand, by Lemma 5, $Q_1 < Q_2$ on the same $(f + a_1)$-th line. Hence we reached to a contradiction.

Thus $P_2$ must be a left member. And we proceed inductively. Then by $(a_1, a_2) = 1$, we see that all line of $T_3$ must contain a left member. However $\sigma(0) \in D_0$ means that the $0$-th line of $T_3$ has no left member. Thus we reached to a contradiction.

A similar reasoning works for a right member. (In this case, $\sigma(P_{v_2 - 1}(0))$ plays a role of $\sigma(0)$.)

Now it is an easy deduction to prove Proposition 2.

7. Finally we shall settle the problem remarked in §1.
In this section, we consider frequently the suffix modulo $v_1$. And we omit special comments at each place.

We take numbers $\{c_h: h \in H\}$ as (4). And we put

$e_i = \text{the cardinality of } c_h \text{ such that } c_h = v_i - i$.

Definition 9. We introduce the following equivalence relation to the set $\{<f_1, f_2, \ldots, f_{v_1}>: f_i \in \mathbb{N} \cup \{0\}\}$.

$<f_1, f_2, \ldots, f_{v_1}>$ is equivalent to $<f'_1, f'_2, \ldots, f'_{v_1}>$ if and only if there exists $u \in \mathbb{N}$ such that $f_{i + u} = f'_i$ for $1 \leq i \leq v_1$.

Then the relation of $e_i$'s and $g_i$'s (of §3) is given by the following
Lemma 7. (i) $g_i = \sum e_s$ where the sum is taken for $s$ such that $(i - 1)a_2 + 1 \leq s \leq ia_2$.

(ii) The equivalence classes $<g_1, g_2, \ldots, g_{v_1}>$ and those of $<e_1, e_2, \ldots, e_{v_1}>$ correspond bijectively.

Proof. By Proposition 1 and (9), we see that $g_i$ is the cardinality of $(h, m) \in H \times M$ such that

$$(v_i - i)a_2 \leq c_h + mv_i \leq (v_i - i + 1)a_2 - 1.$$  

We count that as follows. We take a sequence of $a_2v_i$ terms by arranging $a_2$ times the sequence $e_1, e_2, \ldots, e_{v_1}$ and we divide it into a disjoint union of $v_i$ subsets by cutting each $a_2$ terms from beginning to end. Then it is easy to see that the sum of the $e$'s of the $i$-th set is equal to $g_i$. This proves (i).

Now by $(a_2, v_i) = 1$, the set $\{ia_2: 1 \leq i \leq v_i\}$ is a complete residue set modulo $v_i$. This fact leads to (ii).

Proposition 3. We take $\{c_h: h \in H\}$ and $\{e_i: 1 \leq i \leq v_1\}$ as above. Then

(i) the residue set of an ECF of form (2) is equivalent to one of the following sets.

$$\{b^{(i)}_1: 1 \leq i \leq v_i\} = \{ia_2 + \sum e_s: 1 \leq i \leq v_1\}$$

and

$$\{b^{(j)}_2: 1 \leq j \leq v_2\} = \{-ht - c_h + a_2 - 1: h \in H\}$$

where the sum is taken such that $1 \leq s \leq (i - 1)a_2$.

(ii) The cardinality of the non-equivalent ones among them is given by $(\sum F(d)\phi(d))/v_i$ where the sum is taken for all $d | (v_i, v_2)$, and $F(d) = ((v_1 + v_2)/d - 1)!/(v_1/d - 1)!/(v_2/d)!$, and $\phi(d)$ is the Euler's function.

Proof. As noted in §3, $b_i = a_i$ and $b^{(i+1)}_i - b^{(i)}_i \equiv a_i + g_i \pmod{q}$. Thus (i) follows from Lemma 7.

For (ii), we note that $<e_1, e_2, \ldots, e_{v_1}>$ is characterized by the following relations.

$$(16) \quad e_1 + e_2 + \ldots + e_{v_1} = v_2 \text{ and } e_i \geq 0 \quad (1 \leq i \leq v_1).$$

Now in order to count the cardinality of the equivalence classes, we apply the famous theorem of Burnside (cf. [1], Th. 5.2).

We put $\tau(i) = i + u$ for $1 \leq i \leq v_i$, and count the cardinality of $<e_1, e_2, \ldots, e_{v_1}>$ which is invariant by $\tau$. We put $(u, v_i) = w$ and $v_i = wd$. Then for an invariant set of $e$'s, we have
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(17) \[ e_1 + e_2 + \ldots + e_w = v_2/d. \]

Hence \( d \mid (v_1, v_2) \). Note that the cardinality of the solutions of (17) is given by \( F(d) \). And the number of such \( u \) is \( \phi(d) \). Thus we have the conclusion of Proposition.

References