Disjoint sequences generated by
the bracket function III

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1. Introduction. In this paper, we continue the investigation of the problem which was initiated in II of [3]. We refer to that paper by II in the following.

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ mean as usual. We denote by $\mathbb{N}$ the set $\mathbb{N} \cup \{0\}$. We put $S(q, a, b) = \{\lfloor (qn + b)/a \rfloor : n \in \mathbb{Z}\}$ where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. We take $q_i, a_i \in \mathbb{N}$ with $(q_i, a_i) = 1 \ (i = 1, 2)$. And we consider the following problem:

Determine all $(e_1, e_2) \in \mathbb{N}^2$ for which $e_1 + e_2$ sequences such as $S(q_i, a_i, b_i^{(j)}) \ (1 \leq j \leq e_1)$ and $S(q_2, a_2, b_2^{(k)}) \ (1 \leq k \leq e_2)$ can be made mutually disjoint by taking suitable $b$'s.

As shown in II, this problem is reduced to the following one:

Determine all $(v_1, v_2) \in \mathbb{N}^2$ for which $v_1 + v_2$ sequences such as

\[
S(q, a_1, b_i^{(j)}) \ (1 \leq j \leq v_1) \quad \text{and} \quad S(q, a_2, b_2^{(k)}) \ (1 \leq k \leq v_2)
\]

can be made disjoint by taking suitable $b$'s. (We put $q = (q_1, q_2)$.)

We call $(v_1, v_2)$ as a $d$-pair, if the sequences of (1) can be made disjoint. Here we introduce the following order in $\mathbb{N}^2$:

\[
(2) \quad (m, n) \leq (m', n') \quad \text{if and only if} \quad m \leq m' \quad \text{and} \quad n \leq n'.
\]

Then our problem is to determine all the maximal $d$-pairs with respect to the order (2), for a given $q, a_1, a_2 \in \mathbb{N}$ such as $(q, a_i) = (q, a_2) = 1$.

In II, we gave answers of this problem for some cases, but there remains several open cases. In this paper, we treat only the case

\[
(3) \quad (a_1, a_2) = 1.
\]

Since the case $(a_1, a_2) > 1$ satisfies similar but somewhat different phenomena, we treat the case in another place.

Then as shown in I of [3], $(1, 1)$ is a $d$-pair for the case (3) if and only if

\[
(4) \quad xa_1 + ya_2 = q \quad \text{holds with some} \quad (x, y) \in \mathbb{N}^2.
\]

Now we take $(x_0, y_0)$ from the solutions of (4), for which $1 \leq y_0 \leq a_1$. And we put $h = \lfloor x_0/a_2 \rfloor$. Then we have $(x_j, y_j) \ (0 \leq j \leq h)$ which are the solutions of (4) by putting
As shown in II, the pairs \((x_j, y_j)\) are the maximal d-pairs. We say a d-pair \((v_1, v_2)\) is non-trivial if there exist no \(j\) for which \((v_1, v_2) \leq (x_j, y_j)\).

And our problem is to determine all the non-trivial d-pairs. Here as shown in II, non-trivial d-pairs appear only in the case \(h \geq 1\). Thus we assume henceforth \(h \geq 1\).

For the case, we separate two cases.

(Case A) \(x_i a_i + y_i a_2 \geq a_i a_2\), (Case B) \(x_h a_i + y_0 a_2 < a_i a_2\).

Case A is rather simple, and we gave a complete answer in II. Namely under the assumption \(a_i, a_2 \geq 2\), we have
\[
\{(a_{2j}, a_i(h-j-1)) : 1 \leq j \leq h\}
\]
is the set of all the non-trivial maximal d-pairs. (If either \(a_i\) or \(a_2 = 1\), we have Case A and there exist no non-trivial d-pairs.)

On the other hand, the situation is more complicated in Case B. In this case, by interchanging \(a_i\) and \(a_2\) (if necessary), we assume \(a_2 > 2x_h\).

We put \(\tilde{y} = a_i - y_0\) and define the following numbers:
\[
a_i = e\tilde{y} + f \text{ with } e, f \in \mathbb{Z} \text{ and } 0 \leq f \leq \tilde{y} - 1.
\]
And we put \(\tilde{f} = \tilde{y} - f\). Note that \((q, a_i) = 1\) implies
\[
(a_i, \tilde{y}) = 1.
\]
Thus \(f = 0\) implies \(\tilde{y} = 1\).

We gave in II the unique maximal non-trivial d-pair \((2x_i, 2y_0)\) under the assumption \(e = h = 1\). Since d-pairs satisfy a kind of periodical nature with respect to \(h\), the case \(h = 1\) is essential for our problem. But in II, we could not give any decisive answer for the case \(e \geq 2\). In this paper we determine all the d-pairs for \(h = 1\), and propose a conjecture for the case \(h \geq 2\).

We separate Case B to two cases:

\((R)\) : regular case \(a_2 \geq x_h(e+1)\), \((S)\) : singular case \(a_2 < x_h(e+1)\).

**Theorem 1.** Notations being as above, we assume \(h = 1\), \(e \geq 2\). For \((R)\)-case, we have the following non-trivial d-pairs.

\((x_i e, a_i)\) and \((x_i (e+1), y_0 + f)\) (if \(f > 0\)).

And they are the all of the non-trivial maximal d-pairs.

For \((S)\)-case, we put \(w = a_2 - x_i e\). Then as shown in (23) of II, \(w > 0\). Here we define two numbers \(R_1, R_2\) by
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(8) \( R_1 = \min \{ wY - x_i F : F/Y \equiv f/y \} \),
     \( Y \in \mathbb{N}, F \in \bar{N} \)

(9) \( R_2 = \min \{ W\bar{y} - Xf : W/X \equiv w/x_i \} \).
     \( X, W \in \mathbb{N} \)

Then we see \( R_1, R_2 \in \mathbb{N} \). And we take \((Y_0, F_0)\) and \((X_0, W_0)\) the smallest pair which attains \( R_1 \) and \( R_2 \) respectively. And in case \( R_1, R_2 \equiv 2 \), we define the pair \((\bar{R}_1, \bar{R}_2)\) by

\[
(9) \quad \bar{R}_1 = (X_0 + Y_0)w - (F_0 + W_0)x_i, \quad \bar{R}_2 = (F_0 + W_0)\bar{y} - (X_0 + Y_0)f.
\]

Theorem 2. We assume \( h = 1, e \geq 2 \). For \((S)\)-case, we have \((a_2 - R_1, a_1)\), \((a_2, a_1 - R_2)\) (if \( f > 0 \)) and \((a_2 - \bar{R}_1, a_1 - \bar{R}_2)\) (if \( R_1, R_2 \equiv 2 \)) as the non-trivial d-pairs. And they are the all of maximal ones.

Finally we propose the following conjecture for \( h \geq 2 \).

Conjecture. For a given triple \( q, a_1, a_2 \) with \((q, a_1) = (q, a_2) = (a_1, a_2) = 1 \), the set of all the non-trivial maximal d-pairs is

\[
\{ (v_1 + a_2j, v_2 + a_1(h - j - 1)) : 0 \leq j \leq h - 1 \}
\]

where the pair \((v_1, v_2)\) is taken to be \((2x_h, 2y_0)\) in case \( e = 1 \). And for the case \( e \geq 2 \), they are taken from the pairs given in Theorems 1 and 2 respectively.

We think this conjecture to be very plausible. But we cannot get yet a complete proof of it. The aim of this paper is to prove Theorems 1 and 2.

2. Definitions and notations. We explain the notations used in II, and add some more to them. The assertions stated in this section are such given proofs in II. Thus we give no proof of them. Note that we assume \( h = 1 \) and \( e \geq 2 \).

(i) For \( f \in \mathbb{Z} \) and \( g \in \bar{N} \), we write \([f, f + g] = \{f, f + 1, \ldots, f + g\}\). This set is called a segment of \( \mathbb{Z} \) whose length is \( g + 1 \).

(ii) For \( m \in \mathbb{N} \), \( \sigma_m \) denotes the map from \( \mathbb{Z} \) to \( \mathbb{C} \) defined by \( \sigma_m(r) = \exp(2\pi ir/m) \) for \( r \in \mathbb{Z} \). We put \( C(m) = \sigma_m(\mathbb{Z}) \). For \( P \subseteq C(m) \), we say the coordinates of \( P \) is \( r \) if \( \sigma_m(r) = P \) and \( 0 \leq r \leq m - 1 \). For two points \( P, Q \subseteq C(m) \) whose coordinates are \( r, s \) respectively, we define their distance by \( \min(|r - s|, m - |r - s|) \). For \( S \subseteq C(m) \) and \( t \in \mathbb{Z} \), we define \( t(S) = \sigma_m(t \sigma_m^{-1}(S)) \). And \( |S| \) means the cardinality of \( S \). In this paper we use two maps \( \sigma_q \) and \( \sigma_i \) frequently. Thus we denote them simply by \( \sigma \) and \( \sigma_i \).
respectively.

(iii) The $\sigma_m$ image of a segment of $\mathbb{Z}$ is called a segment of $C(m)$. The length of it is defined as its cardinality. We call a segment of length $k$ simply a $k$-segment. In this paper we frequently treat many segments of the same length simultaneously. To distinguish them, we define the coordinates of the segment $\sigma_m([a, b])$ by the coordinates of $\sigma_m(b)$.

We take $t \in \mathbb{Z}$ such that
$$a_1 t \equiv a_2 \pmod{q}$$
and in the following consider it as fixed.

As shown in II, our problem is equivalent to the following one:

(*) Determine all the maximal pairs $(v_1, v_2)$ such that there exist disjoint $a_1$-segments $s_1, s_2, \ldots, s_v$ of $C(q)$ for which $C(q) - t\langle \bigcup_{i=1}^{v} s_i \rangle$ contains $v_2$ disjoint $a_2$-segments.

We consider (*) in the following. We put

$$\bigcup_{i=1}^{v} s_i = A$$
and $R(A) = C(q) - t\langle A \rangle$.

Since the property (*) is invariant by a translation of $A$ in $C(q)$, we assume that
$$s_i = \sigma([-a_1 + 1, 0])$$.

**Definition 1.** For $n \in \mathbb{Z}$, we take $r \in \mathbb{Z}$ so that $n \equiv -y_0 r \pmod{a_1}$ and $0 \leq r \leq a_i - 1$. We write $r = \omega(n)$. We put $L = \{\omega(n) : 1 \leq n \leq y_0\}$ and $S = \{\omega(n) : y_0 + 1 \leq n \leq a_1\}$. Furthermore we put $\rho(r) = n$ if $\omega(n) = r$ and $1 \leq n \leq a_1$.

We put

$$P(r) = x_0 r + [y_0 r / a_1]a_2$$
for $r \in \mathbb{Z}$.

$$\bar{b}(r) = \sigma([P(r), P(r + 1)])$$
and $b(r) = \bar{b}(r) - \sigma(P(r) \cup P(r + 1))$
where $0 \leq r \leq a_1 - 1$.

**Lemma 1.** The coordinates set of $t\langle s_i \rangle$ is $\{P(r) : 0 \leq r \leq a_1 - 1\}$. And the points are arranged on $C(q)$ by the order of $r$. And we have

$$|b(r)| = \begin{cases} 2a_2 + x_1 - 1 & \text{for } r \in L, \\ a_2 + x_i - 1 & \text{for } r \in S. \end{cases}$$

(iv) For $P \in b(r)$, we introduce new coordinates which is called the $r$-coordinates. Namely if the coordinates of $P$ is $P(r) + u$, we define the $r$-coordinates of $P$ by $u$. Then the range of $u$ is $[1, x_i + 2a_2 - 1]$ for $r \in L$.
and \([1, x_i + a_2 - 1]\) for \(r \in S\).

For an \(a_i\)-segment \(s\) with \(s \cap s_i = \emptyset\), we have
\[
|t(s) \cap b(r)| = \begin{cases} 
2 \text{ or } 1 & \text{for } r \in L, \\
1 \text{ or } 0 & \text{for } r \in S.
\end{cases}
\]

**Definition 2.** We take an \(a_i\)-segment of \(C(q)\) for which \(s \cap s_i = \emptyset\). Then \(s\) is said to be *standard* if \(|t(s) \cap b(r)| = 1\) for all \(0 \leq r \leq a_1 - 1\).

**Definition 3.** We put \(n(m) = my_0 + [x_0m/a_2]a_1 + 1\) and \(s(z, m) = \sigma([n(m) + z - a_i, n(m) + z - 1])\) where \(1 \leq z \leq a_i - 1, 1 \leq m \leq a_2 - 1\). We put \(\tau(m) = a_2(1 - (x_0m/a_2))\) where \((x) = x - \lfloor x \rfloor\). For \(0 \leq m \leq a_2 - 1\), we put \(u(z, m) = \sigma([n(m) + z, n(m) + z + a_i - 1])\). In that the ranges of \(z\) are given by \([0, y_0]\) for \(m\) with \(x_i \leq \tau(m) \leq a_2\) and \([0, y_0 + a_i]\) for \(m\) with \(1 \leq \tau(m) \leq x_i - 1\).

**Lemma 2.** (i) We have \(t(n(m) + z - 1) - P(w(z)) = z - (m) \pmod{q}\) for \(1 \leq z \leq a_1 - 1\) and \(1 \leq m \leq a_2 - 1\).

(ii) The set \(\{s(z, m) : 1 \leq z \leq a_i - 1, 1 \leq m \leq a_2 - 1\}\) is the set of all the non-standard \(a_i\)-segments. And the set \(\{u(z, m)\}\) of Definition 3 is the set of all the standard \(a_i\)-segments.

3. \((\beta)\)-problem. We first proceed the same way adopted in II to prove the maximality of \((2x_i, 2y_0)\) for the case \(e = h = 1\). Let \((v_1, v_2)\) be a non-trivial \(d\)-pair. Then we have
\[
x_i < v_i < x_i + a_2 \text{ and } y_0 < v_2 < y_0 + a_i.
\]
As noted above, we consider (*). Let \(A\) and \(R(A)\) be as in (10). We put \(A_i = A - s_i\). And we count the cardinality of the disjoint \(a_2\)-segments contained in \(R(A)\) by separating \(R(A)\) to each \(b(r) \cap R(A)\). We put
\[
N(r) = \# \text{ of disjoint } a_2\text{-segments contained in } R(A) \cap b(r).
\]
Then by (11), (12) and (13), we see that \(N(r) = 1\) or 0. We say \(r\) is *good* if \(N(r) = 1\), and otherwise say \(r\) is *overflowed*. Let
\[
d = \# \text{ of non-standard } a_i\text{-segments contained in } A_i.
\]
Then as shown in II, (13) implies \(v_i - d \leq x_i\). On the other hand by Lemma 2 (ii), we can take at most one disjoint \(a_i\)-segment from each \(m\). Thus we have \(v_i - x_i \leq d \leq a_2 - 1\).

**Lemma 3.** We take \(s(z, m)\). Then the \(r\)-coordinates of \(t(s(z, m)) \cap b(r)\) are given as follows:

(i) Assume \(r \in L\). Then
\[
\begin{cases}
\tau(m) + a_2 + x_i & \text{for } 1 \leq z \leq \rho(r) - 1, \\
\tau(m) \text{ and } \tau(m) + a_2 + x_i & \text{for } \rho(r) \leq z \leq \rho(r) + \gamma - 1, \\
\tau(m) & \text{for } \rho(r) + \gamma \leq z \leq a_1 - 1.
\end{cases}
\]

(ii) Assume \( r \in S \). Then
\[
\begin{cases}
\tau(m) + x_i & \text{for } 1 \leq z \leq \rho(r) - y_0 - 1, \\
\text{empty} & \text{for } \rho(r) - y_0 \leq z \leq \rho(r) - 1, \\
\tau(m) & \text{for } \rho(r) \leq z \leq a_1 - 1.
\end{cases}
\]

Proof. Easy calculations using Lemma 2 (i).

For \( 1 \leq z \leq a_1 - 1 \), we put
\[\beta_z = \# \text{ of } a_1\text{-segments } s(z, m) \in A_i,\]
\[\beta_0 - 1 = \# \text{ of standard } a_1\text{-segments in } A_i.\]
Then we have (by including \( s_i \))
\[(14) \quad \beta_0 + \beta_1 + \ldots + \beta_{a_1-1} = v_1.\]
We consider a necessary condition for \( N(r) = 1 \). First take \( r \in S \). Then we see by (11) that \( N(r) = 1 \) implies the inequality
\[|t\langle A\rangle \cap b(r)| \leq x_i - 1.\]
Thus by Lemma 3 (ii), we obtain
\[\beta_0 - 1 + \beta_{\rho(r)} + \ldots + \beta_{a_1-1} + \beta_1 + \ldots + \beta_{\rho(r)-y_0-1} \leq x_i - 1.\]
On the other hand for \( r \in L \), we consider the upper limit of the cardinality of \( s(z, m) \in A_i \) for which
\[|t\langle s(z, m)\rangle \cap b(r)| = 2.\]
If the cardinality is \( \geq 1 \), we take \( s(z, m) \) as one of them. And let \( P \) and \( Q \) be the two points of \( t\langle s(z, m)\rangle \cap b(r) \). Since the distance of \( P \) and \( Q \) is \( a_2 + x_i \), we see by (11) that \( N(r) = 1 \) implies that the \( a_2\)-segment must be taken between \( P \) and \( Q \). And if there is another \( s(z', m') \in A_i \) with (16), there is exactly one point of \( t\langle s(z', m')\rangle \) between \( P \) and \( Q \). Hence using Lemma 3 (i), we see \( N(r) = 1 \) implies
\[\beta_{\rho(r)} + \ldots + \beta_{\rho(r)+1} \leq x_i.\]

Definition 4. We take \( \beta_r \) \( (0 \leq r \leq a_1 - 1) \) from \( \tilde{N} \) which satisfy (14). We attach \( \beta_r \) to \( \sigma_1(r) \) of \( C(a_i) \). Here we say that a \((\beta)\)-distribution whose sum = \( v_1 \) is given. For a \( \tilde{y}\)-segment \( y \) of \( C(a_i) \), we put
\[\tilde{V}(y) = \text{sum of } \beta_z \text{ with } \sigma_1(z) \in y.\]
We denote \( y(n) \) the \( \tilde{y}\)-segment whose coordinates is \( n \).

By using these terminology, (15) and (17) become
\[\tilde{V}(y) \leq x_i \text{ where } \begin{cases}
y = y(\rho(r) - y_0 - 1) & \text{for } r \in S, \\
y = y(\rho(r) + \gamma - 1) & \text{for } r \in L.
\end{cases}\]
Note that if we make \( r \) run through \( 0 \leq r \leq a_i - 1 \) in (18), all the \( y \)-segments on \( C(a_i) \) appears exactly once. Thus for a given \( (\beta) \)-distribution, we put
\[
\tilde{v}_2 = \# \text{ of } y \text{-segments } y \text{ of } C(a_i) \text{ for which } \tilde{V}(y) \leq x_1.
\]
And we consider the following problem:

(\( \beta \)) Determine the maximum value of \( \tilde{v}_2 \), for a given quadruple \((a_i, \tilde{y}, x_i, v_i)\).

Since (18) is a necessary condition for \( N(r) = 1 \), we obtain only an upper bound for maximal non-trivial \( d \)-pairs by solving (\( \beta \)). But in some cases, this upper bounds are attained. And moreover we think (\( \beta \)) itself is an interesting extremal problem in combinatorial number theory.

4. A solution of (\( \beta \))-problem. In this section we give a solution of (\( \beta \))-problem. To state the result, we define the following terminology.

**Definition 5.** For a given \( (\beta) \)-distribution on \( C(a_i) \), we say a \( y \)-segment \( y \) which satisfies (18) is \( (\beta) \)-good. And otherwise, we say \( y \) is \( (\beta) \)-overflowed. A point of \( C(a_i) \) is called a place. We call the place \( \sigma_1(z) \) for which positive \( \beta_2 \) is attached as a positive place.

**Definition 6.** Take \( a, b, c, d \in \mathbb{N} \) and the set \( U \subset \mathbb{Z} \) for which \( |U| = a \). We arrange the numbers of \( U \) as follows.
\[
(19) \quad u_0, u_1, u_2, \ldots, u_{a-1}.
\]
And we take a monotonously non-decreasing sequence of integers such as
\[
(20) \quad 0 = c_0 \leq c_1 \leq c_2 \leq \ldots \leq c_{a-1} \leq c.
\]
We say \( U \) is an \( F(a, b, c) \)-sequence (mod \( d \)) (or simply an \( F \)-sequence), if by suitably chosen two sequences (19) and (20), the relation
\[
u_i \equiv u_0 + ib - c_i \pmod{d} \quad \text{for } 1 \leq i \leq a - 1
\]
holds.

**Remark.** \( F \)-sequences have already appeared in III of [2]. There we considered the problem to list up all the possible choices of the residue set \( \{b_{1(j)} : 1 \leq j \leq v_1\} \) and \( \{b_{2(k)} : 1 \leq k \leq v_2\} \) of (1), under the conditions \((a_1, a_2) = 1\) and \( a_1v_1 + a_2v_2 = q \). The main result obtained there is that (1) makes an eventually covering family (ECF) if and only if the residue set \( \{b_2\} \) is an \( F(v_2, -t, v_1 - 1) \)-sequence (mod \( q \)). From this \( \{b_2\} \), the set \( \{b_{1(j)}\} \) is uniquely determined, and it makes also an \( F \)-sequence.

By (7), we take \( \tilde{Y} \in \mathbb{Z} \) such that \( \tilde{Y}_y \equiv 1 \pmod{a_i} \), and in the following
consider it as fixed.

**Definition 7.** We take \(\sigma_1([[(i-1)y+1, iy]])\) and attach \(i\) to this \(\bar{y}\)-segment. We arrange the \(\bar{y}\)-segment of \(C(a_i)\) by the order of \(i\). By (7), all \(\bar{y}\)-segment appears and the \(i\)-th segment is same with the \((i+a_i)\)-th one. We call this cycle the **cycle of \(\bar{y}\)-segments** of \(C(a_i)\).

**Lemma 4.** Let \(a_i, \bar{y}, e, f\) be as in (6), and \((\beta)\) be a \((\beta)\)-distribution on \(C(a_i)\) whose sum = \(e+1\). Assume that the distances of any two adjacent positive places are \(\leq y\). Let \(S\) be the set of numbers attached to \(\bar{y}\)-segment \(y\) for which \(\bar{V}(y) = 2\). Then \(S\) is an \(F(\bar{f}, \bar{Y}, e)\)-sequence \((\text{mod } a_i)\).

**Proof.** First assume \(f=0\). Then (7) implies \(\bar{y}=1\) and \(a_i=e\). By the assumption of Lemma, all places of \(C(a_i)\) are positive. And exactly one place has two points. Since \(\bar{f}=1\), we have the assertion of Lemma.

Thus we assume \(f>0\). Since the property considered is invariant by a translation on \(C(a_i)\), we assume that \(\sigma_1(0)\) is a positive place. And note that \(f>0\) implies, by the assumption of Lemma, that there must be exactly \(e+1\) positive places. We arrange the coordinates of them as follows:

\[
a_i = c_0 > c_1 > c_2 > \ldots > c_e.
\]

(Namely we consider here the coordinates of \(\sigma_1(0)\) as \(a_i\).)

We put \(d_i = (e+1-i)\bar{y} - c_i\). Then \(d_0 = \bar{f}\). And by the assumption of Lemma, we have

\[
\bar{f} = d_0 \geq d_1 \geq d_2 \geq \ldots \geq d_e.
\]

We put \(d_{e+1} = 0\) and consider the set 

\[I = \{i: d_i > d_{i+1}, 0 \leq i \leq e\}.\]

Let \(T\) be the set of \(\bar{y}\)-segments which contain two positive places. Then we see that the coordinates set of \(T\) is

\[
(\bar{21}) \quad [0, d_0 - d_i - 1] \cup_{i \in I} [(e+1-i)\bar{y} - d_i, (e+1-i)\bar{y} - d_{i+1} - 1]
\]

where the first set appears in case \(0 \in I\). Thus we see that \(|T| = \bar{f}\). And we arrange the coordinates of \((\bar{21})\), taking from \(i=0, 1, 2, \ldots\).

\(0, 1, \ldots, d_0 - d_i - 1, e\bar{y} - d_i, \ldots, e\bar{y} - d_2 - 1, \ldots, \bar{y} - 1\).

Their general form is \((e+1-i)\bar{y} - d_i + g, 0 \leq g \leq d_i - d_{i+1} - 1\). To obtain \(S\), we multiply these values by \(\bar{Y}\). Note that \(\bar{Y}\bar{f} = e+1 \pmod{a_i}\) and \(\bar{Y}((e+1-i)\bar{y} - d_i + g) = (e+1-i) - (d_i - g)\bar{Y}\). Thus by (6), we have \(S = \{(\bar{f} - d_i + g)\bar{Y} - i: i \in I\}\). Since \(0 \leq i \leq e\), we have the conclusion of Lemma.

**Lemma 5.** For \(1 \leq m \leq \bar{f} - 1\), we take \(s \in [1, \bar{f} - 1]\) for which \(m \equiv -ys \pmod{\bar{f}}\). Then the coordinates of \(\sigma_1(\bar{Y}m)\) is given by
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\[ -s + \left\lceil \left( m + \frac{\bar{y} s}{f} \right) \right\rceil (e + 1). \]

And the value of (22) increases monotonously as \( s \) increases.

Proof. By (7), we may consider the difference of the \( \bar{y} \) multiple of (22) and \( m \) (mod \( a_i \)). Thus by (6) we have

\[ m + \bar{y} s - \left\lceil \left( m + \frac{\bar{y} s}{f} \right) \right\rceil (e + 1) \bar{y} \equiv m + \bar{y} s - \bar{f} \left\lceil \left( m + \frac{\bar{y} s}{f} \right) \right\rceil \pmod{a_i}. \]

Since \( \bar{f} \equiv m + \bar{y} s \), we see the difference \( \equiv 0 \pmod{a_i} \). Note that the values of (22) takes 0 for \( s = 0 \), and \( a_i \) for \( s = f \). Here \( \bar{y} \geq f \) implies the last assertion of Lemma. Thus (22) gives the coordinates of \( \sigma_i(\bar{y} m) \).

Lemma 6. For a \((\beta)\)-distribution on \( C(a_i) \) whose sum is

\[ (e + 1)x_i + 1, \]

we have \( \bar{v}_2 \leq \bar{y}_0 \). And in the case the equality holds, the set of numbers attached to the overflowed segments makes an \( F(\bar{y}, \bar{Y}, 0) \)-sequence (mod \( a_i \)).

Proof. If \( f = 0 \), then \( \bar{y} = 1 \) and \( a_i = e \). Thus (23) implies that at least one positive place must be overflowed. And if we attach all \( (e + 1)x_i + 1 \) points on one place, we have \( \bar{v}_2 = a_i - 1 = y_0 \).

Henceforth we assume \( f > 0 \), and the cardinality of the overflowed \( \bar{y} \)-segments is \( \leq \bar{y} \). We consider the cycle of \( \bar{y} \)-segments. Note that there exist no \( e + 1 \) serial good \( \bar{y} \)-segments in the cycle. Since if not the case, the union of these \( \bar{y} \)-segments covers \( C(a_i) \), and it contradicts with (18) and (23). On the other hand by our assumption, there are at least \( y_0 \) good \( \bar{y} \)-segments. Thus by (6), there happens \( e \) serial good \( \bar{y} \)-segments in the cycle at least \( f \) times. We take one of them. Since the property of Lemma does not change by translating the numbers on the cycle, we assume that the serial numbers are \( [1, e] \). Then the union of the places contained in these segments is \( \sigma_i([1, e\bar{y}]) \).

Now by the same reasoning used above, we see that the \( \bar{y} \)-segments whose places contain \( \sigma_i([e\bar{y} + 1, a_i]) \) must be overflowed. The coordinates of these \( \bar{y} \)-segments are \( [0, \bar{f}] \). By using Lemma 5, we arrange these \( \bar{y} \)-segments by the order of their numbers. Then we have

\[ 0, e + 1, -s + \left\lceil \frac{\bar{y} s}{f} \right\rceil + 1, (e + 1) \quad (1 \leq s \leq \bar{f} - 1). \]

If \( f = 1 \), then \( \bar{y} = \bar{f} + 1 \). Hence we have obtained the desired overflowed \( \bar{y} \)-segments. And since (24) is \( \bar{Y} \) multiples (mod \( a_i \)) of \( [0, \bar{f}] \), we have the last assertion of Lemma.

Thus we assume \( f \geq 2 \). We put \( \bar{y} = kf + m \) with \( 0 \leq m \leq \bar{f} - 1 \). First assume \( m \geq 1 \). We consider the \( \sigma_i \)-image of (24). We see by easy calculations that the distances of adjacent two points on \( C(a_i) \) are composed of
three kinds. Namely they are $e+1$, $(k+1)(e+1) - 1$ which appears $m-1$ times and $k(e+1) - 1$ which appears $\bar{f} - m + 1$ times. By the same reasoning used above, there does not exist $e+1$ serial good $\bar{y}$-segments in each interval. Thus there must be $k$ and $k-1$ overflowed $\bar{y}$-segments for the latter two kinds of intervals. Thus the least cardinality of the overflowed $\bar{y}$-segments is

$$k(m-1) + (\bar{f} - m + 1)(k-1) + \bar{f} + 1 = \bar{y}.$$  

In case $m = 0$, (7) implies $\bar{f} = 1$. And (24) has only the former two numbers. For the remained serial numbers $[e+2, a_1-1]$, the same reasoning used above works and we have the desired result.

For the latter assertion of Lemma for $f \geq 2$, we note that from $e$ serial good $\bar{y}$-segments, we get $\bar{f} + 1$ serial coordinates of the overflowed $\bar{y}$-segments. And there happens at least $f$ times. Since $\bar{y} = f + \bar{f}$, we conclude that in order that the cardinality of the union of these overflowed $\bar{y}$-segments is just $\bar{y}$, the only possible case is that the coordinates varies one by one for each $e$ serial good $\bar{y}$-segments, and the union of the coordinates makes a $\bar{y}$-segment on $C(a_i)$. Thus we see the number set of these $\bar{y}$-segments makes an $F(\bar{y}, \bar{Y}, 0)$-sequence $(\bmod a_i)$.

Lemma 6 implies that, in considering non-trivial d-pairs, we may assume $v_1 \leq x_i(e+1)$. On the other hand we showed in II that $v_1 \leq a_2$ and $v_2 \leq a_1$. And also it is shown in II that

$$(\min(a_2, x_i(e+1)), a_0 - \bar{f})$$

for $f > 0$ and $(x_i e, a_i)$ are non-trivial d-pairs. Thus in search for non-trivial maximal d-pairs, we may restrict the range of $(v_1, v_2)$ to

$$x_i e \leq v_1 \leq \min(x_i(e+1), a_2) \text{ and } a_1 - \bar{f} \leq v_2 \leq a_1.$$  

5. ($\tau$)-problem. As noted repeatedly, the solution of ($\beta$)-problem gives only an upper bound for non-trivial maximal d-pairs. To obtain the equivalent problem, we must study the $r$-coordinates of $b(r) \cap t\langle A \rangle$ instead of $|b(r) \cap t\langle A \rangle|$.

The behaviour of $t\langle s \rangle$ in $b(r)$ differs fairly by whether $s$ is a standard one or not. Thus in this section we strengthen the difference by using the notation such as $u(\bar{z}, \bar{m})$ and $s(z, m)$.

**Lemma 7.** Take a standard segment $u(\bar{z}, \bar{m})$. Then we have

i) In case $1 \leq \tau(\bar{m}) \leq x_i - 1$, the range of $\bar{z}$ is $[0, y_0 + a_1]$. And the $r$-coordinates of $t\langle u(\bar{z}, \bar{m}) \rangle \cap b(r)$ are
Disjoint sequences generated by the bracket function III

\[
\begin{cases}
\tau(\hat{m}) & \text{for } 0 \leq z \leq \rho(r) - 1, \\
\tau(\hat{m}) + a_2 & \text{for } \rho(r) \leq z \leq \rho(r) + a_1 - 1, \\
\tau(\hat{m}) + 2a_2 & \text{for } \rho(r) + a_1 \leq z \leq y_0 + a_1.
\end{cases}
\]

(ii) In case \( \tau(\hat{m}) \geq x_1 \), the range of \( z \) is \([0, y_0]\). And the r-coordinates of \( t(u(\hat{z}, \hat{m})) \cap b(r) \) is \( \tau(\hat{m}) + a_2 \) for \( \rho(r) \leq z \leq y_0 \), and \( \tau(\hat{m}) \) for \( 0 \leq z \leq \rho(r) - 1 \).

Proof. Easy calculations using Lemma 2 (i).

Let \( A \) (or \( A_i \)) be a set of union of \( a_i \)-segments which satisfies (*) with (25). Assume \( A_i \) contains \( u(\hat{z}, \hat{m}) \) with \( x_i \leq \tau(\hat{m}) \). Then by Lemma 7 (ii) and (11), we see that all \( r \in S \) is overflowed. Thus we have only a trivial d-pair, and for \( u(\hat{z}, \hat{m}) \) in \( A_i \) we have

\[ 1 \leq \tau(\hat{m}) \leq x_i - 1. \]

And also by (i) of Lemma 7 and \( t_a = a_2 \) (mod q), we see that we can take at most one \( u(\hat{z}, \hat{m}) \) from each \( \hat{m} \).

We arrange the non-standard \( a_i \)-segments \( s(z, m) \in A_i \) by the order of \( z \) as follows:

\[ 1 \leq z_1 \leq z_2 \leq \ldots \leq z_d \leq a_i - 1. \]

And we put \( m_i \) for which \( s(z_i, m_i) \in A_i \). We take a fixed \( r \) and study the distribution of r-coordinates set of \( b(r) \cap t(A) \).

(a) First take \( r \in S \). Note that \( y_0 + 1 \leq \rho(r) \leq a_1 \).

We consider the following two segments on \( C(a_i) \).

\[ W_i = \sigma([1, \rho(r) - y_0 - 1]), \ W_{ii} = \sigma([\rho(r), a_i - 1]). \]

(In case \( \rho(r) = y_0 + 1, \ W_i = \emptyset \) and \( W_{ii} = \emptyset \) for \( \rho(r) = a_i \).) We denote by \( \tau_i \) in symbolic those \( \tau(m_i) \) for which \( \sigma_i(z_i) \in W_i \) and use \( \tau_{ii} \) for \( \tau(m_i) \) with \( \sigma_i(z_i) \in W_{ii} \).

And for the standard segment \( u(\hat{z}, \hat{m}) \in A_i \), we separate \( \hat{z} \) to two parts. We use also a similar symbolic notation, and put

\[ \hat{\tau}_i \text{ for } \hat{\tau}(\hat{m}) \text{ with } \hat{z} \in [0, \rho(r) - 1], \]
\[ \hat{\tau}_{ii} \text{ for } \hat{\tau}(\hat{m}) \text{ with } \hat{z} \in [\rho(r), a_i + y_0]. \]

Then by Lemmas 3 and 7, we have the set of r-coordinates of \( b(r) \cap t(A) \) as follows:

(26) \[ 0, \hat{\tau}_i, \tau_{ii}, \tau_i + x_1, \hat{\tau}_{ii} + a_2, a_2 + x_1. \]

Hence if \( N(r) = 1 \), the r-coordinates of the \( a_2 \)-segment taken from \( R(A) \) must lie between \( \max(0, \hat{\tau}_i, \tau_{ii}) \) and \( \min(\tau_i + x_1, \hat{\tau}_{ii} + a_2, a_2 + x_1) \). Thus the criterion for \( N(r) = 1 \) with respect to \( r \in S \) is

(27) \[ \min(\tau_i + x_1, \hat{\tau}_{ii} + a_2, a_2 + x_1) - \max(0, \hat{\tau}_i, \tau_{ii}) \geq a_2 + 1. \]

We put \( u_i \) the cardinality of \( r \in S \) such as (27) holds.
(b) For \( r \in L \), we consider the following three segments on \( C(a_i) \):

\[
W_1 = \sigma_1([1, \rho(r) - 1]), \quad W_2 = \sigma_1([\rho(r), \rho(r) + \tilde{y} - 1]),
\]

\[
W_3 = \sigma_1([\rho(r) + \tilde{y}, a_i - 1]).
\]

We define \( \tau_i, \tau_{i1}, \tau_{i2} \) as in the similar way with (a). Namely

\[
\tau_i = \{ \tau(m_i) : z_i \in W_i \} \text{ etc.}
\]

For \( u(\hat{z}, \hat{m}) \in A_i \), we separate \( \hat{z} \) to three parts and denote

\[
\hat{\tau}_i \text{ for } \hat{z} \in [0, \rho(r) - 1], \quad \hat{\tau}_{i1} \text{ for } \hat{z} \in [\rho(r), \rho(r) + a_i - 1],
\]

\[
\hat{\tau}_{i2} \text{ for } \hat{z} \in [\rho(r) + a_i, \infty).
\]

Then by Lemmas 3 and 7, we have the \( r \)-coordinates of \( b(r) \cap t(A) \):

\[
(28) \quad 0, \hat{\tau}_i, \tau_{i1}, \tau_{i2} + a_2, \tau_{i1} + a_2 + x_1, \tau_i + a_2 + x_i + a_2, 2a_2 + x_i.
\]

From (28) we see that if \( N(r) = 1 \), it is necessary that

\[
(29) \quad \min(\tau_{i1} + a_2 + x_i) = \max(\tau_{i1} + a_2).
\]

We put \( u_2 \) the cardinality of \( r \in L \) such as (29) is satisfied. And we consider the maximum value of \( u_1 + u_2 \). After that we show this maximum value is attained as \( v_2 \).

Now as in § 4, we interpret our problem to a nature of points distribution on \( C(a_i) \).

Take a \( \tilde{y} \)-segment \( y \) of \( C(a_i) \) for which \( y \nsubseteq \sigma_1(0) \). And we define

\[
(30) \quad V(y) = \{ \text{the variation of } \tau(m_i) \text{ for which } \sigma(z_i) \in y \} + 1.
\]

(If there exist no such \( z_i \), we put \( V(y) = 1 \).)

By (30), (29) is interpreted as \( V(y) \leq x_i \) for \( y = y(\rho(r) + \tilde{y} - 1) \).

Here we note the following three facts.

(L-1) For a given \( d \), to make the value \( u_2 \) maximal, we may assume that \( \tau \)'s are arranged monotonously. Namely

\[
(31) \quad \tau(m_i) < \tau(m_2) < \cdots < \tau(m_d) \text{ or } \tau(m_1) > \tau(m_2) > \cdots > \tau(m_d).
\]

(L-2) The value \( u_2 \) is invariant by translating the values of \( \tau \)'s simultaneously, or by reflecting in (31).

(L-3) If \( | \tau(m_i) - \tau(m_d) | \) is made smaller, we obtain larger (at least not smaller) value of \( u_2 \).

Now we consider \( u_1 \). We note the following three facts.

(S-1) For \( u(\hat{z}, \hat{m}) \in A_i \), we have \( 0 \leq \hat{z} \leq y_0 \) or \( a_i \leq \hat{z} \). Thus \( \hat{\tau}_i \) and \( \hat{\tau}_{i1} \) in (26) does not alter for all \( r \in S \). Since if we assume that \( u = u(\hat{z}, \hat{m}) \in A_i \) for which \( y_0 + 1 \leq \hat{z} \leq a_i - 1 \), \( \hat{\tau}_{i1} + a_2 \) of (28) appears for all \( r \in L \). And \( \tau(m) \leq x_1 - 1 \) implies that it lies nearly at the centre of \( b(r) \). This fact makes difficult to have \( N(r) = 1 \) for \( r \in L \). And we have only a \( d \)-pair which does not satisfy (25). To ascertain that we may proceed as follows.

(i) Let list up the \( a_i \)-segments \( s \) such that \( R((u, s, s_j)) \) contains \( a_i - \hat{f} \) (or more) disjoint \( a_2 \)-segments. Then we see that the coordinates of \( s \)
must lie in a segments near to \( \sigma(n(m)) \) with \( 1 \leq \tau(m) \leq x_i - 1 \), and \( m \equiv m \).

(ii) And it is easily seen that we can take at most two disjoint ai-segments from each \( m \). Thus we have \( v_i \leq 2x_i - 2 \).

(S-2) To make \( u_i \) larger, we may assume by L-3 that \( \hat{r} \)'s are taken so that \( \hat{r}_{ab} + a_2 = [a_2 + x_i - c_1, a_2 + x_i - 1] \) and \( \hat{r}_i = [1, c_2] \) for which \( c_i + c_2 = \beta_0 - 1 \). And take \( \tau \)'s so that \( \tau_{ab} > \hat{r}_i \) and \( \tau_i + x_i < \hat{r}_{ab} + a_2 \).

(S-3) To make \( u_i \) larger, we must take \( \tau_1 \) of (26) large, and \( \tau_2 \) small. And arrange them monotonously.

Summing up the above discussions, and L-2, we assume henceforth that the distribution of \( \tau \) is as follows

\[
\hat{r} = [1, \beta_0 - 1] < \tau(m_d) < \tau(m_{d-1}) < \ldots < \tau(m_1) \leq a_2 - 1.
\]

Now we consider \( s_i \) as a standard ai-segment, and attach all standard segment of \( A \) to \( \sigma_i(0) \).

We define \( V(y) \) for \( y \ni \sigma_i(0) \) as follows:

We separate \( y \) into two parts.

\[
y_1 = y \cap \sigma_i([1, \bar{y} - 1]) \quad \text{and} \quad y_2 = y \cap \sigma_i([a_i - \bar{y} + 1, a_i]).
\]

If \( y_1 \) contains a positive place, we put

\[
V(y) = \max_{\sigma_i(z_s) \in y_2} (\tau(m_s), \beta_0 - 1) - \min_{\sigma_i(z_t) \in y_1} \tau(m_t) + a_2 + 1.
\]

Otherwise we put

\[
V(y) = \max_{\sigma_i(z_s) \in y_2} (\tau(m_s), \beta_0 - 1) + 1.
\]

By using these definitions, (27) is interpreted as

\[
V(y) \leq x_i \quad \text{for} \quad y = y(\rho(r) - \rho_0 - 1).
\]

To clear up our situation, we formulate our problem (\( \tau \)):

We take two sequences composed of \( v_i \) integers such that

\[
1 \leq z_i \leq z_2 \leq \ldots \leq z_{v_i} = a_i,
\]

\[
a_2 - 1 \geq \tau_1 \geq \tau_2 > \ldots > \tau_{v_i} = 0.
\]

And attach \( z_i(1 \leq r \leq v_i) \) to \( \sigma_i(r) \), and make the pair \((z_i, \tau_i)\). We say in this situation a \((\tau)\)-distribution is given. And we define \( V(y) \) for a \( \bar{y} \)-segment of \( C(a_i) \) by (30), (33) and (34), where

\[
\beta_0 = \# \text{ of } i \text{ with } z_i = a_i.
\]

And we put

\[
v_2 = \# \text{ of } \bar{y} \text{-segments } y \text{ which satisfies } V(y) \leq x_i.
\]

(\( \tau \)) For a given quadruple \((a_i, a_2, x_i, \bar{y})\), we consider all the \((\tau)\)-distribution of (35). (Note that by its definition \( a_2 \geq v_i \).) Then determine all the maximal pairs \((v_1, v_2) \in \mathbb{N}^2\).

We show that \((\tau)\)-problem is equivalent to our original problem. It is
sufficient to ascertain the existence of a \(d\)-pair for a solution \((v_1, v_2)\) of \((\tau)\). We put \(\chi\) a map from \([1, a_2 - 1]\) to \([1, a_2 - 1]\) defined by \(\chi(s) = t\) if \(\tau(t) = s\). And we take \(A\) from (35) as follows:

\[
A = \bigcup_{z_i \in a_i} s(z_i, \chi(\tau_i)) \bigcup_{z_i \in a_i} u(0, \chi(\tau_i)) \bigcup s_i
\]

To see that \(A\) has the desired property (*), we note that

(i) By taking \(\hat{z} = 0\), \(\tau_{\Pi}\) of (28) does not appear for all \(r \in L\). And since \(\tau_{\Pi} < \tau_{\Pi} < \tau_i\), we see that (29) is a sufficient condition for \(N(r) = 1\).

(ii) By (26) and (28), \(t(A)\) is a disjoint set. Thus \(A\) is also a disjoint union.

Thus in the following \((\tau)\)-problem plays a central role.

**Definition 8.** We say a \(\tilde{y}\)-segment \(y\) is good in case \(V(y) \leq x_i\) holds. Otherwise we say it overflowed. Note that \((\tau)\)-distribution contains a \((\beta)\)-distribution. And in general \(V(y) \geq \tilde{V}(y)\). We call \(a_2 - v_i\) the total leap of the \((\tau)\)-distribution.

6. \(R_1, R_2\) and \((\tilde{R}_1, \tilde{R}_2)\). As noted in §4, the \(d\)-pairs given in Theorem 1 are already appeared in II. But the pairs mentioned in Theorem 2 are the new ones. Preliminary to treat them, we study in this section the properties of \(R\)'s, \((\tilde{R}_1, \tilde{R}_2)\) and of \((X_0, W_0)\) and \((Y_0, F_0)\). We assume in this section \(f > 0\).

We put \(R = a_1 a_2 - (a_1 x_1 + a_2 y_0)\). Since we are treating Case B, \(R\) must be in \(N\). And as easily seen we have

\[
R = \tilde{y}w - x_1 f.
\]

We consider \((S)\)-case. Namely \(x_i > w\). Thus in (8), we have

\[
F/Y \leq f/\tilde{y} < w/x_i \leq W/X.
\]

By taking \((Y, F) = (1, 0)\) and \((X, W) = (1, 1)\) in (8), we have

\[
R_i \leq w \text{ and } R_2 \leq f.
\]

Note that (37) implies \(R_1, R_2 \in N\). Thus we take the minimum pair \((Y_0, F_0)\) and \((X_0, W_0)\) which attain \(R_1\) and \(R_2\) respectively.

Henceforth we interpret our problem to the nature of lattice points in \(Z^2\). We denote by \(|\triangle ABC|\) the area of \(\triangle ABC\), and by \(|PQ|\) the distance of \(P\) and \(Q\). \(P + Q\) means the addition as vectors in \(Z^2\).

We put \(A = (x_i, w), B = (\tilde{y}, f), P_i = (Y_0, F_0)\) and \(P_2 = (X_0, W_0)\).

We denote as

\[
D_1 = \{ (t, u) \in Z^2 : t, u \in N \text{ and } u/t \geq w/x_i \},
\]

\[
D_2 = \{ (t, u) \in Z^2 : t, u \in N \text{ and } w/x_i > u/t > f/\tilde{y} \}.
\]
Disjoint sequences generated by the bracket function III

\[ D_3 = \{(t, u) \in \mathbb{Z}^2 : t \in \mathbb{N}, u \in \mathbb{N} \text{ and } f/\bar{y} \geq u/t\}. \]

Then the condition imposed to \( R_1 \) (resp. \( R_2 \)) is that \(|OAP_1|\) (resp. \(|OBP_2|\)) takes the minimum value where \( P_1 \) (resp. \( P_2 \)) runs through \( D_3 \) (resp. \( D_1 \)).

We consider the set \( T = \{(S_1, S_2) \} \subset \mathbb{N}^2 \) which satisfies

(i) \( S_1 < R_1 \) and \( S_2 < R_2 \),

(ii) With suitably taken \((g, g_i) \in \mathbb{N}^2\), we have

\[ S_1 = gw - g_1x_1 \text{ and } S_2 = g_2y - gf. \]

As easily seen, \( T = \emptyset \) if either \( R_1 \) or \( R_2 = 1 \).

**Lemma 8.** If \( T \neq \emptyset \), then \( T \) has the minimum pair.

**Proof.** We prove that \( T \) is linearly ordered by (2). Let take two pairs \((S_1, S_2)\) and \((S'_1, S'_2)\) in \( T \), which are got by \( G = (g, g_i) \) and \( G' = (g', g'_i) \) in (39) respectively. Then by the minimality of \( R_1 \) and \( R_2 \), we see \( G \) and \( G' \) are in \( D_2 \). Assume that no order relation holds between \((S_1, S_2)\) and \((S'_1, S'_2)\). Without loss of generality, we assume \( S_1 < S'_1 \) and \( S_2 > S'_2 \). Note that (38) implies \( g \neq g'_i \), and either \( G < G' \) or \( G > G' \) holds.

First assume \( G < G' \). We take \( \bar{G} = G' - G \). Note that \( S_2 > S'_2 \) implies \( \text{grad. of } \bar{O}\bar{G} < f/\bar{y} \). Thus \( \bar{G} \in D_1 \). We consider \(|OAG|\). Then by \( G \in D_2 \), we have

\[ f/\bar{y} < \text{grad. of } \bar{G} \bar{G}' < w/x_1. \]

Thus \(|OAG| < |OAG'| \). Since \( 2|OAG'| < R_1 \) and \( \bar{G} \in D_3 \), we reach to a contradiction. In case \( G' < G \), we reach a contradiction with the minimality of \( R_2 \).

We put \( \bar{P} = P_1 + P_2 \), and define \((\bar{R}_1, \bar{R}_2)\) by (9).

**Lemma 9.** Assume that \( T \neq \emptyset \). Then \((\bar{R}_1, \bar{R}_2)\) is the minimum element of \( T \).

**Proof.** First we show that the parallelogram \( OP_1P_2 \) contains no lattice points inside or on the sides except for its vertices. We assume \( X_0 \geq Y_0 \). We use the following Figure 1.

First note that if \((0, 1)\) is in \( OL_a \), then \((1, 1)\) lies as an inner point of \( \triangle OL_e P_2 \). And it contradicts (38). Thus by the definition of \( P_2 \) and \( X_0 \), \( \triangle OL_e P_2 \) contains no lattice points except for \( O \) and \( P_2 \). Thus \( \triangle OL_e P_2 \) contains no lattice points except for \( O \) and \( P_2 \). Since if it contains a lattice point \( Q_1 \), then we take lattice point \( Q_2 \) in \( \triangle OL_e P_2 \) so that \( OQ_1P_2Q_2 \) makes a parallelogram. Then \( Q_2 \in \triangle OL_e P_2 \), and we see \( Q_2 = O \) or \( P_2 \).

Now by the definition of \( P_1 \) and the assumption \( X_0 \geq Y_0 \), we see that
the lattice points contained in \( \triangle OP_1P_2 \) are its vertices only. And if there exists a lattice point \( Q_3 \) in \( \triangle P_1P_2\tilde{P} \), we take \( Q_4 \) so that \( P_1Q_2P_2Q_4 \) makes a parallelogram. Then \( Q_4 \in \triangle OP_1P_2 \). Thus we have the aimed conclusion.

In case that \( Y_0 > X_0 \), we take \( L_3 \) in the \( u \)-axis so that \( PIL_3 \parallel l_i \). Noting that the grad of \( l_i < 1 \) and \( Y_0 \geq 2 \), we see that \( (1, 0) \notin \triangle OL_3P_1 \). Hence we obtain the same conclusion by a similar reasoning used above. Thus we have

\[
(40) \quad Y_0W_0 - X_0F_0 = 1.
\]

Now we return to the assertion of Lemma. By (9) we see that \( \tilde{R}_i = R_i + (X_0w - W_0x_i) \). Thus we have \( \tilde{R}_i \leq R_i \). Note that \( (x_i, w) = 1 \). Hence the minimality of \( (X_0, W_0) \) implies that the equality holds if and only if \( P_2 = A \). Thus by (40) we have \( R_i = 1 \). This implies \( T = \emptyset \). Namely the assumption of Lemma includes \( \tilde{R}_i < R_i \) and \( \tilde{R}_2 < R_2 \). If we assume \( \tilde{P} \in D_1 \) (or \( \tilde{P} \in D_3 \)), we get a contradiction with the minimality of \( R_2 \) (resp. \( R_1 \)). Thus we have \( (\tilde{R}_1, \tilde{R}_2) \in T \).

Assume that \( (\tilde{R}_1, \tilde{R}_2) \) is not minimum, then by Lemma 8, there exists a smaller element in \( T \). Let \( G = (g, g_1) \) be the pair of (39), which corresponds to this element. We have

\[
(41) \quad 0 < |OAG| \leq |OA\tilde{P}| \text{ and } 0 < |OBG| \leq |OB\tilde{P}|.
\]

Since \( G \in D_2 \), (41) implies that \( G \) is contained in the parallelogram \( OP_2\tilde{P}P_1 \). Being \( G \) cannot be a vertex, we reach a contradiction with our former
For \( Q = (t, u) \) with \( t > 0 \) and \( u \geq 0 \), we put
\[
\lambda(Q) = \lambda(t, u) = et + u.
\]

**Lemma 10.** We have the following three inequalities.

(i) \( \lambda(P_1) \leq \min(a_i, a_2 - R_i) \),
(ii) \( \lambda(P_2) \leq \min(a_i, a_2) \)
(iii) If \( T \neq \emptyset \), then \( \lambda(\bar{P}) \leq \min(a_i, a_2 - \bar{R}_i) \).

**Proof.** First we consider the value of \( \lambda(t, u) \) for
\[
E - \{(t, u) : 0 \leq u/t \leq 1, \frac{u}{t} \leq 1\}.
\]
We put \( \lambda(t, u) = A(t) \) for \( (t, u) \in E \). Then we have \( 1/\sqrt{2} \leq t \leq 1 \). And
\[
A(1/\sqrt{2}) = (1 + e)/\sqrt{2}. \quad \text{And for } 1/\sqrt{2} \leq t \leq e/\sqrt{1 + e^2}, \quad A(t) \text{ increases monotonously as } t \text{ increases.}
\]
And \( A(t) \) decreases monotonously as \( t \) increases. And \( A(1) = e \). Thus for \( Q, Q_2 \in E \), we have
\[
\lambda(Q_2)/\lambda(Q_1) \leq \lambda(2/(e + 1)) \text{ for } e = 3 \quad \text{and} \quad \frac{\sqrt{5}}{2} \text{ for } e = 2.
\]
Now returning to the assertion of Lemma, we note that
(i) \( a_i = \lambda(B) \) and \( a_2 = \lambda(A) \). And \( A, B, P_1, P_2 \) and \( \bar{P} \) are all contained in \( \{(t, u) : 0 \leq u/t \leq 1\} \). (About \( P_2 \), we note that \( (1, 1) \) lies on the boundary. Thus the minimality of \( R_2 \) and \( X_0 \) implies our assertion.)
(ii) By (38), we see that \( a_2 - \bar{R}_i > a_2 - R_i > e x_i \). Thus we have
\[
a_2 - \bar{R}_i > a_2 - R_i \geq \frac{e}{e + 1}.
\]
(iii) From (44), we have \( 2e/(e + 1) > \lambda(Q_1)/\lambda(Q_2) \) for every \( Q, Q_2 \in E \).
By these three facts, we see that if
\[
2|OP_1| \quad \text{(resp. } 2|OP_2|, 2|OP|) \leq |OA|,
\]
we have
\[
\lambda(P_1) \quad \text{(resp. } \lambda(P_2), \lambda(\bar{P})) < a_2 - R_i \quad \text{(resp. } a_2, a_2 - \bar{R}_i)\).
\]
And also the corresponding results hold with respect to \( |OP|'s \) and \( |OB| \) and \( a_i \).

Now we proceed to the proof of Lemma. We use Figure 2. Our plan of Proof is as follows.

(a) We try to get (45) by comparing the area of triangles which appear in Figure 2. Here note that by (40), \( |OP,P_2| = |OP,\bar{P}| = |OP_2,\bar{P}| = 1/2 \). And the area of other triangles are half integers.
(b) For the case that fails (a), there is a relation between the points
of Fig. 2. Using it, we deduce the conclusion.

We explain the process taking the inequality \( \lambda(\overline{P}) \leq a_2 - R_1 \).

(a) Since \( \overline{P} \) appears in case \( R_1, R_2 \geq 2 \), we have \( P_1 \cong A \) and \( P_2 \cong B \). We compare \( |OP_2A| \) and \( |OPP_2| \) (=1/2).

If \( |OP_2A| \geq 1 \), then by the relations
\[
2|OP_2A| = |OA||OP_2| \sin \angle P_2OA \quad \text{and} \quad \angle P_2OA < \angle P_2OP,
\]
we have \( |OA| > 2|OP_2| \).

(b) If \( |OP_2A| = 1/2 \), step (a) fails. But in this case, \( P_1, \overline{P} \) and \( A \) lies on a line. Thus we have
\[
A = P_1 + \alpha(\overline{P} - P_1) = P_1 + \alpha P_2.
\]
So we have \( a_2 = \lambda(P_1) + \alpha \lambda(P_2) \). By (40), we have \( \alpha = R_2 \geq 2 \). Thus we have the desired inequality.

Very similar reasonings work for other inequalities.

7. (E)-process. For a given \( \tau \)-distribution, we arrange the positive places starting from \( P_0 = \sigma_i(0) \) to the negative direction of \( C(a_i) \) as follows.

(46) \( P_0, P_1, \ldots, P_{m-1} \).

Note that we may assume for the maximal \( \tau \)-distribution that \( \tau \)'s attached to the same positive place are taken as a serial numbers. Thus we assume it henceforth. And we define (and call it a leap):

\[
\begin{align*}
L(P_i, P_{i+1}) &= \min_{\sigma_1(z_0) = P_{i+1}} \max_{\sigma_1(z_i) = P_i} \tau_i - 1 \quad (0 \leq i \leq m - 1), \\
L(P_{m-1}, P_0) &= a_2 - \max_{\sigma_1(z_i) = P_{m-1}} \tau_i - 1.
\end{align*}
\]

We consider the indices of (46) cyclicly modulo \( m \). And we put
\[
L(P_u, P_{u+s}) = \text{the sum of } L(P_{u+i}, P_{u+i+1}) \text{ for } 0 \leq i \leq s - 1.
\]

And we put
\[
V(P_u, P_{u+s}) = \sum_{i=0}^{s} \beta_{u+i} + L(P_u, P_{u+s}).
\]

We take a \( \bar{y} \)-segment \( y \) of \( C(a_i) \). Let \( y = \sigma_i([c - \bar{y} + 1, c]) \). We consider the places of \( y \) by the following order:
\[
\sigma_1(c), \sigma_1(c - 1), \ldots, \sigma_1(c - \bar{y} + 1).
\]

Let \( P_i \) be the first positive place we met, and \( P_i \) be the last one. Then as
easily seen we have
\[ V(y) = V(P_s, P_t). \]
Thus we henceforth study \( V(P, P_{0-s}) \) as the main target.

We consider the following process:

(E) We start from a positive place \( P_{i_0} \) and proceed to the negative direction on \( C(a_i) \). Take \( P_{i_0} \) such that
\[ V(P_{i_0}, P_{i_0} > x_i) \text{ and } V(P_{i_0}, P_{i_0} < x_{i+1}). \]
We denote this situation by \( P_{i_0} \rightarrow P_{i_1} \). Note here that for a \((\tau)\)-distribution which corresponds to a non-trivial d-pair, it must hold \( \beta_i \leq x_i \) for all \( 0 \leq i \leq a_i - 1 \). Since if \( \beta_i > x_i \), the \( \gamma \)-segments which contain \( \sigma_i \) are all overflowed. Thus we have \( y_2 \leq y_2 \). By this fact we have \( P_{i_0} \equiv P_{i_1} \). And we continue the process by starting from \( P_{i_1} \), and so on. After that we have a chain of positive places on \( C(a_i) \) such as
\[ G(r) \rightarrow P_{i_1} \rightarrow P_{i_2} \rightarrow \ldots. \]
Since the cardinality of positive places is finite, we obtain a cycle in the chain. Let \( E(\tau) \) be one of them. We consider it in the following as fixed.
\[ E(\tau) \rightarrow P_{i_1} \rightarrow P_{i_2} \rightarrow \ldots \rightarrow P_{i_k} \rightarrow P_{i_1} \rightarrow \ldots. \]
Assume that all \( P_j \)'s which appear in \( E(\tau) \) are all different. We call \( k \) the number of steps of \( E(\tau) \). We put \( g \) the number which is defined as follows. By starting from \( P_{i_1} \) of \( E(\tau) \), we come back to \( P_{i_1} \) after going around \( C(a_i) \) just \( g \) times. Then we have
\[ (48) \sum_{u=1}^{k} V(P_{i_u}, P_{i_u-1}) + \sum_{u=1}^{k} L(P_{i_u-1}, P_{i_u}) = ga_2. \]
We put the latter term of the left hand side of (48) as \( L[E(\tau)] \). Since \( P_j \)'s are all different, we have
\[ (49) L[E(\tau)] \leq a_2 - v_1. \]

8. Attainability of the d-pairs of Theorem 2. We construct the aimed \((\tau)\)-distribution by making so that \( E(\tau) \) is the total chain \( G(\tau) \). Namely we make \((\tau)\) with \( k \) positive places. Since there is much similarity in the method to construct the three d-pairs given in Theorem 2, we use the same terminology which is to be interpreted by the following table:

<table>
<thead>
<tr>
<th>case</th>
<th>d-pairs</th>
<th>( k )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>((a_2 - R_1, a_1))</td>
<td>( \lambda(P_1) )</td>
<td>( R_1 )</td>
<td>0</td>
<td>( Y_0 )</td>
</tr>
<tr>
<td>(b)</td>
<td>((a_2, a_1 - R_2))</td>
<td>( \lambda(P_2) )</td>
<td>0</td>
<td>( R_2 )</td>
<td>( X_0 )</td>
</tr>
<tr>
<td>(c)</td>
<td>((a_2 - R_1, a_1 - R_2))</td>
<td>( \lambda(\overline{P}) )</td>
<td>( \overline{R}_1 )</td>
<td>( \overline{R}_2 )</td>
<td>( X_0 + Y_0 )</td>
</tr>
</tbody>
</table>
We first determine the coordinates of the positive places. We define $u_i$, for the former two cases.

(a) $u_i = X_0 - \theta Y_0$ with $1 \leq u_i \leq Y_0$, $\theta \in \mathbb{N}$

(b) $u_i = Y_0 - \theta X_0$ with $1 \leq u_i \leq X_0$, $\theta \in \mathbb{N}$

And we put $s = Y_0 f - y F_0$. Here $s = 2|OP_iB|$. Hence $s \in \mathbb{N}$. Using these terms, we define $\xi(m)$ ($0 \leq m \leq k - 1$) respectively for three cases of Table 1 as follows:

(a) $m(R_2 + \theta s) + \left[ \frac{mu_i}{Y_0} \right] s$,

(b) $m(s + \theta R_2) + \left[ \frac{(mu_i - 1)/X_0}{y} \right] o + R_2$,

(c) $ms + \left[ \frac{(mY_0 - 1)/(X_0 + Y_0)}{y} \right] + 1)R_2$.

Lemma 11. The positive places on $C(a_i)$ whose coordinates are given by (50) satisfy the following property:

The cardinality of the $y$-segments which contain $g + 1$ positive places is $r_2$. And all other $y$-segments contain at most $g$ positive places.

Proof. The reasoning of the proof is very similar for three cases. Thus we show it by treating Case (b). We assume $f > 0$. Namely $y \geq 2$.

First note that either $s$ or $\theta$ is positive. Since if not the case, we have $B = P_i$ and $Y_0 < X_0$. Hence we reach to a contradiction with (40), by noting $y \geq 2$. Thus $\xi(m)$ increases monotonously as $m$ increases. Now we ascertain that (50) gives a coordinates set on $C(a_i)$. Since $\xi(0) = 0$, we calculate $\xi(k - 1)$. For that we consider $\xi(eX_0)$ and $\xi(W_0)$. Here using (40), we have

(51) $W_0 u_1 - 1 \equiv W_0 Y_0 - 1 \equiv 0$ (mod $X_0$).

By (51), we have $\xi(W_0) = f + R_2$. It is easy to see $\xi(eX_0) = e\bar{y}$. Now again by (51), we have

(52) $\xi(k) = \xi(eX_0) + \xi(W_0) = a_1 + R_2$.

From this value we have $\xi(k - 1) = a_1 - (\theta R_2 + s)$. Hence (50) gives a coordinates set.

Now we note the relation

(53) $\xi(u + X_0) - \xi(u) = \bar{y}$ for all $u \in \mathbb{Z}$.

Hence by (52), we have $a_1 - \xi(k - X_0) = \bar{y} - R_2$. Thus the $y$-segments whose coordinates are $[0, R_2 - 1]$ contains $X_0 + 1$ positive places. Since their cardinality is $R_2$, it is sufficient to ascertain that all other $y$-segments contains $X_0$ positive places.

For that we note by (51), we have $u_i k \equiv 1$ (mod $X_0$). Thus by using $(u_i, X_0) = 1$, we see that the relation

$\sigma_1(\xi(u + k)) = \sigma_1(\xi(u))$ for $1 \leq u \leq X_0 - 1$
Disjoint sequences generated by the bracket function III

holds. Thus we obtained the aimed assertion by (53).

Starting from $\sigma_1(0)$ and proceeding to the negative direction on $C(a_i)$, we put the $k$ places given in Lemma 11 as follows:

(M) $M_0(=\sigma_1(0)), M_1, M_2, \ldots, M_{k-1}$.

As usual we consider the indices of $M$ by modulo $k$. And to treat the leaps, we put the number $i + 1/2$ to the intermediate part of $M_i$ and $M_{i+1}$. We put $\tilde{a}_2 = a_2 - r_1$. And we distribute $\tilde{a}_2$ points and $r_1$ leaps on $M$ by the following rules:

(P) Put the $u$-th point on $M_{ug}$ $(0 \leq u \leq \tilde{a}_2 - 1)$,

(L) Put the $j$-th leap at the intermediate part whose number is $(\tilde{a}_2 - 1)g + 1/2 + j(g - 1)$ $(1 \leq j \leq r_1)$.

And we count the points and leaps given at each places.

**Lemma 12.** By the above process, $M$ satisfies

(54) $V(M_i, M_{i+g-1}) \leq x_i$ for all $0 \leq i \leq k - 1$.

**Proof.** The reasoning used for (a) and (c) is entirely same. Note that we have

(55) $a_2g - r_1 = kx_i$.

And by Lemma 10, we see that each places has at least one points. Hence all place of $M$ is positive. And (55) means that the final leap is put at between $M_{k-g}$ and $M_{k-g+1}$. We show in our case, all relations of (54) are the equality.

We consider $V(M_{u+1}, M_{u+g}) - V(M_u, M_{u+g-1})$. Then by its definition, the difference is

$(\beta_{u+g} - \beta_u) + L(M_{u+g-1}, M_{u+g}) - L(M_u, M_{u+1})$.

We consider the process (P) first. If $M_u$ has a point, then $M_{u+g}$ has the next point. Next at the join from (P) to (L), if $M_u$ has a point then a leap is put between $M_{u+g-1}$ and $M_{u+g}$. And after that, leaps are put to $u + g - 1/2$ and $u + 1/2$ simultaneously. And as noted above, this process is nicely jointed at the final step. Thus we see $V(M_i, M_{i+g-1})$ is a constant. By (55), it must be $x_i$.

Next we treat (b). In this case we have

(56) $a_2g = kx_i - (x_iW_0 - X_0w)$.

Here $x_iW_0 - X_0w = 2\mid OAP_i \mid \in \tilde{N}$. First we show in this case the value of $V(M_i, M_{i+g-1})$ differs at most one. To show it, we map the indices of $M$ to $C(k)$. For a $g$-segment $g$ of $C(k)$, we put

$P(g) =$ the sum of # of points put on $M_i$ with $\sigma_k(i) \in g$ by (P).
Since \( r_i = 0 \) for (b), we have \( V(M_i, M_{i+g-1}) = P(g) \) for \( g = g(i+g-1) \). Now we consider the variation of \( P(g) \) by operating \((P)\) with \( u = 0, 1, \cdots \). Then at the \( u \)-th step, we see easily that

\[
P(g) = P(g') + 1 \text{ for } \text{Cood} (g) \in [0, \text{Res}(u+1)g - 1],
\]

\[
\text{Cood} (g') \in [\text{Res}(u+1)g, k-1],
\]

where we put \( \text{Res} m = v \) for \( m \in \mathbb{N} \) if \( 1 \leq v \leq k \) and \( m \equiv v \) (mod \( k \)). Now by (56), we see that \( P(g) \) is

\[
x_i - \left[ \frac{(x_iW_0 - X_0w)}{k} \right] \text{ or } x_i - \left[ \frac{(x_iW_0 - X_0w)}{k} \right] - 1.
\]

We combine the above two Lemmas, and obtain a \((\tau)\)-distribution which corresponds to the d-pair \((a_2 - r_1, a_1 - r_2)\).

9. Maximality of the d-pairs. We prove in this section the maximality of the d-pairs given in Theorems 1 and 2. We start with general remarks. By (25), we may consider the pairs such that

\[
(57) \quad v_i \geq x_i e + 1.
\]

For a given \((\tau)\)-distribution, we take a cycle \( E(\tau) \). Let it be as

\[
E(\tau) \quad P_{i_1} \rightarrow P_{i_2} \rightarrow \cdots \rightarrow P_{i_h} \rightarrow P_{i_1} \rightarrow \cdots
\]

We define the rotation number \( g \) as defined in §7.

Here we consider the cardinality of steps for each one round of \( C(a_i) \) by operating \((E)\)-process to \( E(\tau) \). We note two facts:

(i) By (47) and (57), we need at least \( e + 1 \) steps to go around \( C(a_i) \).

(ii) On the other hand, we must reach or pass over the starting place by \( e + 1 \) steps. Since if it is not the case, we see by (47) that there must be at least \( ((e+1)\bar{y} + 1) - a_i = \bar{f} + 1 \) overflowed \( \bar{y} \)-segments. It contradicts to (25).

Hence we have the relation

\[
(58) \quad k = g_i(e+1) + g_2 e \text{ with } g_1 + g_2 = g \text{ and } g_i \in \mathbb{N}.
\]

And we have by (47) the following inequality:

\[
\{\text{the sum of } V(P_{j_0}, P_{j_{u-1}}) \text{ for } 1 \leq u \leq k\} \leq kx_i.
\]

We denote \( L[\text{E}(\tau)] \) simply by \( L \). By (48), we have

\[
a_2g - kx_i \leq L.
\]

Then by substituting (58), we have

\[
(59) \quad gw - g_i x_i \leq L.
\]

Lemma 13. For \((R)\)-case, we take a \((\tau)\)-distribution with

\[
x_i(e+1) \geq v_i \geq x_i e + 1.
\]

Then we have at least \( \bar{f} \) overflowed \( \bar{y} \)-segments. And in the extremal case,
the numbers put to these \(\tilde{y}\)-segments make an \(F(\bar{f}, \bar{Y}, e)\)-sequence (mod \(a_i\)).

**Proof.** First we prove that the only possible type for (R)-case is \(g = g_1 = 1\). Assume \(g \geq 2\). Then \(g_2 > 0\). Since if \(g_2 = 0\), as noted above, we must actually pass over the starting place. And \(E(r)\) cannot be a cycle.

We put \(a_2 = x_i(e + 1) + \alpha\ (\alpha \geq 0)\). Then by (59), we have \(L \geq g_2 x_i + \alpha g\).

On the other hand \(L\) must be \(\leq a_2 - v_i\). Hence we have

\[g_2 x_i + \alpha g \leq x_i + \alpha - 1.\]

Since \(g_2 \geq 1\), we reach a contradiction. Thus we have \(g = g_1 = 1\). Namely \(k = e + 1\). And we obtain the assertion of Lemma by Lemma 4.

**Lemma 14.** For (S)-case, the d-pairs given in Theorem 2 are the all of non-trivial maximal d-pairs.

**Proof.** Since the non-triviality is clear, we consider about their maximality. As easily seen, \((a_2, a_1)\) cannot be a d-pair. And \(v_1 \leq a_2\) and \(v_2 \leq a_1\).

Thus there exist maximal d-pairs of type \((a_2 - u_i, a_1)\) and \((a_2, a_1 - u_2)\) with \(u_i, u_2 \in N\). Now we show \(u_i = R_i\) and \(u_2 = R_2\).

(a) First we treat \((a_2 - u_i, a_1)\). In this case, all \(\tilde{y}\)-segments must be good. Thus in \(E(r)\), the distance of \(P_{ju}\) and \(P_{ju + 1}\) must be \(\geq \tilde{y}\). And as the total inequality, we have \(k\tilde{y} \leq g a_i\). By substituting the values given in (58), we have \(g_i \tilde{y} \leq g f\). Namely \(g_i / g \leq f / \tilde{y}\). On the other hand by (59), we have \(g_2 - g_i x_i \leq L \leq u_i\). Thus by considering \(Y = g\) and \(F = g_i\), we have the desired assertion. (Note that in our discussion \(g_i\) must be in \(N\). But by allowing \(F = 0\), we make the pair \((x_i e, a_i)\) to join in the competition. That saves the case that there is no pair with \(v_1 \geq x_i e + 1\) and \(v_2 = a_1\).)

(b) \((a_2, a_1 - u_2)\). In this case there are no leaps. Thus by (59) we have \(g \leq g_i x_i \leq 0\). Assume that the distance of \(P_{ju}\) and \(P_{ju + 1}\) is \(\tilde{y} - m\) with \(m \in N\). Then the \(\tilde{y}\)-segments which contain these places must be overflowing, and their cardinality is \(m\). Thus we have

\[u_2 \geq k\tilde{y} - a_i g = g_i \tilde{y} - f g.\]

We have the desired assertion by considering \(g = X\) and \(g_i = W\) in (8).

(c) Finally we consider maximal d-pairs of type \((a_2 - u_i, a_1 - u_2)\) with \(0 < u_i < R_i\) and \(0 < u_2 < R_2\). By the same reasoning used in (a) and (b), we have

\[u_i \geq g - x_i g_1\]\n\[u_2 \geq g_2 \tilde{y} - f g.\]

We put \((S_i, S_2) = (g - x_i g_i, g_i \tilde{y} - f g)\). We consider \((g, g_i) \in N^2\). By the discussion used in §6, we see that \((g, g_i)\) cannot be contained in \(D_1 \cup D_3\). Thus we have \((S_i, S_2)\) is contained in \(T\) of §6. Now by Lemma 8, the minimum value of \(S_1\) and \(S_2\) are attained simultaneously. And by Lemma
9, they are $\bar{R}_1$ and $\bar{R}_2$.

Now collecting the results of §8 and §9, we obtain easily a proof of Theorems 1 and 2. (N.B. If $f = 0$, we have $y_0 = a_i - 1$, and $R_2 = 1$ for (S)-case. Hence $(a_2, a_i - 1)$ is a trivial d-pair.)

10. Concluding remarks. We conclude this paper with a few remarks.

(1) For the case $h \geq 2$, the existence of the d-pairs given in Conjecture in §1 can be fairly easily ascertained. But we cannot overcome a gap in the proof of their maximality. For the sake we have to investigate the nature of standard segments more closely. (In case $h = 1$, the main difficulty comes from the non-standard ones.)

(2) In Lemma 13, we clarified the structure of the overflowed $\bar{y}$-segments. But for (S)-case, we have not studied the corresponding results. So far as judged from examples, there exists a similar structure theory for (S)-case also. And on the other hand, there exists a similar structure theory for leaps. Of course, the original problem has a symmetry with $a_i$ and $a_2$. This symmetry is broken by the condition $a_2 > 2x_{h}$. But there is some dual property between $r_1$ and $r_2$ for (S)-case.

(3) In this paper, we restricted our study of ($\beta$)-problem only for the necessary parts to solve our problem. But we think ($\beta$)-problem to be interesting as an extremal problem in combinatorial number theory. We hope to return to ($\beta$) and the problem mentioned in (2) in another place, with this framework.

References