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Disjoint sequences generated by the bracket function $V$

Ryozo Morikawa

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1. Introduction. We revisit in this short paper the problem of disjoint triples of rational Beatty sequences.

Let $\mathbb{Z}$ and $\mathbb{N}$ mean as usual. For $q, a \in \mathbb{N}$ and $b \in \mathbb{Z}$, we put

$$S(q, a, b) = \{\lfloor(qn+b)/a\rfloor : n \in \mathbb{Z}\}$$

where $\lfloor x \rfloor$ means the greatest integer $\leq x$. We take $(q_i, a_i) \in \mathbb{N}^2 (1 \leq i \leq 3)$ for which

(1) $$(q_i, a_i) = 1 \text{ and } (a_i, a_j) = 1 \text{ for all } 1 \leq i \neq j \leq 3.$$ 

Now we consider the following two problems.

Problem A. Obtain a criterion to decide whether three sequences $S(q_i, a_i, b_i) (1 \leq i \leq 3)$ can be made disjoint by taking suitable $b_i \in \mathbb{Z}$ ($1 \leq i \leq 3$).

Problem B. To list up all the $(b_1, b_2, b_3) \in \mathbb{Z}^3$ for which three sequences $S(q_i, a_i, b_i) (1 \leq i \leq 3)$ are mutually disjoint.

About Problem A, we obtained a simple if and only if criterion in I, IV of [1]. (In the following we refer the paper in [1] by its number.) But we neglected Problem B there. At a glance, Problem B seems to be insignificant, and rather a complicated one. But in many cases, Problem B becomes indispensable to solve other problems.

We started in IV to study Problem A for disjoint quadruples of Beatty sequences. The result given there is an insufficient one. But we recently obtained a fairly satisfactory method to treat that problem. On the method, it is essential to solve Problem A for triples.

Since there remain some difficulties to round off the theory of disjoint quadruples, we give here an answer for Problem B of triples. Roughly speaking, we reduce Problem B to list up all lattice points of a parallelepiped which lie on certain lines. (cf. Proposition 1, § 4.)

In this paper, we treat Problem B for somewhat restricted type of triples.
But we think the interesting part of the problem is not spoiled. And the discussions given here are sufficient to treat disjoint triples of any type.

2. In the first half of the paper, we follow the same line of investigations as of I and IV. But we note that some notations are changed for the convenience of applications to the theory of disjoint quadruples.

As noted in §1, we do not consider the problem in its full generality. The first assumption is

\[(q_i, q_j) = q \text{ for all } 1 \leq i \neq j \leq 3,\]

Then by Theorem 1 of I, our problem is equivalent to that of sequences \( S(q, a_i, b_i) (1 \leq i \leq 3) \). Thus we consider this triple, and frequently denote \( S(q, a_i, b_i) \) by \( S_i \).

Assume that the triple \( (S_1, S_2, S_3) \) are mutually disjoint. Then by Theorem 1 of I, we obtain the following relation.

\[
\begin{pmatrix}
x_2 & y_1 & 0 \\
0 & y_3 & z_2 \\
x_3 & 0 & z_1
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
= \begin{pmatrix} q \\ q \\ q \end{pmatrix}
\]

where \( x's, y's \) and \( z's \) are in \( \mathbb{N} \). Hereafter we assume

\((\#)\) For given \( q, a_i (1 \leq i \leq 3), (3) \) is the unique relation, if all \( x's, y's \) and \( z's \) are required to be in \( \mathbb{N} \).

We introduce the following numbers \( f, d \) and \( F \) as follows.

\[x_2 y_3 z_1 + x_3 y_1 z_2 = qf, (x_2, x_3) = d \text{ and } dF = f.\]

Then by Lemma 2 of IV, \( f \) and \( F \) are in \( \mathbb{N} \).

By translating all \( S_i (1 \leq i \leq 3) \) simultaneously, we may assume

\[b_3 = -1.\]

And we take \( \hat{a}_i \in \mathbb{Z} \) for which \( \hat{a}_i a_i \equiv 1 (\text{mod } q) \), and we consider them to be fixed in the following.

Now by \((\#)\) and Proposition 1 of I, we see \( S_1 \cap S_3 = S_2 \cap S_3 = \phi \) if and only if \( b_1 \) and \( b_2 \) are of the following forms.

\[\begin{cases}
b_1 \equiv m_1 + \hat{a}_3 a_1 n_1 \pmod{q}, & 0 \leq m_1 \leq z_1 - 1, 0 \leq n_1 \leq x_3 - 1, \\
b_2 \equiv m_2 + \hat{a}_3 a_2 n_2 \pmod{q}, & 0 \leq m_2 \leq z_2 - 1, 0 \leq n_2 \leq y_3 - 1.
\end{cases}\]

And by \((\#)\) and Theorem 2 of I, \( S_1 \cap S_2 = \phi \) if and only if

\[a_1 b_2 - a_2 b_1 \in \{a_1 [0, x_2 - 1] + a_2 [1, y_1] \} \pmod{q}.\]

Substituting (5), we have

\[\{-a_1 \hat{a}_2 [-m_2, x_2 - 1 - m_2] - a_1 \hat{a}_3 (n_1 - n_2) \} \cap [1 + m_1, m_1 + y_1] \neq \phi \pmod{q}.\]

By using the relation \( \hat{a}_2 z_2 + \hat{a}_3 y_3 \equiv 0 (\text{mod } q) \), we have

\[\{-a_1 \hat{a}_2 [z_2 - m_2, x_2 + z_2 - m_2 - 1] - a_1 \hat{a}_3 (y_3 + n_1 - n_2) \} \cap [1 + m_1, m_1 \]
Here we note that if we take \( \hat{m} \) and \( \hat{n} \) so that \( \hat{m} \in [z_2 - m_2, x_2 + z_2 - m_2 - 1] \) and \( \hat{n} = y_3 + n_1 - n_2 \), we have
\[
1 \leq \hat{m} \leq x_2 + z_2 - 1 \quad \text{and} \quad 1 \leq \hat{n} \leq y_3 + x_3 - 1.
\]

3. To study (6), we define several numbers as follows:

For \((\hat{m}, \hat{n}) \in \mathbb{Z}^2\), we take \( r, s \in \mathbb{Z} \) such that
\[
\hat{m} \equiv -a_2 r \pmod{x_2} \quad \text{and} \quad \hat{n} \equiv -a_3 s \pmod{x_3}.
\]

We put \( x_2 = dX_2 \) and \( x_3 = dX_3 \). And we take \( \lambda \in \mathbb{Z} \) for which
\[
-\lambda = rX_2 + sX_3 \pmod{dX_2 X_3}
\]
\[
(-dX_2 X_3 + F)/2 < \lambda \leq (dX_2 X_3 + F)/2.
\]

Finally we put
\[
\chi (\hat{m}, \hat{n}) = (y_1 X_3 \hat{m} + z_1 X_2 \hat{n} - q \lambda) / dX_2 X_3.
\]

**Lemma 1.** \( \chi (\hat{m}, \hat{n}) \equiv -a_1 \hat{a}_2 \hat{m} - a_1 \hat{a}_3 \hat{n} \pmod{q} \).

**Proof.** We refer to the proof of Lemma 1 of IV.

**Lemma 2.** We put \( (F \hat{m} - z_2 \lambda) / X_2 = a \) and \( (F \hat{n} - y_3 \lambda) / X_3 = b \).

Then \( a \) and \( b \) are integers. And we have
\[
\chi (\hat{m}, \hat{n}) = (ay_1 + bz_1) / f.
\]

**Proof.** We refer to the proof of Lemma 2 of IV.

**Lemma 3.** Take \((\hat{m}, \hat{n}) \in \mathbb{Z}^2 \) which satisfy (7). Assume that
\[
2f < x_2 x_3 \quad \text{and} \quad a_i \geq 4 \quad (1 \leq i \leq 3).
\]

Then the relation \( \chi (\hat{m}, \hat{n}) \in [1, y_1 + z_1 - 1] \pmod{q} \) implies \( 1 \leq \chi (\hat{m}, \hat{n}) \leq y_1 + z_1 - 1 \).

**Proof.** It is sufficient to prove \(-q + y_1 + z_1 \leq \chi \leq q \).

First we prove \( \chi \leq q \). Note that \( \lambda \geq (-dX_2 X_3 + F) / 2 \) and \( \hat{m} \leq x_2 + z_2 - 1, \hat{n} \leq x_3 + y_3 - 1 \). Thus by (9) we have
\[
\chi \leq q/2 + y_1 (x_2 + z_2 - 1) / z_2 + z_1 (x_3 + y_3 - 1) / x_3 - fq/2x_2 x_3.
\]

Thus by (4), we have \( q - \chi \geq (1 - f/x_2 x_3) q/2 - (y_1 + z_1) \).

By (10), we see easily this value \( \geq 0 \).

A similar reasoning works for \(-q + y_1 + z_1 \leq \chi \).

The condition given in (10) exclude only very particular cases. Thus in the following we assume (10) to be satisfied.

**Lemma 4.** Assume that three numbers \( \hat{m}, \hat{n} \) and \( \chi (= \chi (\hat{m}, \hat{n})) \) are
given for which

(11) \[ 1 \leq \chi \leq y + z - 1, \ 1 \leq \hat{m} \leq x + z - 1, \ 1 \leq \hat{n} \leq x + y - 1. \]

We take \( m, n \) which satisfy the following (12).

\[
\begin{align*}
\text{Max} (\chi - y, 0) & \leq m_1 \leq \text{Min} (\chi - 1, z - 1), \\
\text{Max} (\hat{m} - \hat{n}, 0) & \leq m_2 \leq \text{Min} (x + z - \hat{m} - 1, z - 1), \\
\text{Max} (\hat{n} - y, 0) & \leq n_1 \leq \text{Min} (x - 1, \hat{n} - 1), \\
n_2 & = n_1 + y - \hat{n}.
\end{align*}
\]

By substituting these values in (5), we obtain mutually disjoint sequences

\[ S(q, a_1, b_1), S(q, a_2, b_2) \text{ and } S(q, a_3, -1). \]

Proof. For a triple \((\chi, \hat{m}, \hat{n})\) which satisfies (11), we take \( m, n \) which satisfy (12).

Then it is easy to see that

(i) \( m, n \) satisfy the inequalities of (5).

(ii) \( \hat{m} \leq m_1 + x - m_1 + y - y \)

By (5) and (6), \((S_1, S_2, S_3)\) is a disjoint triple.

By Lemmas 3 and 4, the problem to list up disjoint triples for given \( q, a_i \)

\( 1 \leq i \leq 3 \) is reduced to the problem to list up \((\chi, \hat{m}, \hat{n})\) which satisfies

(13)

\[ \hat{a}_1 \chi + \hat{a}_2 \hat{m} + \hat{a}_3 \hat{n} \equiv 0 \pmod{q} \]

Now we take all \((a, b, c)\) which satisfy the following (13);

\[
\begin{align*}
f & \equiv ay_1 + bz_1 \equiv (y_1 + z_1 - 1) f, \\
f & \equiv ax_2 + cz_2 \equiv (x_2 + z_2 - 1) f, \\
f & \equiv bx_3 + cy_3 \equiv (x_3 + y_3 - 1) f, \\
ay_1 + bz_1 & \equiv ax_2 + cz_2 \equiv bx_3 + cy_3 \equiv 0 \pmod{f}.
\end{align*}
\]

Lemma 5. We assume \((q, f) = 1\). Then the relation (13) gives a one to one correspondence between \((\chi, \hat{m}, \hat{n})\) which satisfies (13) and \((a, b, c)\) which satisfies (13).

Proof. First note that the determinant of the mapping (13) is \( f \). From

\((\chi, \hat{m}, \hat{n})\), we obtain \((a, b, c)\) by the process stated above. Since \((\chi, \hat{m}, \hat{n})\)

\( \in \mathbb{Z}^3 \), we obtain the congruence relation of (13).

Conversely, we start from \((a, b, c)\) with (13). Then by (13), we obtain
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$(\chi, \hat{m}, \hat{n}) \in \mathbb{Z}^3$ with the inequalities given in (8). Thus it is sufficient to ascertain the congruence relation. Since $(q, f) = 1$, we consider $(\hat{a}_1 \chi + \hat{a}_2 \hat{m} + \hat{a}_3 \hat{n}) f$. As easily seen, it is $a (\hat{a}_1 y_1 + \hat{a}_2 x_2) + b (\hat{a}_1 z_1 + \hat{a}_3 x_3) + c (\hat{a}_2 z_2 + \hat{a}_3 y_3)$. By (3), this value $\equiv 0 \pmod{q}$.

Here we note the following fact;

By (8) we see $(x, \hat{m}, \hat{n})$ has a symmetrical nature (in spite of their different appearances). Thus to apply Lemma 3, it is sufficient if one of the three inequalities $f < 2x_1 x_3$, $f < 2y_1 y_3$, $f < 2z_1 z_2$ is satisfied. As easily seen by (3), the relation $q > 16f^2$ implies at least one of them.

Now by summing the above discussions, we obtain

**Proposition 1.** For given $q, a_i (1 \leq i \leq 3)$, we assume $(\#)$, $(q, f) = 1$, $a_i \geq 4 (1 \leq i \leq 3)$ and $q > 16f^2$. Then the problem to list up all the disjoint triple $(S(q, a_1, b_1), S(q, a_2, b_2), S(q, a_3, b_3))$ is equivalent to list all $(a, b, c)$ which satisfies (*).

We close this paper with a remark. As shown in Lemma 3, we obtain a class of $(b_1, b_2, -1)$ from $(x, \hat{m}, \hat{n})$. Conversely starting from $(b_1, b_2, -1)$ of a disjoint triple, we obtain a unique corresponding $(x, \hat{m}, \hat{n})$. We discuss this property in the forthcoming paper.

**References**