Disjoint sequences generated by
the bracket function

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1. Introduction. Let \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{N} \) have the usual meanings. For \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), let \( S(\alpha, \beta) \) denote the sequence of the form \( \lfloor \alpha n + \beta \rfloor \) where \( \lfloor x \rfloor \) denotes the greatest integer \( \leq x \), and \( n \) runs through \( \mathbb{N} \).

We take a family \( \{S(\alpha_1, \beta_i) : 1 \leq i \leq k\} \) of these sequences. And our main concern is to investigate criterions for disjointness of these sequences.

For the understandings of the background of the problem and current investigations, we refer to [1], especially page 19–20, and [6]. As pointed there, the situation is fairly clarified if some \( \alpha_i \) of the family is irrational. For example, a criterion for disjointness of \( S(\alpha_1, \beta_1) \) and \( S(\alpha_2, \beta_2) \) with \( \alpha_1/\alpha_2 \notin \mathbb{Q} \) was given by Th. Skolem (cf. [2] or [6]), and a criterion for the union of \( S(\alpha_i, \beta_i) \) \( 1 \leq i \leq k \) to cover every sufficiently large positive integers exactly once was given by R. L. Graham [3]. Thus in the following, we treat mainly the less understood case that all \( \alpha_i \) in the family are rational numbers.

For the case, we remark the following easy facts:

Let \( \alpha = q/a \) where \( q, a \in \mathbb{N} \) and \( (q, a) = 1 \), then

(i) The effect of \( \beta \in \mathbb{R} \) on the sequence \( \lfloor \alpha n + \beta \rfloor \) depends only on \( \lfloor a \beta \rfloor \). Hence without changing the sequence \( \lfloor \alpha n + \beta \rfloor \), \( \beta \) can be replaced by a rational number \( \frac{b}{a} \) whose denominator is \( a \).

(ii) If all \( \alpha_i \in \mathbb{Q} \), then \( S(\alpha_i, \beta_i) \) are mutually disjoint if and only if the sets \( \{ \lfloor \alpha_i n + \beta_i \rfloor : n \in \mathbb{Z} \} \) are mutually disjoint \( (1 \leq i \leq k) \).

So we write \( S(q, a, b) = \{ \lfloor (qn + b)/a \rfloor : n \in \mathbb{Z} \} \) and seek in the following a criterion for disjointness of \( S(q_i, a_i, b_i) \) \( (1 \leq i \leq k) \). In this paper we treat the following three problems.

(I) About a criterion for disjointness of two sequences \( S(q_i, a_i, b_i) \) \( i = 1, 2 \), there are some investigations which treat only fairly restrictive cases (cf. [2], [6]). We announced such criterion in [4], and gave a rough sketch of the proof. The details of it will be given here. The result is as
follows:

We take $S(q_i, a_i, b_i)$ $i = 1, 2$ with $(q_i, a_i) = 1$. We put
\begin{equation}
(q_1, q_2) = q, (a_1, a_2) = a, a_i = au_i \ (i = 1, 2).
\end{equation}

**Theorem 1.** Notations being as above, consider $q_i, a_i$ and $a_2$ as given. Two sets $S(q_i, a_i, b_i)$ and $S(q_2, a_2, b_2)$ are disjoint with suitable two integers $b_i$ and $b_2$ if and only if the equation
\begin{equation}
xu_1 + yu_2 = q - 2u_1u_2(a - 1)
\end{equation}
holds with some $(x, y) \in \mathbb{N} \times \mathbb{N}$.

In case this condition is satisfied, we take a solution $(x_0, y_0)$ of (2) such that $1 \leq y_0 \leq u_i$.

**Theorem 2.** Assume that $q_i, a_i$ $i = 1, 2$ satisfy the condition of Theorem 1. Then $S(q_i, a_i, b_i)$ $(i = 1, 2)$ are disjoint if and only if
\begin{equation}
u_1b_2 - u_2b_1 \in (E_1 \cup E_2) \ (\text{mod } q)
\end{equation}

where
\begin{align*}
E_1 &= \{u_1X + u_2Y + u_1u_2(a - 1) : 0 \leq X \leq x_0 - 1, 1 \leq Y \leq y_0\}, \\
E_2 &= \{u_1X + u_2Y + u_1u_2(a - 1) : 0 \leq X \leq x_0 - u_2 - 1, y_0 + 1 \leq Y \leq u_1\}
\end{align*}
(In case $x_0 \leq u_2$, we define $E_2 = \emptyset$.)

(II) It seems that a criterion for disjointness of two sequences $S(\alpha_i, \beta_i)$ $i = 1, 2$ for the case $\alpha_i \in \mathbb{Q}$ and $\alpha_i/\alpha_2 \in \mathbb{Q}$ has hitherto not ascertained (cf. [2]). Hence we give it.

We put $\alpha_i/\alpha_2 = m_i/m_2$ where $m_i, m_2 \in \mathbb{N}$ and $(m_i, m_2) = 1$. Let $\mu$ denote the number $\alpha_i/m_i \ (= \alpha_i/m_2)$, and we denote by $\|x\|$ the distance to the nearest integer of $x$ from $x$.

**Theorem 3.** Notations being as above, two sequences $S(\alpha_i, \beta_i)$ $i = 1, 2$ are disjoint if and only if the inequality
\begin{equation}
\mu \| (\beta_1 - \beta_2) \mu \| \geq 1
\end{equation}
holds.

We remark here that the following result is obtained in [7]:
Under the situations stated above, (3) is also a necessary and sufficient condition for $S(\alpha_i, \beta_i) \cap S(\alpha_2, \beta_2)$ to be a finite set.
Here as easily seen in our situations,
$S(\alpha_i, \beta_i) \cap S(\alpha_2, \beta_2) = \emptyset \Leftrightarrow S(\alpha_i, \beta_i) \cap S(\alpha_2, \beta_2)$ is a finite set.
Thus the result is essentially same. But the theme of [7] and the method used for the proof are fairy different from ours.

(III) About a disjointness criterion for general \( S(q_i, a_i, b_i) \) \( 1 \leq i \leq k \), we may apply the result of (I). But then we must treat \( k(k-1)/2 \) relations simultaneously. And it brings a new complicated problem. In the following, we treat the case \( k = 3 \) under the following assumptions:

(4) \((q_i, a_i) = 1 \) for \( 1 \leq i \leq 3 \). And \((q_i, q_j) = q, (a_i, a_j) = 1 \) for all \( 1 \leq i < j \leq 3 \).

Assume that \( S(q_i, a_i, b_i) \) \( (1 \leq i \leq 3) \) are mutually disjoint. Then by Theorem 1, the following relations hold.

\[
\begin{aligned}
&\begin{cases}
x_1a_1 + y_1a_2 = q, \\
x_2a_2 + y_2a_3 = q, \text{ where } (x_i, y_i) \in \mathbb{N} \times \mathbb{N} \ (1 \leq i \leq 3), \\
x_3a_3 + y_3a_1 = q,
\end{cases}
\end{aligned}
\]

From (5), we have

\[
\begin{aligned}
&\begin{cases}
a_1 = \frac{q}{x_1x_2x_3 + y_1y_2y_3} \\
a_2 = \frac{x_2x_3 + y_1y_2 - x_3y_1}{x_1x_2x_3 + y_1y_2y_3} \\
a_3 = \frac{x_3x_1 + y_2y_3 - x_1y_2}{x_1x_2x_3 + y_1y_2y_3}
\end{cases}
\end{aligned}
\]

Now the relation \((a_i, a_j) = 1 \) \((1 \leq i < j \leq 3)\) implies

(7) \( qf = x_1x_2x_3 + y_1y_2y_3 \) with \( f \in \mathbb{N} \).

**Theorem 4.** Notations being as above, the sequences \( S(q_i, a_i, b_i) \) \( i = 1, 2, 3 \) are disjoint with suitable \( b_i \)'s if and only if \( f \geq 2 \) with a suitable solution system \((x_i, y_i)\) \((1 \leq i \leq 3)\).

The object of this paper is to give proofs of above Theorems. Finally we remark that these results are applied to the theory of “eventually covering families” (cf. [5]).

2. We explain the notations used:

(i) For \( f \in \mathbb{Z} \) and \( h \in \mathbb{N} \) or \( 0 \), we write \([f, f + h] = \{f, f + 1, \ldots, f + h\} \).

This set is called a *segment* of \( \mathbb{Z} \) of *length* \( h + 1 \).

(ii) For \( m \in \mathbb{N} \) and \( S \subset \mathbb{Z} \), we define

\[
\rho_m(S) = \{z \in \mathbb{Z} : z \equiv s \ (\text{mod} \ m) \text{ with } s \in S\}.
\]

We denote \( \sigma_m \) the map from \( \mathbb{Z} \) into \( \mathbb{C} \) defined by

\[
\sigma_m(r) = \exp(2\pi ir/m) \text{ for } r \in \mathbb{Z}.
\]

We put \( C(m) = \sigma_m(\mathbb{Z}) \). The \( \sigma_m \)-image of a segment of \( \mathbb{Z} \) is called a *segment* of \( C(m) \), the *length* of which is defined as its cardinality.
(iii) Let $q_i$, $a_i$, $u_i$ ($i = 1, 2$) and $q$ be given as in (1). We define the numbers as follows.

$$q_i = q d_i \quad (i = 1, 2), \quad Qq = q_1 q_2.$$ 
And we take $(c, w) \in \mathbb{N} \times \mathbb{N}$ such that

$$\tag{8} c q_i = w i i - 1.$$ 
Take $t \in \mathbb{N}$ such that

$$\tag{9} t u_i \equiv u_2 \pmod{q}.$$ 
In the following, we consider these numbers as fixed.

3. Let $q_i$, $a_i$, $b_i$, $u_i$ ($i = 1, 2$), $q$ and $a$ be as in §1. Besides these, we fix the number $b_i$ to be $-1$, and investigate a criterion for $b_2$ such that

$$S(q_i, a_i, -1) \cap S(q_2, a_2, b_2) = \emptyset.$$ 
Put $A = S(q_i, a_i, -1)$ and $B = S(q_2, a_2, b_2)$. And we put

$$A_j = S(q_i u_i, a_i, q_i c_j - 1) \quad (0 \leq j \leq u_i - 1)$$ 
where $c$ is the number given in (8). Then by $(c, u_i) = 1$, we see easily that $A$ is the union of all $A_j$ ($0 \leq j \leq u_i - 1$).

We put $b(-1) = \{ b_2 \in \mathbb{Z} : A \cap B = \emptyset \}$ and

$$v_j = \{ b_2 \in \mathbb{Z} : A_j \cap B = \emptyset \} \quad (0 \leq j \leq u_i - 1).$$

**Lemma 1.** $v_0 = \rho_q(\lfloor -a_2, (a - 1) u_2 - 1 \rfloor)$ and $v_j = v_0 + j t \quad (1 \leq j \leq u_i - 1)$ where $t$ is defined in (9).

**Proof.** In the proof of Lemma, we use frequently the following easy fact: For two numbers $\alpha, \beta \in \mathbb{R}$,

$$\tag{*} [\alpha] = [\beta] \Leftrightarrow [\alpha] \leq \beta < [\alpha] + 1.$$ 
Now we divide to three cases.

(i) ($a = a_2 = 1$) In this case $a_i = u_i \quad (i = 1, 2)$ (We use always $a_i$ for $u_i$) and $q_2 = q$. Since $A_0 = \{ q_i n - 1 : n \in \mathbb{Z} \}$, we see by (*), and noting $q_2 \mid q_i$,

$$b_2 \in v_0 \Leftrightarrow b_2 \in \rho_q(\lfloor -a_2, -1 \rfloor).$$ 
For $1 \leq j \leq a_i - 1$, we have

$$[q_i n + (q_i c_j - 1)/a_i] = [q_i n + w j - (j + 1)/a_i] = q_i n + w j - 1$$ 
where $w$ is given in (8). Now by (8) and (9), we have

$$\pmod{q_i n + w j - 1} a_2 \equiv a_i t (w j - 1) \equiv tj - a_2 \pmod{q}.$$ 
Hence by (*), we see that

$$b_2 \in v_j \Leftrightarrow b_2 \in \rho_q(\lfloor -a_2 + tj, -1 + tj \rfloor).$$

(ii) ($a = 1, a_2 > 1$) In this case, we put $A_0(r) = \{ q_i n + q_i r - 1 : n \in \mathbb{Z} \}$

$(0 \leq r \leq d_2 - 1)$. Then it is easy to see that $A_0$ is the union of all $A_0(r)$.
Since \( q_2 \mid Q \), we have by (*),
\[
\{ b_2 : A_0(r) \cap B = \emptyset \} = \rho_{q_2}( [ q_1 a_2 r - a_2, q_1 a_2 r - 1 ] ).
\]
Now \( v_0 \) is the union of all these sets of \( 0 \leq r \leq d_2 - 1 \). Note that \( \{ d_1 a_2 r : 0 \leq r \leq d_2 - 1 \} \) is a complete set of residues modulo \( d_2 \). Hence considering the relation \( q_1 a_2 r = q(d_1 a_2 r) \) and \( q_2 = q d_2 \), we have
\[
\bigcup_{r=0}^{d_2-1} (\rho_{q_2}( [ q_1 a_2 r - a_2, q_1 a_2 r - 1 ] )) = \rho_q( [ -a_2, -1 ] ) ( = v_0 ).
\]
For \( 1 \leq j \leq a_1 - 1 \), as in (i), we have \( A_j = S(q_i, 1, wj - 1) \). And we proceed in the same way as above, i.e. dividing \( A_j \) into subsets such as
\[
A_j(r) = \{ Qn + q_i r + wj - 1 : n \in \mathbb{Z} \} (0 \leq r \leq d_2 - 1)
\]
and consider the union of the sets \( \{ b_2 : A_j(r) \cap B = \emptyset \} \) for all \( 1 \leq r \leq d_2 - 1 \). Here the same reasoning works and we obtain
\[
v_j = \rho_q( [ -a_2 + tj, -1 + tj ] ).
\]

(iii) \((a > 1)\) In this case \( A_0 = \{ [q_i n/a - 1/a_1] : n \in \mathbb{Z} \} \). Note that \( [q_i n/a - 1/a_1] = [ (q_i n - 1)/a ] \). We have the inequality
\[
(q_i n - a)/a \leq \lfloor (q_i n - 1)/a \rfloor \leq (q_i n - 1)/a.
\]
About (10), we remark that
(a) The left-hand side equality of (10) is attained by \( n \equiv 0 \) (mod \( a \)),
and the right-hand side equality is attained by \( q_i n \equiv 1 \) (mod \( a \)).
(b) \((q_i n - 1)/a - (q_i n - a)/a = (a - 1)/a < 1\).
Hence if \( d_2 = 1 \) (i.e. \( q_2 = q \)), we have by (*),
\[
v_0 = \rho_q( [ -a_2, u_2(a - 1) - 1 ] ).
\]
For the case \( d_2 > 1 \), we obtain the same relation by putting
\[
A_0(r) = \{ [ (Qn + q_i r - 1)/a ] : n \in \mathbb{Z} \} (0 \leq r \leq d_2 - 1)
\]
and proceed in the same way as in (ii).
For \( 1 \leq j \leq u_i - 1 \), we have
\[
[q_i n/a + (q_i c j - 1)/a_i] = [q_i n/a + (w u_i j - j - 1)/a_i] = [ (q_i n + w j - 1)/a ].
\]
Now by a similar reasoning as above, we obtain the equality
\[
v_j = \rho_q( [ -a_2 + tj, (a - 1)u_2 + tj - 1 ] ).
\]
Lemma 1 implies that \( \sigma(q(v_j)) \) is a segment of \( C(q) \). We denote its starting point by \( P_j \) (\( = \sigma_q(-a_2 + tj) \)).

Since \((u_i, u_0) = 1\), we can take \( x_0, y_0 \in \mathbb{Z} \) such that
\[
q - 2 u_i u_2 (a - 1) = x_0 u_i + y_0 u_2 \quad \text{and} \quad 1 \leq y_0 \leq u_i.
\]
We put \( \bar{y} = u_i - y_0 \). Note that in case \( u_i > 1 \), we have \( \bar{y} > 0 \) and \((u_i, \bar{y}) = 1\).
**Definition 1.** For \( j \in [0, u_1 - 1] \), we take \( r \in [0, u_1 - 1] \) such that \( j \equiv yr \pmod{u_i} \). We denote \( r = \tau(j) \). And we put
\[
P(r) = r(x_0 + (2a - 1)u_2) - [yr/u_i]u_2 - a_2.
\]

**Lemma 2.** \( P_j = q_i(P(r)) \) where \( r = \tau(j) \).

**Proof.** We consider the difference modulo \( q \) of \( -(a_2 + tj)u_1 \) and \( u_1P(r) \). Then by (9) and (11), we see easily that the difference is modulo \( q \) to
\[
u_2(j - r(u_1 - y_0) + [yr/u_i]u_1) = 0.
\]
Since \( (q, u_i) = 1 \), we have the conclusion of Lemma.

**Lemma 3.** If \( x_0 > u_2 \), then \( \sigma_q(b(-1)) \) is composed of \( y_0 \) segments of \( C(q) \) with the equal length \( x_0 \) and of \( y \) segments of equal length \( x_0 - u_2 \). If \( 1 \leq x_0 \leq u_2 \), then \( \sigma_q(b(-1)) \) is composed of \( y_0 \) segments with the equal length \( x_0 \). If \( x_0 \leq 0 \), then \( b(-1) = \phi \).

**Proof.** We assume that \( x_0 > 0 \), then Lemma 2 implies that \( P_j \)'s are arranged on \( C(q) \) by the order of \( \tau(j) (= r) \). And note that the difference of \( P(r) \) and \( P(r + 1) \) is
\[
(a) \ x_0 + (2a - 1)u_2 \text{ if } [yr/u_i] = [y(r + 1)/u_i],
(b) \ x_0 + 2(a - 1)u_2 \text{ if } [yr/u_i] < [y(r + 1)/u_i].
\]
As easily seen, the cardinalities of \( r \) of (a) and of (b) are \( y_0 \) and \( y \) respectively. (Note that \( P(u_i) = P(0) + q \).)

On the other hand, the length of \( \sigma_q(v_j) \) is \( u_2(2a - 1) \). Thus the structure of \( \sigma_q(b(-1)) \) depends on the order of magnitude of
\[
u_2(2a - 1), \ x_0 + u_2(2a - 1) \text{ and } x_0 + 2u_2(a - 1).
\]
Since \( x_0 + 2u_2(a - 1) > u_2(2a - 1) \) means \( x_0 > u_2 \), and the relation \( x_0 + 2u_2(a - 1) \leq u_2(2a - 1) < x_0 + u_2(2a - 1) \) means \( 1 \leq x_0 \leq u_2 \), the assertion of Lemma for \( x_0 > 0 \) follows from the above facts.

Finally if \( x_0 \leq 0 \), then the union of all \( \sigma_q(v_j) \) \( (0 \leq j \leq u_1 - 1) \) is \( C(q) \). This means \( b(-1) = \phi \).

Now above three Lemmas lead to the following

**Proposition 1.** Assume that \( q_i, a_i (i = 1, 2) \) satisfy the condition of Theorem 1. Let the pair \( (x_0, y_0) \) be defined as in §1. Then
\[
S(q_1, a_1, -1) \cap S(q_2, a_2, b_2) = \phi \iff b_2 \in \rho_q(G_1 \cup G_2)
\]
where
\[
G_1 = \{a_2 - u_2 + X + mt : X \in [0, x_0 - 1], m \in [0, y_0 - 1] \}.
\]
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\[ \mathcal{G}_2 = \{-x_0 - u_2(a - 1) + X + nt : X \in [0, x_0 - u_2 - 1], n \in [0, u_t - y_0 - 1] \}. \]

(If \( x_0 \leq u_2 \), we put \( \mathcal{G}_2 = \phi \).)

**Proof.** The structure of \( \sigma_q(b(-1)) \) is given by Lemma 3. We see that the longer segments lie at the head of \( \sigma_q(v_j) \) (\( 0 \leq j \leq y_0 - 1 \)). Thus they are obtained by translating \( t_j \) (\( 0 \leq j \leq y_0 - 1 \)) the segment which lies at the head of \( \sigma_q(v_0) \), and that segment is

\[ \sigma_q( [a_2 - u_2, a_2 - u_2 + x_0 - 1] ). \]

For the shorter segments (if not empty), they are obtained by operating the \( t \)-translation successively to the segment which lies at the head of \( \sigma_q(v_y_0) \). By noting (9) and (11), we see that the segment is

\[ \sigma_q( [-x_0 - u_2(a - 1), -a_2 - 1] ). \]

Since \( b(-1) = q_q(b(-1)) \), we obtain the conclusion of Proposition.

4. Now we proceed to prove Theorems 1 and 2. We put

\[ b(b_i) = \{ b_2 \in \mathbb{Z} : S(q_i, a_i, b_i) \cap S(q_2, a_2, b_2) = \phi \}. \]

**Lemma 4.** \( b(b_i) = b(-1) + (b_i + 1)t \).

**Proof.** Note that the integers of \( S(q_i, a_i, b_i) \) are composed of some congruence classes modulo \( q_i \), \( i = 1, 2 \). Thus we may consider those to be composed of congruence classes modulo \( Q \) simultaneously. We take \( s_i \in \mathbb{Z} \) such that \( s_i a_i \equiv 1 \) (mod \( q_i \)). And take \( s_2 \) such that both \( s_2 a_2 \equiv 1 \) (mod \( q_2 \)) and \( (s_2, Q) = 1 \). (Since \( Q = q_2 d_1 \) and \( (q_2, d_1) = 1 \), such choice is always possible.) Thus we have

\[ S(q_i, a_i, b_i) = S(q_i, a_i, a_i s_i (b_i + 1) - 1) = S(q_i, a_i, -1) + s_i (b_i + 1). \]

Let \( \bar{a}_2 \in \mathbb{Z} \) such that \( \bar{a}_2 s_2 \equiv 1 \) (mod \( Q \)). We put \( \bar{b}_2 = \bar{a}_2 s_i (b_i + 1) \). Then \( \bar{b}_2 s_2 \equiv s_i (b_i + 1) \) (mod \( Q \)). Hence noting \( \bar{a}_2 \equiv a_2 \) (mod \( q_2 \)), we obtain

\[ S(q_2, a_2, b_2 - \bar{b}_2) = S(q_2, a_2, b_2) - s_i (b_i + 1). \]

By (13) and (14), we have \( b(b_i) = b(-1) + \bar{a}_2 s_i (b_i + 1) \).

Here note that \( b(-1) \) is composed of congruence classes modulo \( q \). And by (9), we have \( \bar{a}_2 s_i \equiv t \) (mod \( q \)). Hence we may replace \( \bar{a}_2 s_i (b_i + 1) \) with \( t(b_i + 1) \).

Now Theorem 1 follows from Proposition 1 and Lemma 4. Thus it remains to prove Theorem 2. By Lemma 4, we obtain

\[ u_t b(b_i) - u_2 b_i = u_t b(-1) + u_2. \]
The structure of $b(-1)$ is given in Proposition 1. By (9), we obtain the set $E_i$ from $G_i$ putting $Y = m + 1$. And similarly by (9) and (11), we obtain $E_2$ from $G_2$ by putting $Y = y_0 + n + 1$.

5. In this section we treat Theorem 3. We assume that $\alpha_i, \beta_i, m_i (i = 1, 2)$ and $\mu$ be as in (II) of §1. Now we put

$$S(j) = S(m_1 m_2 \mu, m_1 (j - m_2) \mu + \beta_i) \quad (1 \leq j \leq m_2).$$

Then it is easy to see that $S(\alpha_i, \beta_i)$ is the union of all $S(j)$. Hence we investigate the condition for $\beta_2$ such that $S(j)$ and $S(\alpha_2, \beta_2)$ to be disjoint.

Take first the set $S(m_2)$. Note that by the well known theorem of Kronecker, $\mu \in \mathbb{Q}$ implies that the set

$$\{(m_1 m_2 \mu n (\text{mod} \ 1) : n \in \mathbb{N}\}$$

is dense in $[0, 1]$. Hence about the inequalities

$$( (m_1 m_2 \mu n + \beta_i - 1) < \lfloor (m_1 m_2 \mu n + \beta_i) \rfloor \leq (m_1 m_2 \mu n + \beta_i),$$

both the right-hand and the left-hand side inequalities are approximable infinitesimally by taking a suitable $n \in \mathbb{N}$.

Since $m_2 | m_1 m_2$ and by (*), we can deduce easily from above fact that

$$S(m_2) \cap S(\alpha_2, \beta_2) = \emptyset \iff \beta_2 \equiv (\beta_i - 1, \beta_i + 1) \pmod{m_2 \mu}. $$

For general $j$, the same reasoning works and we obtain that

$$S(j) \cap S(\alpha_2, \beta_2) = \emptyset \iff \beta_2 \equiv (m_1 j \mu + \beta_i - 1, m_1 j \mu + \beta_i + 1) \pmod{m_2 \mu}. $$

Note here that the set $\{m_1 j : 1 \leq j \leq m_2\}$ is a complete set of residues modulo $m_2$. Hence it is easy to see that

$$\bigcup_{j=1}^{m_2} \{(m_1 j \mu + \beta_i - 1, m_1 j \mu + \beta_i + 1) \pmod{m_2 \mu} = (\beta_i - 1, \beta_i + 1) \pmod{\mu}. $$

Now putting together above facts, we reach to

$$S(\alpha_1, \beta_1) \cap S(\alpha_2, \beta_2) = \emptyset \iff \beta_2 \equiv (\beta_i - 1, \beta_i + 1) \pmod{\mu}. $$

It is an easy deduction to formulate this conclusion in the form given in Theorem 3.

6. Now we proceed to a proof of Theorem 4, and give in this section preliminary discussions.

Let $q_i, a_i$, and $q$ satisfy (4). We say $S(q_i, a_i, b_i) (i = 1, 2, 3)$ which are mutually disjoint simply a disjoint triple. For a disjoint triple, we take from solutions $(x_i, y_i) (1 \leq i \leq 3)$ of (5) so that

$$(15) \quad 1 \leq y_i \leq a_i \quad (1 \leq i \leq 3). $$

We use frequently the following fact

$$(16) \quad (x_1, a_2) = (x_2, a_3) = (x_3, a_1) = (y_1, a_i) = (y_2, a_2) = (y_3, a_3) = 1.$$
which follows from (4) and (5).

We take $\hat{a}_i \in \mathbb{N}$ such that $\hat{a}_i a_i \equiv 1 \pmod{q}$, and put $c_i = \hat{a}_i b_i \ (1 \leq i \leq 3)$. We consider $\hat{a}_i$'s as fixed.

Now by Theorem 2, we have

\begin{align*}
\begin{cases}
    c_2 - c_1 \equiv \{\hat{a}_2 X_1 + \hat{a}_1 Y_1\} \pmod{q}, \\
    c_3 - c_2 \equiv \{\hat{a}_3 X_2 + \hat{a}_2 Y_2\} \pmod{q}, \\
    c_1 - c_3 \equiv \{\hat{a}_1 X_3 + \hat{a}_3 Y_3\} \pmod{q},
\end{cases}
\end{align*}

(17)

where $X_i$ and $Y_i$ run through the range given in Theorem 2. Here we remark that the sum of three terms of the left-hand side of (17) is 0. And as easily seen, if we can take three elements from the right-hand side of (17) respectively such that the sum of them $\equiv 0 \pmod{q}$, we can make a disjoint triple by determining $c_i$'s (or $b_i$'s) from these elements.

Hence for the existence of a disjoint triple, it is enough if we investigate the condition for the existence of $X_i$ and $Y_i \ (1 \leq i \leq 3)$ such as

\begin{align*}
\hat{a}_i (X_i + Y_i) + \hat{a}_2 (X_1 + Y_2) + \hat{a}_3 (X_2 + Y_3) \equiv 0 \pmod{q},
\end{align*}

(18)

where $X_i$ and $Y_i$ belong to the sets described in Theorem 2. We treat (18) in the following form

\begin{align*}
(X_3 + Y_3) \equiv -a_1 \hat{a}_2 (X_1 + Y_2) - a_1 \hat{a}_3 (X_2 + Y_3) \pmod{q}.
\end{align*}

(19)

Definition 2. For $m \in \mathbb{Z}$, we define $r \in [0, x_1 - 1]$ such that $m \equiv a_2 r \pmod{x_1}$. And for $n \in \mathbb{Z}$, we define $s \in [0, y_3 - 1]$ such that $n \equiv -a_3 s \pmod{y_3}$. We denote $\tau_1(m) = r$ and $\tau_2(n) = s$. And we put

\begin{align*}
\chi(m, n) = \{r/x_1 + s/y_3\} q + y_1 m/x_1 + x_3 n/y_3
\end{align*}

wher $\{x\}$ means $x - [x]$ (As easily seen $\chi(m, n) \in \mathbb{Z}$.)

Lemma 5. $(-a_1 \hat{a}_2 m - a_1 \hat{a}_3 n) \equiv \chi(m, n) \pmod{q}$.

Proof. As the proof of Lemma 2, we see that the following relations hold (by multiplying $a_2$ and $a_3$ respectively),

\begin{align*}
    -a_1 \hat{a}_3 m &\equiv r a_2 + \left[\left(r a_2/x_1\right) + \left[\left(m - 1\right)/x_1\right] + 1\right] y_1 \pmod{q}, \\
    -a_1 \hat{a}_3 n &\equiv s a_2 + \left[\left(s a_2/y_3\right) + \left[\left(n - 1\right)/y_3\right] + 1\right] x_3 \pmod{q},
\end{align*}

Now add two relations. Then by (5), we obtain the conclusion of Lemma.

7. We start to prove Theorem 4. In the following, we are interested in the existence criterion only, and do not try to list up all those triples.

First we seek a solution of (17) by taking

\begin{align*}
X_i \in [0, x_i - 1] \text{ and } Y_i \in [1, y_i] \text{ for all } i = 1, 2, 3.
\end{align*}

(21)

In that case, our problem is to investigate whether the relation
Lemma 6. If \((x_1, y_3) > 1\), then (22) holds with a suitable \((m, n)\).

Proof. We put \((x_1, y_3) = d\), and take \(m_0 \in [1, x_1]\) such that \(\tau_1(m_0) = x_1/d\). And take \(n_0 \in [1, y_3]\) such that \(\tau_2(n_0) = y_3 - y_3/d\). Then we have
\[\chi(m_0, n_0) = y_1 m_0 / x_1 + x_3 n_0 / y_3.\]
Note that by their definition, \(m_0 \nmid x_1\) and \(n_0 \nmid y_3\). Hence we have \(\chi(m_0, n_0) \in [1, x_3 + y_1 - 1]\).

By Lemma 6, we assume in the following
\[(23)\quad (x_1, y_3) = 1.\]

Definition 3. For \(h \in [0, x_1 y_3 - 1]\), we take \((R, S)\) by
\[y_3 R + x_1 S \equiv h \pmod{x_1 y_3}\] and \(R \in [0, x_1 - 1], S \in [0, y_3 - 1]\).
By (23), the pair \((R, S)\) is determined uniquely from \(h\). We call it simply the \(h\)-pair. We put
\[M_1(h) = \{m : \tau_1(m) = R, m \in [1, x_1 + y_2 - 1]\},\]
\[M_2(h) = \{n : \tau_2(n) = S, n \in [1, x_2 + y_3 - 1]\}.\]
We put \(M_h = M_1(h) \times M_2(h)\). We take \((m_0, n_0) \in M_h\) such that \(m_0 \in [1, x_1]\) and \(n_0 \in [1, y_3]\), and call it the initial pair of \(M_h\). Finally we take \((m_i, n_i) \in M_h\) such that \(m_i \in [y_2, x_1 + y_2 - 1]\) and \(n_i \in [x_2, x_2 + y_3 - 1]\), and call it the last pair of \(M_h\).

By using above terminology, we have
\[(24)\quad \chi(M_h) \subseteq [\chi(m_0, n_0), \chi(m_1, n_1)]\].

Lemma 7. If \(f \geq 2\), then (22) has a solution.

Proof. We take \(h = x_1 y_3 - 1\). Then for the initial pair \((m_0, n_0)\) of \(M_h\), we have
\[q(1 - 1/x_1 y_3) < \chi(m_0, n_0) \leq q(1 - 1/x_1 y_3) + y_1 + x_3.\]
On the other hand, for the last pair \((m_i, n_i)\) of \(M_h\), we put \(m_i = y_2 + w_i\) and \(n_i = x_2 + w_2\). Then by (7), we have
\[\chi(m_i, n_i) = q(1 + (f - 1)/x_1 y_3) + y_1 w_i / x_1 + x_3 w_2 / y_3.\]
Since \(f \geq 2\), we have \(\chi(m_i, n_i) > q\).
Note here that the difference of any adjacent two members of \(\chi(M_h)\) is \(\leq 2\).
Max($x_3, y_1$). Hence by (24), we conclude that
\[
\chi(M_h) \cap [1, x_3 + y_1 - 1] \pmod{q} \neq \emptyset.
\]

Thus we have proved the existence of disjoint triples for the case $f \geq 2$ (with the solution system of (15)). In the following we assume $f = 1$. Then we have
\[
(25) \quad q = x_1x_2x_3 + y_1y_2y_3, \quad a_1 = x_2x_3 + y_1y_2 - x_3y_1,
\]
\[
a_2 = x_3x_1 + y_2y_3 - x_1y_2 \text{ and } a_3 = x_1x_2 + y_3y_1 - x_2y_3.
\]

**Lemma 8.** *If $f = 1$, then $(x_1, y_3) = 1$. And for $(m, n)$ of $M_h$, we have $x_1 | (m + y_2h)$ and $y_3 | (n + x_2h)$.*

**Proof.** Put $(x_1, y_3) = d$, then by (25), we have $d | q$ and $d | a_2$. Hence $d = 1$. Now we take $(m, n) \in M_h$ and let $(R, S)$ be the $h$-pair. Note that by (25), $a_2 = y_2y_3 \pmod{x_1}$. Hence by Definition 3, we have $m \equiv -y_2h \pmod{x_1}$. The second relation follows by a similar reasoning.

**Lemma 9.** *In case $f = 1$, $(x_3 + y_1) \leq \chi(m, n) \leq q$ where $m \in [1, x_1 + y_2 - 1]$ and $n \in [1, x_2 + y_3 - 1]$.*

**Proof.** We take $h \in [0, x_1y_3 - 1]$, and consider it as fixed. Let $(m_0, n_0)$ and $(m_1, n_1)$ be the initial pair and the last pair of $M_h$ respectively. Then by (25), we have
\[
x_1y_3 \chi(m_0, n_0) = h(x_1x_2x_3 + y_1y_2y_3) + y_1y_3m_0 + x_1x_3n_0 = x_1x_3(hx_2 + n_0) + y_1y_3(hy_2 + m_0).
\]
Now by Lemma 8, we have $\chi(m_0, n_0) \geq x_3 + y_1$.

For $(m_1, n_1)$, we put $m_1 = y_2 + w_1$ and $n_1 = x_2 + w_2$. Then we have
\[
x_1y_3 \chi(m_1, n_1) = y_1y_3((h + 1)y_2 + w_1) + x_1x_3((h + 1)x_2 + w_2).
\]
By Lemma 8, we have $x_1 | ((h + 1)y_2 + w_1)$ and $y_3 | ((h + 1)x_2 + w_2)$. Hence noting that $w_1 \in [0, x_1 - 1]$, $w_2 \in [0, y_3 - 1]$ and $(h + 1) \leq x_1y_3$, we have $\chi(m_1, n_1) \leq (x_1x_2x_3 + y_1y_2y_3) = q$. Now the assertion of Lemma follows from (24).

8. Lemma 9 implies that in case $f = 1$, (22) has no solution. Hence we must check the other possible ranges of $X_i$'s and $Y_i$'s. Then by Theorem 2, we may assume for that case $x_i > a_2$ (We operate a suitable permutation to $i$, if necessary). Then from (25), we have
\[
(26) \quad x_3 \leq y_2.
\]
First we consider the case that
\[ X_1 \in [0, x_1 - a_2 - 1], \quad Y_1 \in [y_1 + 1, a_1], \] and \( X_2, X_3, Y_2 \) and \( Y_3 \) are as (21).
Namely our problem is to study the possibility of the relation
\[ (27) \quad \chi(m, n) \equiv [y_1 + 1, x_3 + a_1 - 1] \pmod{q} \]
with \( m \in [1, x_1 + y_2 - a_2 - 1] \) and \( n \in [1, x_2 + y_3 - 1] \).

**Lemma 10.** If \( x_3 < y_2 \), then (27) holds with a suitable \((m, n)\).

**Proof.** We put \( y_2 - x_3 = z \geq 1 \). Now we take \( h = [a_2/y_2] \). And let \((R, S)\) be the \( h\)-pair. We first ascertain the existence of \( m_0 \) such as \( \tau_1(m_0) = R \). Since \( a_2 \leq y_2 \) and \((a_2, y_2) = 1\), we have
\[ (28) \quad a_2 - y_2 + 1 \leq y_2 h \leq a_2 < x_3. \]
Now noting Lemma 8, we see that \( m_0 \) can be taken (uniquely) from the range given in (27). We take \((m_0, n_0) \in M_h\) such that \( 1 \leq n_0 \leq y_3 \). By Lemma 8, we put \( h y_2 + m_0 = h_1 x_1 \) and \( h x_3 + n_0 = h_2 y_3 \). Then as noted above, \( h_1 = 1 \). On the other hand for \( h_2 \), note that \( a_2 < y_2 y_3 \) implies \( h < y_3 \). Hence we have \( h_2 = [h x_3/y_3] + 1 \leq x_2 \). Since \( \chi(m_0, n_0) = y_1 + h_2 x_1 \), by above facts and by noting \( a_1 = x_2 x_3 + z y_1 \), we reach to the conclusion that
\[ \chi(m_0, n_0) \equiv [y_1 + 1, x_3 + a_1 - 1]. \]

Finally, we treat the remained case (i.e. \( x_3 = y_2 \)). By (25), we have \( a_1 = x_2 x_3 \) and \( a_2 = y_2 y_3 \). Since \( (a_1, a_2) = 1 \), we have \( x_3 = y_2 = 1 \). Thus in this case, (5) becomes
\[ (29) \quad \begin{cases} x_1 a_1 + y_1 a_2 = q, \\ a_1 a_2 + a_3 = q, \text{ and } x_1 > a_2. \end{cases} \]

**Lemma 11.** If \( q, a_1 \) and \( a_2 \) satisfy (29), there exists no disjoint triple.

**Proof.** About this case, instead of (19), we consider
\[ (X_2 + Y_3) \equiv -a_1 a_3 (X_3 + Y_1) - a_2 a_3 (X_1 + Y_2) \pmod{q}. \]
Since \( -a_1 a_3 \equiv a_2 \pmod{q} \) and \( -a_2 a_3 \equiv a_1 \pmod{q} \), our relation to be considered is
\[ (X_2 + Y_3) \equiv a_2 (X_3 + Y_1) + a_1 (X_1 + Y_2) \pmod{q}. \]
Note that by (29), we see that \( a_1 > a_2 \), \( a_2 \). Thus we have \( x_2 = a_1 \) and \( y_3 = a_2 \). Now it is an easy deduction to obtain the assertion of Lemma.

Here we note the following facts:
We assume that \( f = 1 \) for the solution system \((x_i, y_i)\) \((1 \leq i \leq 3)\) which satisfies (15), and that there exists any other solution system. Then by taking a suitable permutation to \( i = 1, 2, 3 \) (if necessary), we may take a solution such as

\[(x_i - a, y_i + a), (x_2, y_2) \text{ and } (x_3, y_3)\]

Then we see easily that if \( f = 1 \) for this solution system, we have \( x_3 = y_2 \). Namely we have (29). And it is easy to see that in this case we have \( f = 1 \) for all other solution systems.

Now by collecting the discussions of §7 and 8, we conclude the assertion of Theorem 4.

References

[3] R. L. Graham, Covering the positive integers by disjoint sets of the form \( \{[n\alpha + \beta]: n = 1, 2 \ldots \} \), J. Comb. Th. (A), 15 (1973), 354–358.