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<td>Citation</td>
<td>長崎大学教養部紀要 [自然科学篇] 1989, 30(1), p.1-10</td>
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<tr>
<td>Issue Date</td>
<td>1989-07</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/16598">http://hdl.handle.net/10069/16598</a></td>
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Disjoint sequences generated by the bracket function IV

Dedicated to Professor Michio Kuga on his 60th birthday

Ryozo MORIKAWA

(Received April 28, 1989)

1. Introduction. For $q, a \in \mathbb{N}$ and $b \in \mathbb{Z}$, we put

$$S(q, a, b) = \{ \left\lfloor \frac{qn + b}{a} \right\rfloor : n \in \mathbb{Z} \}.$$

In this paper, we treat the following two problems:

(I) Take $q, a \in \mathbb{N}$ such that $q = a_i$ for $1 \leq i \leq 3$. Our problem is to obtain a criterion for that three sequences $S(q_i, a_i, b_i)$ can be made mutually disjoint by taking suitable $b_i$'s. We gave such criterion in [1] under certain additional assumptions. In the first half of this paper, we give a general answer for the problem, and give a proof.

We state here the result. We put

(2) $$S(q, q_2, q_3) = \frac{q}{q_2}, \quad S(q_2, q_3) = \frac{q_2}{q_3}, \quad S(q_1, q_3) = \frac{q_1}{q_3},$$

(3) $$x_i a_i + y_i a_2 = q_1, \quad x_2 a_2 + y_2 a_3 = q_2, \quad x_3 a_3 + y_3 a_1 = q_3,$$

with $(x_i, y_i) \in \mathbb{N}^2$.

From (3), we have

(4) $$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{q}{x_1 x_2 x_3 + y_1 y_2 y_3} \begin{pmatrix} t_1 x_2 x_3 + t_3 y_1 y_2 - t_2 x_3 y_1 \\ t_2 x_3 x_1 + t_1 y_2 y_3 - t_3 x_1 y_2 \\ t_3 x_1 x_2 + t_2 y_3 y_1 - t_1 x_2 y_3 \end{pmatrix}.$$

By considering (1), we have

(5) $$q f = x_1 x_2 x_3 + y_1 y_2 y_3$$

with $f \in \mathbb{N}$.

Theorem 1. Take $q, a \in \mathbb{N}$ such that $(q_i, a_i) = (a_i, a_j) = 1$ for $1 \leq i \neq j \leq 3$. Then three sequences $S(q_i, a_i, b_i)$ can be made mutually disjoint by taking suitable $b_i$'s if and only if there exists a solution system $(x_i, y_i)(1 \leq i \leq 3)$ of (3) such that $f \geq 2$. 
The statement of Theorem 1 is same to that of Theorem 4 of [1]. But in [1], we treated the problem under the assumption $q_1 = q_2 = q_3$. The proof of Theorem 1 is also similar to that of Theorem 1 of [1]. But we follow a somewhat different way for the convenience of application to (II).

We think the assertion of Theorem 1 is noticeable for the following two points.

(a) Since $x_1x_2x_3 + y_1y_2y_3$ is the determinant of the matrix of coefficients of (3), Theorem 1 has an atomosphere of the well known Minkowski's Theorem. And it is remarkable that the value $f$ controls the existence of the solution completely.

(b) Theorem 1 shows the fact that in case the criteria for pairwisely disjointness are satisfied, there exists a disjoint triple except for the extremal case $f = 1$.

(II) It is natural to ask whether the properties (a) and (b) remain true for the case of disjoint quadruples. This is the theme of the latter half of this paper. A criterion for disjointness of quadruples $S(q_i, a_i, b_i; 1 \leq i \leq 4$ is given in Theorem 2. But it is rather an insufficient one, and we cannot give any decisive answer to the above question. Thus we put off a full discussion of the problem to a forthcoming paper. Instead of it, we give some numerical examples which suggest curious phenomena which do not appear in case of disjoint triples.

2. We start to prove Theorem 1. Let $q_i, a_i (1 \leq i \leq 3)$ be as in §1. We take $\hat{a}_1 \in \mathbb{Z}$ such that

$$\hat{a}_1 a_1 \equiv 1 \pmod{q_1}$$

and consider it to be fixed in the following.

Since our problem does not change by the simultaneous translation of $S(q_i, a_i, b_i) (= S_i)$, we assume $b_i = -1$. We take the solution system of (3) such that

$$1 \leq y_1 \leq a_1, 1 \leq y_2 \leq a_2 \text{ and } 1 \leq x_3 \leq a_1.$$

Now Proposition 1 of [1] shows that if

$$b_2 \equiv m_2 + \hat{a}_1 a_2 n_2 \pmod{q_1} \text{ with } 0 \leq m_2 \leq x_1 - 1, 0 \leq n_2 \leq y_1 - 1,$$

$$b_3 \equiv m_3 + \hat{a}_1 a_3 n_3 \pmod{q_3} \text{ with } 0 \leq m_3 \leq y_3 - 1, 0 \leq n_3 \leq x_3 - 1,$$

then $S_1 \cap S_2 = S_1 \cap S_3 = \emptyset$. (We first consider $G_1$ only.)

For $S_2 \cap S_3 = \emptyset$, we use Theorem 2 of [1]. Thus we transform the properties of $b_2$ and $b_3$ given in (8) to that of modulo $q_2$. We put

$$q_2 = d_2 \hat{q}_1 \text{ and } q_3 = d_3 \hat{q}_3.$$
Then we have
\[ b_2 \equiv m_2 + a_1 a_2 n_2 + w_2 q_1 \pmod{q_2} \quad \text{with} \quad 0 \leq w_2 \leq d_2 - 1, \]
\[ b_3 \equiv m_3 + a_1 a_3 n_3 + w_3 q_3 \pmod{q_3} \quad \text{with} \quad 0 \leq w_3 \leq d_3 - 1. \]

Now by Theorem 2 of [1], we see that if
\[ a_2 b_3 - a_3 b_2 \in \{ a_2 x + a_3 y : 0 \leq x \leq x_2 - 1, 1 \leq y \leq y_2 \} \pmod{q_2}, \]
then \( S_2 \cap S_3 = \emptyset \). (We first consider \( E_1 \).)

We take \( \hat{a}_3 \in \mathbb{Z} \) such that \( \hat{a}_3 a_3 \equiv 1 \pmod{q_2} \) and consider it to be fixed in the following.

By easy calculations, we see that (10) is equivalent to the following
\[ \{ -a_2 \hat{a}_3 m - a_2 \hat{a}_1 n + a_2 \hat{a}_3 w_3 q_3 - w_2 q_1 \} \cap [1, x_1 + y_2 - 1] \not\equiv \phi \pmod{q_2}, \]
where \(-y_3 + 1 \leq m \leq x_2 - 1\) and \(-x_3 + 1 \leq n \leq y_1 - 1\)
and \(0 \leq w_i \leq d_i - 1\) (i = 2, 3).

Now (11) implies
\[ \{ -a_2 \hat{a}_3 m - a_2 \hat{a}_1 n \} \cap [1, x_1 + y_2 - 1] \not\equiv \phi \pmod{q}. \]
On the other hand by the Chinese remainder theorem, we see that (12) implies (11). Thus we study (12) in the following.

We use frequently the fact
\[ (y_i, a_i) = (x_i, a_{i+1}) = 1 \quad \text{for all} \quad 1 \leq i \leq 3, \]
which follows from (1) and (3). (We consider \( i \) cyclicly.)

For \( m, n \in \mathbb{Z} \), we define \( r \) and \( s \) as follows.
\[ \begin{cases} m \equiv -a_3 r \pmod{x_2} & \text{and} \quad 0 \leq r \leq x_2 - 1, \\ n \equiv -a_1 s \pmod{y_1} & \text{and} \quad 0 \leq s \leq y_1 - 1. \end{cases} \]

By (13), \( r \) and \( s \) are determined uniquely from \( m \) and \( n \). Now we put
\[ \chi(m, n) = \{ t_2 r/x_2 + t_1 s/y_1 \} q + y_2 m/x_2 + x_1 n/y_1, \]
where \( \{ x \} \) means \( x - \lfloor x \rfloor \).

**Lemma 1.** \( \chi(m, n) \in \mathbb{Z} \) and satisfies the following relation.
\[ \chi(m, n) \equiv -a_2 \hat{a}_3 m - a_2 \hat{a}_1 n \pmod{q}. \]

**Proof.** First we note the following two relations.
\[ \begin{cases} -a_2 \hat{a}_3 m \equiv r a_3 + (\lfloor r a_3/x_2 \rfloor + \lfloor (m - 1)/x_2 \rfloor + 1) y_2 \pmod{q}, \\ -a_2 \hat{a}_1 n \equiv s a_2 + (\lfloor s a_2/y_1 \rfloor + \lfloor (n - 1)/y_1 \rfloor + 1) x_1 \pmod{q}. \end{cases} \]

We prove the former one. Since \( (q, a_3) = 1 \), we multiply the relation by \( a_3 \) and consider the difference of two sides. Then we have
\[-a_2 m - r a_3 a_3 - (\lfloor r a_3/x_2 \rfloor + \lfloor (m - 1)/x_2 \rfloor + 1) y_2 a_3.\]

Using the relation \( y_2 a_3 \equiv -x_2 a_2 \pmod{q} \), we have
\[ a_2 (-m - r a_3 + x_2 (\lfloor r a_3/x_2 \rfloor + \lfloor (m - 1)/x_2 \rfloor + 1)). \]

By (14), we see the value = 0. A similar reasoning works for the latter
relation of (17).

Now we add two relations of (17). Then we have
\[
\frac{r(a_2x_2 + a_3y_2)}{x_2 + y_2} \left( \frac{m - 1}{x_2} \right) + 1 - \left\{ \frac{ra_3}{x_2} \right\}
+ \frac{s(a_2y_1 + a_1x_1)}{y_1 + x_1} \left( \frac{n - 1}{y_1} \right) + 1 - \left\{ \frac{sa_1}{y_1} \right\}.
\]

Hence by (3) and (14), we have
\[
q \left( \frac{rt_2}{x_2} + \frac{st_1}{y_1} \right) + \frac{y_2m}{x_2} + \frac{x_1n}{y_1}.
\]

Considering the value modulo q, we obtain the relation of Lemma. And we see the value is in \(\mathbb{Z}\). Q. E. D.

Now we give another expression of \(\chi\). In order that, we define the following numbers. Namely we put
(18) \((y_1, x_2) = d, x_2 = dX_2 \text{ and } y_1 = dY_1.\)

And we take \(\lambda \in \mathbb{Z}\) such that
(19) \(\lambda \equiv t_2 Y_1 r + t_1 X_2 s \pmod{dX_2 Y_1}\) and \(0 \leq \lambda \leq dX_2 Y_1 - 1.\)

Finally we put
(20) \(F = f/d, u = (Fm + y_3\lambda)/X_2 \text{ and } v = (Fn + x_3\lambda)/Y_1.\)

**Lemma 2.** \(F \in \mathbb{N}\) and \((u, v) \in \mathbb{Z}^2\). And we have
(21) \(\chi = \left( \frac{uy_2 + vx_1}{f} \right)\).

**Proof.** If \(F \notin \mathbb{N}\), there exist a prime number \(p\) and \(a \in \mathbb{N}\) such that \(p^a \mid d\) and \(p^a \nmid f.\) Then by (4) and (5), we have \(p \mid (q, a_1).\) This contradicts (1).

Next we prove \(u \in \mathbb{Z}.\) By (19), we have \(\lambda \equiv t_2 r Y_1 \pmod{X_2}\). And \(a_3F \equiv t_2 Y_1 y_3 \pmod{X_2}\) follows from (4). By using \(m \equiv -a_3 r \pmod{X_2}\), we have the relation \(Fm \equiv -t_2 Y_1 y_3 r \equiv -y_3 \lambda \pmod{X_2}\). Thus \(u \in \mathbb{Z}.\) A similar reasoning works for \(v.\)

From (15), we have \(\chi = \left( \frac{q\lambda + y_2 Y_1 m + x_1 X_2 n}{dX_2 Y_1} \right)\). Now by (5), we have (21).

Q. E. D.

We denote
(22) \(H = \{(m, n) : -y_3 + 1 \leq m \leq x_2 - 1, -x_3 + 1 \leq n \leq y_1 - 1 \}.\)

Then our problem is to seek the pair \((m, n) \in H\) such that
(23) \(\chi(m, n) \in [1, x_1 + y_2 - 1] \pmod{q}\). 

**Lemma 3.** If \(\lambda = 0\) for some \((r, s) \neq (0, 0), (23)\) has a solution pair \((m, n)\).

**Proof.** Let \(\lambda = 0\) for \((r, s) \neq (0, 0).\) Then there exists \((m, n)\) which satisfies (14) and \(0 \leq m \leq x_2 - 1\) and \(0 \leq n \leq y_1 - 1.\) Since \((m, n) \neq (0, 0),\) we see by (15) that the pair satisfies (23).

Q. E. D.

We put
Corollary. If $D > 1$, there exists a disjoint triple.

Proof. If $(r, s)$ runs through the range given in (13), the values of $\lambda$ covers 0 at least twice. Q.E.D

In the following we assume $D = 1$. Since $D = 1$ implies $d = (t_1, y_1) = (t_2, x_2) = 1$, the values of $\lambda$ cover $[0, x_2y_1 - 1]$ exactly once. We put

$$M(h) = \{ (m, n) \in H : \text{the corresponding } \lambda = h \}.$$  

Lemma 4. Assume $D = 1$ and $f \geq 2$. Then there exists a pair $(m, n) \in H$ which satisfies (23).

Proof. We take $(M, N)$ from $M(1)$ such that $0 \leq M \leq x_2 - 1$ and $0 \leq N \leq y_1 - 1$. Then the pairs $(m, n)$ of $M(1)$ are of the form $m = M - w_1x_2$ and $n = N - w_2y_1$ where $w_1, w_2 \in N \cup \{0\}$. We consider the corresponding $(u, v) \in \mathbb{Z}^2$ which is defined in (20).

Here we note the following three facts;

(i) If $u$ decreases $x_2$, the corresponding $u$ decreases $f$.

(ii) The $u$ which corresponds to $M$ is a positive integer.

(iii) $f \geq 2$ implies $(f(-y_3) + y_3)/x_2 < 0$.

By (i) - (iii), we see that there exists $u$ such that $0 \leq u \leq f - 1$. Since $v$ satisfies a similar property, we can deduce easily the assertion of Lemma by using (21). (In case $(u, v) = (0, 0)$, we replace it by $(f, 0)$.) Q.E.D.

3. Thus the remained case to be considered is $D = f = 1$.

Lemma 5. If $f = 1$, there exist no pairs $(m, n) \in H$, which satisfy (23).

Proof. By (22) and the fact $(u, v) \in \mathbb{Z}^2$, we obtain the following two inequalities.

$$\begin{align*}
\lceil y_3(\lambda - 1)/x_2 \rceil + 1 &\leq u \leq 1 + \lceil (y_3 \lambda - 1)/x_2 \rceil, \\
\lceil x_3(\lambda - 1)/y_1 \rceil + 1 &\leq v \leq 1 + \lceil (x_3 \lambda - 1)/y_1 \rceil.
\end{align*}$$

We first assume $1 \leq \lambda \leq x_2y_1 - 1$. Then we see from (26)

$$1 \leq u \leq y_1y_2 \text{ and } 1 \leq v \leq x_2x_3.$$  

Thus by Lemma 2, we have $y_2 + x_1 \leq \chi (m, n) \leq q$.

If $\lambda = 0$, we have

$-\lceil (y_3 - 1)/x_2 \rceil \leq u \leq 0$ and $-\lceil (x_3 - 1)/y_1 \rceil \leq v \leq 0$.

Hence we have $-q + y_2 + x_1 \leq \chi \leq 0$. Q.E.D.
Note here the fact that Lemma 5 does not imply directly the non-existence of disjoint triples. To ascertain that, we must check the other possible solutions of (3). By operating a suitable permutation on i, we may assume
\[ y_3 > a_3. \]
Note that (27), (4) and (5) imply \( t_2 y_1 \leq t_1 x_2 \).

**Lemma 6.** If \( t_2 y_1 = t_1 x_2 \), we have \( t_1 = t_2 = t_3 = y_1 = x_2 = 1 \). And there are no disjoint triples.

**Proof.** Since \( D = 1 \), \( t_2 y_1 = t_1 x_2 \) implies \( t_1 = t_2 = y_1 = x_2 = 1 \). By (4), we see \( a_1 = t_3 y_2 \) and \( a_3 = t_3 x_2 \). Thus (1) implies \( t_3 = 1 \). This case is treated in [1], and we ascertained there the latter assertion of Lemma. Q. E. D.

Thus we put
\[ z = t_1 x_2 - t_2 y_1 \in \mathbb{N}. \]
We take \( b_3 \) from \( G_2 \) of Proposition 1 of [1]. And we take \( b_2 \in G_1 \) and \( a_2 b_3 - a_3 b_2 \) in \( E_1 \). Then we see that if the relation
\[ (29) \quad \chi (m, n) \in [1, x_1 + y_2 - 1](\mod q) \]
holds with
\[ (30) \quad a_3 + 1 \leq m \leq x_2 + y_3 - 1 \quad \text{and} \quad - a_1 + x_3 + 1 \leq n \leq y_1 - 1, \]
there exists a disjoint triple.

**Lemma 7.** Assume (28). Then (29) has a solution pair.

**Proof.** We consider the pair with \( \lambda = 0 \). In the case, \( r = s = 0 \). Thus \( x_2 \mid m \) and \( y_1 \mid n \), and the ratios are \( u \) and \( v \). By (30), we have the following inequalities for \((u, v) \in \mathbb{Z}^2;\)
\[ \begin{cases} t_3 x_1 - \lfloor (y_3 x_2 - 1)/x_2 \rfloor \leq u \leq 1 + \lfloor (y_3 - 1)/x_2 \rfloor, \\ - t_3 y_2 - \lfloor (x_3 (z - 1) - 1)/y_1 \rfloor \leq v \leq 0. \end{cases} \]
Then by Lemma 2, we have
\[ - \lfloor (y_3 x_2 - 1)/x_2 \rfloor y_2 - \lfloor (x_3 (z - 1) - 1)/y_1 \rfloor \leq \chi \leq (1 + \lfloor (y_3 - 1)/x_2 \rfloor) y_2. \]
By noting \( z \geq 1 \), and by the fact that the differences of the adjacent values of \( \chi \leq \max(x_1, y_2) \), we obtain the conclusion of Lemma. Q. E. D.

As a final step of the proof of Theorem 1, we need the following

**Lemma 8.** (i) \( f = 1 \) implies \( D = 1 \).

(ii) Assume \( (27) \) and \( f = 1 \), then \( x_1 x_2 (x_3 + a_1) + (y_3 - a_3) y_1 y_2 = 1 \) if and only if \( t_1 x_2 = t_2 y_1 \).

**Proof.** Assume \( f = 1 \), then \( d = 1 \). If there exists a prime \( p \mid
(x_2, t_2), then (1) implies p \mid y_2. Thus (5) implies p \mid q. On the other hand, we have p \mid a_i \text{ from (4)}. This contradicts (1). By a similar reasoning, we have (y_1, t_1) = 1.

The assertion (ii) follows by easy calculations. Q. E. D.

Now we collect the results of Lemmas, and easily conclude the assertion of Theorem 1.

4. Henthorth we treat the problem of disjoint quadruples. We take q_i, a_i (1 \leq i \leq 4) such that (q_i, a_i) = 1. For simplicity, we assume

\begin{align*}
(q_i, q_j) = q & \text{ and } (a_i, a_j) = 1 \text{ for all } 1 \leq i \neq j \leq 4.
\end{align*}

We denote S(q_i, a_i, b_i) simply by S_i. We take \hat{a}_i \text{ such that } \hat{a}_i a_i \equiv 1 \pmod{q}, and consider it to be fixed in the following. The relation S_i \cap S_j = \emptyset for all 1 \leq i \neq j \leq 4 implies the following 6 relations.

\begin{align*}
\begin{pmatrix}
x_1 & y_1 & 0 & 0 \\
0 & x_2 & y_2 & 0 \\
y_3 & 0 & x_3 & 0 \\
z_1 & 0 & 0 & w_1 \\
0 & z_2 & 0 & w_2 \\
0 & 0 & z_3 & w_3
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
= \begin{pmatrix}
q \\
q \\
q \\
q
\end{pmatrix}
x_j, y_j, w_j, z_j \in \mathbb{N}
\end{align*}

(1 \leq j \leq 3)

We assume that

\begin{align*}
(\#) \ (32)
\end{align*}

is the unique solution system for q and a_i (1 \leq i \leq 4).

By operating a simultaneous translation on S_i, we may assume b_4 = -1. Then by Proposition 1 of [1], we see that S_i \cap S_4 = \emptyset if and only if

\begin{align*}
b_j = m_j + \hat{a}_4 a_i n_i \pmod{q} \text{ with } 0 \leq m_j \leq w_j - 1, 0 \leq n_j \leq z_j - 1.
\end{align*}

(We consider j cyclicly modulo 3.)

Lemma 9. Under the assumption of (\#), S(q_i, a_i, b_i) can be made mutually disjoint by taking suitable b's if and only if there exist pairs (\hat{m}_j, \hat{n}_j) (1 \leq j \leq 3) which satisfy the following two conditions (34) and (35).

\begin{align*}
(34) \quad & \begin{cases}
\hat{n}_j = n_j - n_{j-1} \text{ with } 0 \leq n_j \leq z_j - 1, \\
-m_{j+1} \leq \hat{m}_j \leq x_j - m_{j+1} - 1 \text{ with } 0 \leq m_j \leq w_j - 1.
\end{cases}
\end{align*}

\begin{align*}
(35) \quad & \{ -a_j \hat{a}_{j+1} \hat{m}_j - a_i \hat{a}_4 \hat{n}_j \} \cap [ m_j + 1, m_j + y_j ] \neq \emptyset \pmod{q}.
\end{align*}

Proof. By taking b_j as in (33), we use Theorem 2 of [1]. Then we obtain (35) by a similar calculation used in §2. Q. E. D.

Now as in §2, we introduce the following numbers.
\[
\begin{aligned}
(x_j, z_j) &= d_j, f_j = d_j F_j, x_j = d_j X_j, z_j = d_j Z_j, \\
\hat{m}_j &= -a_{j+1} r_j \pmod{x_j} \text{ with } 0 \leq r_j \leq x_j - 1, \\
\hat{n}_j &= -a_i s_j \pmod{z_j} \text{ with } 0 \leq s_j \leq z_j - 1, \\
\lambda_j &= Z_j r_j + X_j s_j \pmod{d_j X_j Z_j} \text{ with } 0 \leq \lambda_j \leq d_j X_j Z_j - 1, \\
f_j \mathbf{q} &= x_j z_j w_j + y_j z_j w_{j+1}, \\
x_j &= (q \lambda_j + y_j Z_j \hat{m}_j + w_j X_j \hat{n}_j) / d_j X_j Z_j.
\end{aligned}
\]

We treat the conditions given in Lemma 9 by separating into the following three parts.

(A) List up all \((\hat{m}_j, \hat{n}_j)\) such that \(-w_{j+1} + 1 \leq \hat{m}_j \leq x_j - 1, -z_{j+1} + 1 \leq \hat{n}_j \leq z_j - 1\) and \(x_j (\hat{m}_j, \hat{n}_j) \in [1, w_j + y_j - 1] \pmod{q}\).

(N) From the solutions of (A), list up all \((\hat{n}_1, \hat{n}_2, \hat{n}_3)\) such that \(\hat{n}_1 + \hat{n}_2 + \hat{n}_3 = 0\).

(J) From the solutions of (N), we take the corresponding \(\hat{m}_j\) and \(x_j\) \((1 \leq j \leq 3)\). Seek the pair \((\hat{m}_{j-1}, x_j)\) such that \((\hat{m}_{j-1} + x_j) \in [1, x_{j-1} + y_j - 1] \pmod{q}\).

**Lemma 9.** There exist a disjoint quadruple if and only if there exist triples \((\hat{m}_j, \hat{n}_j, x_j) (1 \leq j \leq 3)\) which satisfy (A), (N) and (J).

**Proof.** Only if part is easy. Thus assume that we have triples \((\hat{m}_j, \hat{n}_j, x_j) (1 \leq j \leq 3)\) which satisfy (A), (N) and (J). First we ascertain the fact that \(\hat{n}_j\) can be expressible as in (34). By easy calculations, we see that the cardinality of \((n_1, n_2, n_3)\) is given by \(\text{Min}(\hat{n}_j + z_{j+1}, z_j - \hat{n}_j, z_j)\).

We see also that from \(\hat{m}_j, x_j (1 \leq j \leq 3)\) which satisfy (A) and (J), we can take \(m_j (1 \leq j \leq 3)\) which satisfy (34). Q. E. D.

Note that (A) is the problem to determine all the disjoint triples \((S_j, S_{j+1}, S_4)\). Finally we reformulate our criterion as follows. We define

\[
\begin{aligned}
u_j &= (F_j \hat{m}_j + \lambda_j w_{j+1}) / X_j \text{ and } v_j = (F_j \hat{n}_j + \lambda_j z_{j+1}) / Z_j, \\
u_j / f_j &= \alpha_j, \quad v_j / f_j = \beta_j. \quad \lambda_j / F_j = \gamma_j \text{ and } \mu_j = \beta_j - \gamma_{j-1}.
\end{aligned}
\]

Then the condition (N) becomes
\[(37) \quad \mu_1 z_1 + \mu_2 z_2 + \mu_3 z_3 = 0.\]

And (J) becomes
\[(38) \quad a_j y_j + \mu_j w_j + a_{j-1} x_{j-1} \in [1, x_{j-1} + y_j - 1] \pmod{q}.\]

Now collecting above discussions, we have

**Theorem 2.** We take \(q_i, a_i \in \mathbb{N} (1 \leq i \leq 4)\), which satisfy (31) and (37). Then four sequences \((q_i, a_i, b_i)\) can be made mutually disjoint by taking suitable \(b_i's\) if and only if there exist triples \((\hat{m}_j, \hat{n}_j, x_j) (1 \leq j \leq 3)\) which satisfy (35), (37) and (38).
As noted in §1, the criterion given in Theorem 2 is rather an insufficient one. However it works to study numerical examples. We give some examples which suggest the future possible theory of disjoint quadruples in §5.

Here we state two general remarks.

(i) The main difficulty arises from (N). But if we can take $\mu_1 = \mu_2 = \mu_3 = 0$, then (N) holds trivially. And in the case the condition of (J) becomes simple. Such examples are given in §5.

(ii) It seems plausible that a proposition of the following type holds; Namely if $f_j < K$ ($1 \leq j \leq 3$) hold with a constant $K$, then there exists $c (K)$ such that there exist no disjoint quadruples for $q > c (K)$ (except possibly the case noted in (i)).

5. We give several numerical example:

Example 1. $q = 30011$. $a_1 = 326, a_2 = 271, a_3 = 349, a_4 = 389$. We have $x_1 = 48, x_2 = 94, x_3 = 15, y_1 = 53, y_2 = 13, y_3 = 76, z_1 = 67, z_2 = 105, z_3 = 18, w_1 = 21, w_2 = 4, w_3 = 61$.

(A) $f_1 = 4$. And the solution triples are as follows:

$(\hat{m}_1, \hat{n}_1, x_1) = (15, 62, 8), (26, 19, 16), (37, 43, 45), (37, -24, 24), (11, 24, 29), (11, -43, 8), (22, 48, 58), (22, -19, 37), (22, -86, 16), (33, 5, 66), (33, -62, 45), (7, -14, 50), (7, -81, 29), (18, -57, 58), (29, -100, 66).

$f_2 = 3$. $(\hat{m}_2, \hat{n}_2, x_2) = (72, 82, 7), (83, 41, 10), (11, 64, 7), (22, 23, 10), (-50, 46, 7), (-39, 5, 10).

$f_3 = 3$. $(\hat{m}_3, \hat{n}_3, x_3) = (5, 12, 66), (5, -6, 5), (10, 6, 71), (10, -12, 10), (-16, -37, 127), (-16, -55, 66), (-11, -43, 132), (-11, -61, 71).

(N) As easily seen, the sum $\equiv 0$ for all $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$.

Example 2. $q = 503$. $a_1 = 38, a_2 = 25, a_3 = 23, a_4 = 29$.

$x_1 = 6, x_2 = 10, x_3 = 7, y_1 = y_2 = 11, y_3 = 9, z_1 = 4, z_2 = 17, z_3 = 10, w_1 = 9, w_2 = 2, w_3 = 7$.

(A) $f_1 = 2$. $(\hat{m}_1, \hat{n}_1, x_1) = (3, 2, 10), (3, -2, 1), (1, -11, 19), (1, -1 5, 10).

$f_2 = 3$. $(\hat{m}_2, \hat{n}_2, x_2) = (8, 1, 3), (9, 9, 8), (9, -8, 6), (1, 8, 5), (1, -9, 3), (2, 16, 10), (2, -1, 8), (-6, 15, 7), (-6, -2, 5), (-5, 6, 10).

$f_3 = 2$. $(\hat{m}_3, \hat{n}_3, x_3) = (-1, 3, 8).

(N) $(\hat{n}_1, \hat{n}_2, \hat{n}_3) = (-2, -1, 3), (-11, 8, 3).

(J) For the above solutions of (N), $x_1 + \hat{m}_3$ fails (J).

Example 3. $q = 2003$. $a_1 = 97, a_2 = 67, a_3 = 59, a_4 = 53$. 
(A) \( f_1 = 6. \ (\hat{m}_1, \hat{n}_1, \chi_1); (-5, 6, 16), (-10, 12, 32), (-12, 12, 5), (10, -1, 18), (-15, 5, 34), (-17, 5, 7), (-22, 11, 23), (-20, -2, 36), (-22, -2, 9), (-27, 4, 25). \)

\( f_2 = 2. \ (\hat{m}_2, \hat{n}_2, \chi_2); (-1, -5, 35), (-8, -2, 43), (-8, -5, 9), (-15, 1, 51), (-15, -2, 17), (-22, 1, 25). \)

\( f_3 = 2. \ (\hat{m}_3, \hat{n}_3, \chi_3); (5, -10, 19), (-1, -3, 32), (-1, -10, 2), (-7, 4, 45), (-7, -3, 15), (-13, 4, 28). \)

\( (N) \ (\hat{n}_1, \hat{n}_2, \hat{n}_3); (12, -2, -10), (5, -2, -3), (-2, -2, 4). \) The corresponding \( (\mu_1, \mu_2, \mu_3) = (1/2, 1/6, -1), (0, 0, 0), (-1/2, -1/6, 1) \).

(J) There exist 24 combinations of \( (\hat{m}_j, \chi_j) \), which correspond to \( (N) \). Only two of them satisfy (J). They are \( \hat{m}_1 = -12, \chi_1 = 5, \hat{m}_2 = -15, \chi_2 = 17, \hat{m}_3 = 5, \chi_3 = 19, \) and \( \hat{m}_1 = -20, \chi_1 = 36, \hat{m}_2 = -8, \chi_2 = 43, \hat{m}_3 = -13, \chi_3 = 28. \)

Example 4. \( q = 30011, a_1 = 326, a_2 = 209, a_3 = 389, a_4 = 271. \) Then \( f_1 = 9, f_2 = 37, f_3 = 4. \) This case has fairly large numbers of solutions.

Reference