On eventually covering families generated by
the bracket function IV

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1. Introduction. Let \( Z \) and \( \mathbb{N} \) mean as usual. For \( q, a \in \mathbb{N} \) and \( b \in \mathbb{Z} \), we write \( S(q, a, b) \) the set \( \{ [(qn + b)/a] : n \in \mathbb{Z} \} \) where \([x]\) means the greatest integer \( \leq x \). And a finite family \( \{ S(q_i, a_i, b_i) : 1 \leq i \leq k \} \) is said to be an eventually covering family (ECF) if they are mutually disjoint and the union of them is \( Z \).

We treat in this paper the problem to list up ECF's with \( k = 4 \). The case of \( k = 3 \) is treated in I (We refer the preceding serial papers by their number). Our interest for this problem is arisen besides itself by a trial to seek a key for the conjecture of A. S. Fraenkel, which asserts:

If all \( q_i/a_i \) (\( 1 \leq i \leq k \)) of an ECF are distinct and \( k \geq 3 \), then \( q_i = 2^k - 1 \) and \( a_i = 2^i \) (\( 0 \leq i \leq k - 1 \)).

We consider in this paper the problem not in its full generalities. Namely we assume that

(1) \( q = q_i \) and \( (q, a_i) = 1 \) for \( 1 \leq i \leq 4 \).

The reason of this restriction is mainly due to avoid troublesome complexities. And we think that it does not spoil the essential part of the problem and that the methods introduced in this paper are sufficient to treat the problem to list up all ECF's with \( k = 4 \).

In the following, we write \( S(q, a_i, b_i) \) simply by \( (a_i ; b_i) \). The aim of this paper is to prove the following Theorem.

Theorem. Notations being as above, and we assume (1). Then any ECF \( \{ S(q_i, a_i, b_i) : 1 \leq i \leq 4 \} \) is equivalent to one of the following ones. \( (A, B \in \mathbb{N}, A \nmid B \text{ and } (A, B) = 1) \):

(i) \( q = 4, \) \( (1 ; 0) \cup (1 ; 1) \cup (1 ; 2) \cup (1 ; 3) \),

(ii) \( q = A + 3B, \) \( (A ; -1) \cup (B ; 0) \cup (B ; [q/3] \cup (B ; [2q/3]) \),

(iii) \( q = 2A + 2B, \)

\( (A ; A + B - 1) \cup (A ; -1) \cup (B ; 0) \cup (B ; A + B) \)

or \( (A ; A + B - 1) \cup (A ; -2) \cup (B ; 0) \cup (B ; A + B + 1) \),
2. Now we explain notations and definitions used. We assume for readers to be familiar with I–III.

We denote frequently $S(q, a_i, b_i)$ simply by $S_i$ and call it as an $a_i$–sequence. We use the terminology $C(q)$ and an $a_i$–segment in $C(q)$, which are introduced in II (or III). We say a quadruple $A = (a_1, a_2, a_3, a_4)$ is good if it can be a moduli set of an ECF.

We quote from I the criterion for disjointness of $S(q_i, a_i, b_i)$ $i = 1, 2$, and criterion (♯) from II. And besides these, we quote the following criterion (F) from [2] :

(F) Let $q, a_1, a_2 \text{ and } a_3 \in \mathbb{N}$. We assume that $(q, a_i) = 1$ and $(a_i, a_j) = 1$ for $1 \leq i \neq j \leq 3$. Then three sequences $S(q, a_i, b_i)$ $1 \leq i \leq 3$ can be mutually disjoint by taking suitable $b_i$’s if and only if there exist $(x_i, y_i) \in \mathbb{N}^2$ $(1 \leq i \leq 3)$ such that

\[
\begin{align*}
&x_1a_1 + y_1a_2 = q, \\
x_2a_2 + y_2a_3 = q, \quad \text{and} \quad x_1x_2x_3 + y_1y_2y_3 > q, \\
x_3a_3 + y_3a_1 = q,
\end{align*}
\]

For a proof of (F), we refer to [2].

In referring to these criterions, we use the following notations. ( (D) ; $a_i, a_j$) means that we apply the disjointness criterion of $S_i$ and $S_j$. ( (♯), g) means that we apply (♯) with a multiplier $g$. And ( (F) ; $a_n, a_i, a_j$) means that we apply (F) to $\{S_n, S_i, S_j\}$.

And to list up the possible residues, we use frequently Proposition 2 of I and the result of III.

Now we start from a rather simple case.

Let $\{S(q, a_i, b_i) : 1 \leq i \leq 4\}$ be an ECF. Then by the density relation, we have

\[(2) \quad a_1 + a_2 + a_3 + a_4 = q.\]

(i) If $a_1 = a_2 = a_3 = a_4$, then by (1) and (2) they are 1. And by Proposition 2 of I, we obtain the first one of Theorem.
(ii) In case that three of $a_i$'s are equal, we put $a_1 = a_2 = a_3 = A$ and $a_4 = B$. Then also by Propositions 1 and 2 of I, we obtain the second one of Theorem.

(iii) If there are two equal pairs in $a_i$'s, we put $a_1 = a_2 = A$ and $a_3 = a_4 = B$. Then by (1) and (2), A and B are odd. Now we apply the result of III with $v_1 = v_2 = 2$. Then we obtain (iii) of Theorem, which has two non-equivalent residue sets.

3. Thus in the following, we assume that

(3) there are at least three distinct numbers in $a_i$ ($1 \leq i \leq 4$).

We arrange them as

(4) $a_1 \geq a_2 \geq a_3 \geq a_4$.

Lemma 1. Let $A = (a_1, a_2, a_3, a_4)$ be a good quadruple. Assume that $a_1 > a_2$. Then

(i) in case $(a_1, a_2) = 1$, either $2a_1 + a_2 = q$ or $a_1 + 2a_2 = q$ holds.

(ii) If $(a_1, a_2) > 1$, then $(a_1, a_2) = (2a, a)$, excluding $(3a, a, a, a-1)$, $(3a, a, a, a-2)$, $(6, 4, 4, 3)$ and $(8, 2, 2, 1)$.

Proof. (i) By $(D); (a_1, a_2)$, we have $xa_1 + ya_2 = q$ with $(x, y) \in \mathbb{N}^2$. Now by (2) - (4), we have $q < a_1 + 3a_2$. Hence by (3) and (4), we see that $(x, y)$ is $(2, 1)$ or $(1, 2)$.

(ii) We put $a_3 = a_4$ (i = 1, 2). By $(D); (a_1, a_2)$, we have

(5) $xu_1 + yu_2 + 2(a - 1)u_1u_2 = q$ with $(x, y) \in \mathbb{N}^2$.

By (2) and (3), we have $q < 4a_1$. Thus we have easily from (5),

(6) $(1 - 1/a)u_2 < 2$.

Since $a \geq 2$, we have $u_2 = 1$, 2 or 3.

(A) If $u_2 = 3$, then by (6), $a = 2$. Hence $a_2 = 6$ and $a_1 = 2u_1$. Thus we have from (5)

$$7u_1 + 3 \leq xu_1 + 3y + 6u_1 \leq 2u_1 + 17.$$ 

This contradicts to $u_1 > u_2 = 3$.

(B) Assume $u_2 = 2$. Then $a_2 = 2a$ and from (5), we have

$$u_1 + 2 + 4(a - 1)u_1 < au_1 + 6a.$$ 

Hence we have $3(a - 1)(u_1 - 2) < 4$. By $a \geq 2$ and $u_1 > 2$, the only possible case is $a = 2$ and $u_1 = 3$. Thus we have $A = (6, 4, a_3, a_4)$. Now by (5), we see $q = 12 \geq 5$. Thus by (3), the only possible case is $(6, 4, 4, 3)$.

(C) Let $u_2 = 1$, then $a_2 = a$ and $q < a_1 + 3a$. Thus by (5), we have $(a - 1)(u_1 - 3) < 2$. And $a \geq 2$ implies $u_1 = 2$ or 3 or 4. Substituting these
values in (5), we see under (1) - (4), the possible cases are $u_1 = 2$ or $(3a, a, a, a - 1)$ or $(3a, a, a, a - 2)$ or $(8, 2, 2, 1)$.

**Lemma 2.** Let $A = (a_1, a_2, a_3, a_4)$ be a good quadruple. We assume that $a_1 > a_3$ and $(a_1, a_3) = a > 1$. Then $a_3 = a$, excluding the following ones:

(i) $q = 17$, $A = (6, 5, 4, 2), (6, 4, 4, 3)$ or $(6, 6, 4, 1),$
(ii) $q = 19$, $A = (6, 6, 4, 3)$ or $(6, 5, 4, 4),$
(iii) $q = 29$, $A = (9, 9, 6, 5)$ or $(9, 8, 6, 6)$.

**Proof.** We put $a_i = au_i$ and $a_3 = au_3$. Then by ((D); $a_1, a_3$), we obtain the following

\[ux + yu_3 + 2(a - 1)u_1u_3 = q \text{ with (}x, y\text{) } \in \mathbb{N}^2.\]

Now a similar reasoning used in Lemma 1 for $u_3$ works and we obtain $u_3 = 1$ or $2$. Assume $u_3 = 2$. Then $q < 2a_i + 4a$. Hence from $(5')$, we obtain

\[(2a - 3)(u_1 - 2) < 4.\]

Since $u_1 > 2$ and $(u_1, u_3) = 1$, we have $u_1 = 3$ or $5$. Now it is an easy deduction to obtain those quadruples given in Lemma.

4. In this section, we treat the case that there are (two) same numbers in $a_i$ ($1 \leq i \leq 4$). We start with two Lemmas.

**Lemma 3.** (i) Assume that $(a_i, a_j) = 1$ and $q = a_i + ya_j$ with $y \in \mathbb{N}$. If $a_i \geq y$, we can not take two disjoint $a_i$-sequences which are disjoint with any given $a_j$-sequence.

(ii) Let $(a_i, a_j) = a > 1$, and we put $a_i = au_i$ and $a_j = au_j$. Assume that $q = 2(a - 1)u_iu_j + u_i + u_j$. Then there exists exactly one $a_i$-sequence which is disjoint with a given $a_j$-sequence.

**Proof.** (i) We apply ((#), $a_i$). Then the image of the values of an $a_i$-sequence makes a segment in $C(q)$. And since $ya_i \equiv -a_i \pmod q$, the image of an $a_j$-sequence makes an arithmetic progression in $C(q)$ with difference $y$ of length $a_i$. By $a_i \geq y$, we can take disjoint $a_i$-segments only from the segment of length $q - (y(a_i - 1) + 1) = a_i + y - 1$. Thus we can take exactly one $a_i$-segment.

(ii) By ((D); $a_i, a_j$), we see that the residue of a disjoint $a_i$-sequence is determined by the given $a_j$-sequence.

**Lemma 4.** Assume that $4A + B = q$ with $(4A, B) = 1$ and $A \equiv B$. Then $A = (2A, A, A, B)$ is a good quadruple, and there are two possible choices
for a residue set. Namely

\[(2A; 0) \cup (A; [q/4]) \cup (A; [3q/4]) \cup (B; -1)\]

and \[(2A; [q/2]) \cup (A; 0) \cup (A; [q/2]) \cup (B; -1).\]

Proof. We fix the residue of the B-sequence as \(-1\). Note that \(2A\)-sequence is a union of two \(A\)-sequences. By Propositions 1 and 2 of I, we see that the four \(A\)-sequences have a unique residue set \(\{0, [q/4], [q/2], [3q/4]\}\). We can choose \(\{0, [q/2]\}\) or \(\{[q/4], [3q/4]\}\) as an residue set which comes from a \(2A\)-sequence. And also by Proposition 2 of I, we see that no other choice is possible.

Now we list up good quadruples with (two) same moduli. We divide to three cases.

(i) \(a_i - a_2 > a_3 > a_4\): If \((a_i, a_3) = 1\), then \(q = xa_i + ya_3\) with \((x, y) \in \mathbb{N}^2\). Since \(q = 2a_i + a_3 + a_4\) and \(a_3 > a_4\), we have \(x = 1\). And \(a_3 \geq 2\) implies \(y \leq a_i\). Hence by Lemma 3, there exist no good quadruples. Thus we put \((a_i, a_3) = a > 1\) and \(a_i = a_u\) \((i = 1, 3)\). Then by Lemma 2, possible quadruples are of the form \((a_u, a_u, a, a_i)\) or \((6, 6, 4, 3)\) or \((6, 6, 4, 1)\) or \((9, 9, 6, 5)\). We see the latter three are not good by \(((\#), 6)\), Lemma 3 and Lemma 3 respectively.

Thus we consider \((a_u, a_u, a, a_i)\). By \(((\#), a_u)\), we see that the segment in \(C(q)\) of length \(a + a_i\) must contain an arithmetic progression with difference \(u_i\) of length \(a\). Thus we have \(a + a_i \geq u_i(a - 1) + 1\). Hence \(u_i = 2\) and \(a_i = a - 1\). Namely \(A = (2a, 2a, a, a - 1)\). Here we fix the residue of \((a - 1)\)-sequence to \(-1\), and divide \(S(q, 5a, 0)\) to five \(a\)-sequences. Then by Proposition 2 of I, their residue set is \(\{0, [q/5], [2q/5], [3q/5], [4q/5]\}\). Since four of them come from \(2a\)-sequences, we see that the only possible case is \(a = 2\). And then \(A = (4, 4, 2, 1)\), which has a unique residue set given in (v) of Theorem.

(ii) \(a_i > a_2 > a_3 > a_4\): If \((a_i, a_2) = 1\), by Lemma 1, we have \(2a_i + a_2 = q\) or \(a_1 + 2a_2 = q\). Since \(q = a_1 + 2a_2 + a_3\), only the first case is possible. Now by Lemma 3, we see that there exist no good quadruples. If \((a_i, a_2) = a > 1\), also by Lemma 1, we have \((2a, a, a, a_4)\), \((3a, a, a, a - 1)\), \((3a, a, a, a - 2)\), \((6, 4, 4, 3)\) and \((8, 2, 2, 1)\) as possible quadruples.

For \((2a, a, a, a_i)\), we apply Lemma 4. For \((3a, a, a, a - 1)\), by a similar reasoning used in (i) for \((2a, 2a, a, a - 1)\), we obtain only one good quadruple given in (vi) of Theorem. For the latter three, we see that they
are not good by ( ( ), 3a), Lemma 3 and Lemma 3 respectively.

(iii) \( a_1 > a_2 > a_3 = a_4 \): For the case \((a_1, a_2) = 1\), we obtain by Lemma 1, 
\((2a_3, a_2, a_3, a_4)\) or \((a_1, 2a_3, a_3, a_4)\). We apply Lemma 4 to them. If \((a_1, a_2) = a > 1\), we have \(A = (2a, a, a_3, a_4)\). We apply the result of III with \(v_1 = 2\) and \(v_2 = 3\). Then we obtain two possible non-equivalent residues of three \(a\)-sequences. Since two of them must come from a \(2a\)-sequence, we see the only possible case is \(a = 3\) and \(a_3 = 1\). Namely we obtain \((6, 3, 1, 1)\), which has the residue set given in (vii) of Theorem.

5. Finally we consider the case
\[(7) \quad a_1 > a_2 > a_3 > a_4.\]

We treat the case dividing to four cases.

(i) \((a_1, a_2) = 1\) and \((a_1, a_2) = 1\): By Lemma 1, \(q = a_1 + 2a_2\) or \(q = 2a_1 + a_2\).

(A) Assume \(q = a_1 + 2a_2\). Put \(q = xa_1 + ya_3\) with \((x, y) \subseteq \mathbb{N}^2\). Then \(x = 1\) or \(2\). If \(x = 1\), then \(2a_2 = ya_3\). Thus \((y - 2)a_3 = 2a_4\) and \(a_3 > a_4\) implies \(y = 3\). And we have \(A = (q - 6a_4, 3a_4, 2a_4, a_4)\). By ( ( ), 6a), we see that \(A\) is not good.

Next assume \(x = 2\). Then we see that \(y = 1\). Hence \(A = (a_3 + 2a_4, a_3 + a_4, a_3, a_4)\). Here \((a_1, a_3) = 1\) implies \((a_2, a_3) = 1\). Thus we put \(q = 2a_2 + y_3\) with \((\hat{x}, \hat{y}) \subseteq \mathbb{N}^2\). Then we have \((\hat{x} + \hat{y} - 3)a_3 = (4 - \hat{x})a_4\). By \((a_3, a_4) = 1\), the possible pairs are \((\hat{x}, \hat{y}) = (1, 3)\) or \((1, 4)\). Namely we obtain \((5, 4, 3, 1)\) and \((7, 5, 3, 2)\) respectively. We apply to them ( (F) ; 5, 4, 3) and ( (F) ; 7, 5, 3) respectively. Thus we see that they are not good.

(B) Let \(q = 2a_1 + a_2\). And we put \(xa_1 + ya_3 = q\). Then \(x \leq 2\), but \(x = 2\) is impossible since then \(ya_3 = a_2 < a_1 = a_3 + a_4\). Thus \(x = 1\), and we have \(A = (a_3 + a_4, a_2, a_3, a_4)\) and \(a_2 = (y - 1)a_3 - a_4\). Now by (7), we see \(y = 3\). Hence \(A = (a_3 + a_4, 2a_3 - a_3, a_3, a_4)\). Since \((a_3, a_4) = 1\), we have \((a_2, a_3) = 1\). Putting \(q = 2a_2 + y_a_3\), we have \((2\hat{x} + \hat{y} - 4)a_3 = (\hat{x} + 1)a_4\). Hence \((a_3, a_4) = 1\) implies the possible \((\hat{x}, \hat{y})\) are \((3, 1)\), \((2, 1)\), \((2, 2)\) and \((1, 3)\).

By substituting these values, we obtain only \((7, 5, 4, 3)\) which satisfies \((1)\) and \((7)\). Then ( (F) ; 7, 5, 4) shows that is not good.

(ii) \((a_1, a_2) = 1\) and \((a_1, a_3) = a > 1\): By Lemmas 1 and 2, we have \(A = (a_1, a_2, a, a_4)\) or \((6, 5, 4, 2)\). For \((6, 5, 4, 2)\), we apply ( ( ), 5) and see that is not good. Thus we consider the former one. As easily seen, the relation \(q = 2a_1 + a_2\) is impossible. Thus \(q = a_1 + 2a_2\). And \(A = (a_1, a + a_4, a, a_4)\). By ( (D) ; a_1, a), we obtain \((a - 1) (w_1 - 4) \leq 1\). Thus \(w_1 \leq 5\). Since
must be odd, \( u_i = 3 \) or 5. If \( u_i = 5 \), then \( a = 2 \) and \( a_4 = 1 \). This contradicts to (1). Thus \( u_i = 3 \), and \( A = (3a, a + a_i, a, a_i) \). Now \( (a, a_i) = 1 \) implies \( (a_i, a_i) = 3 \) or 1.

(A) Assume \( (a_i, a_i) = 3 \). Then by \((D) ; a_i, a_i\), we obtain the relation \( q = 5a + 2a_i \geq (a_i/3)4a + a \). Thus \( a_i = 3 \). Namely \( A = (3a, a + 3, a, 3) \). Now again by \((D) ; a_i, a_3\), we have \( a \leq 8 \). Since \( a \) is odd, we obtain \((21, 10, 7, 3)\) and \((15, 8, 5, 3)\). By \((#), 21\) and \((#), 15\) respectively, we see that they are not good.

(B) If \( (a_i, a_i) = 1 \), we put \( q = xa_i + ya_i \) with \((x, y) \in \mathbb{N}^2\). We see easily that \( x = 1 \) and \( 2a = (y - 2)a_i \). Thus \( a_i = 2 \) or 1. If \( a_i = 2 \), \( A = (3a, a + 2, a, 2) \). And \( a_i = 1 \) implies \( A = (3a, a + 1, a, 1) \). Now by \((D) ; a_i, a_i\), we obtain \((9, 4, 3, 1)\) and \((15, 7, 5, 2)\) and \((9, 5, 3, 2)\). We see they are not good by \((#, 9)\), \((F) ; 15, 7, 2\) and \((F) ; 9, 5, 2\) respectively.

(iii) \((a_i, a_2) > 1 \) and \((a_i, a_3) > 1\): By Lemmas 1 and 2, we see that \( A = (aa_3, aa_3/2, a_3, a_i) \). By \((D) ; a_i, a_2\), we obtain \( a_3(a - 4) \geq 2 \). Since \( a_3 \geq 2 \), we have \( a = 3 \) or 4 or 5. Here the case \( a = 5 \) and \( a_3 = 2 \) contradicts to (1). Thus \( A = (3a_3, 3a_3/2, a_3, a_i) \) or \((4a_3, 2a_3, a_3, a_i)\). Now by \((#, 3a_3)\) and \((#, 4a_3)\) respectively, we see the only possible one is \((8, 4, 2, 1)\). It is shown in I that this quadruple has a unique residue set, which is given in \((viii)\) of Theorem.

(iv) \((a_i, a_2) > 1 \) and \((a_i, a_3) = 1\): By Lemma 1, we have \( A = (2a, a, a, a_i) \). Now let \( q = xa_i + ya_3 \) with \((x, y) \in \mathbb{N}^2\). Then as easily seen, \( x = 1 \). Thus \( ya_3 = a + a_i \).

(A) Assume \((a_i, a_i) = 1\), and let \( x_a + y_a = q \). Then \( x = 1 \). Thus \( y_a = ya_3 \). Since \((a_3, a_i) = (a_3, a) = 1\), we have \( a_i | y \). Hence \( q = a_i + za_3a_4 \) with \( z \in \mathbb{N} \). Now \((F) ; a_i, a_3, a_i\) shows that for a good quadruple, \( z \geq 2 \). By \((D) ; 2a, a\), we have \( a_3 + a_i \geq a - 1 \). Namely we have \( za_3a_4 \leq 2(a_3 + a_4) + 1 \). This implies \( a_4 = 1 \). Hence \( q = a_i + za_3 \). Here \( z \geq 3 \), and we have \( a_3 \leq 3 \) and \( a \leq 5 \). Substituting these numbers, we obtain a quadruple \((10, 5, 3, 1)\). We apply it to \((#, 10)\).

(B) \((a_i, a_i) = \hat{a} > 1\). Then by a similar reasoning used in Lemma 1, we have \( a_i = \hat{a} \) and we put \( a_i = a_iu_i \). Then we have \( A = (a_iu_i, a_iu_i/2, a_3, a_4) \) where \( a_3 = a_i(u_i + 2)/2(y - 1) \). From (1) and (7), we see that

\[
a_iu_i \text{ is even, } a_i | 2(y - 1) \text{ and } 2y - 4 < u_i.
\]

We first show \( y \leq 4 \). If \( y \geq 5 \), then \( a_3 \leq a_4(u_i + 2)/8 \). Thus by \((D) ; a_i, a_4\), we have \( a_i(3u_i - 10) \leq 8 \). From (8), we have \( u_i \geq 6 \), and it is impos-
sible.

Next we put \( y = 4 \). Then by \((D) ; a_1, a_2\), we have \( a_4(u_i - 1) < 3 \). From (8), \( u_i > 4 \) and \( a_4 = 1, 2, 3 \) or \( 6 \). It is easy to check that no good quadruples exist.

Thus as the final step of the proof of Theorem, we put \( y = 3 \). Then by \((D) ; a_1, a_2\), we obtain \( a_4(u_i - 6) \leq 4 \). From (8), we see that \( a_i = 1, 2 \) or \( 4 \). By substituting these numbers we obtain \( (10, 5, 3, 1), (8, 4, 3, 2), (16, 8, 5, 2), (20, 10, 7, 4) \) and \( (28, 14, 9, 4) \). We see that they are not good by applying (\#) with multipliers 10, 8, 16, 20 and 28 respectively.

References
