Disjoint sequences generated by the bracket function VI

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1. Introduction. This paper is intended to be the final one of the series of papers with the same title [1]. Thus we survey first the results obtained and the problems remained open in these papers. We refer to the papers in [1] by its number I - V.

The sequences treated in these papers are of the following type, which is called a rational Beatty sequence. Let \( \mathbb{N}, \mathbb{Z} \) mean as usual. We take \( q, a \in \mathbb{N} \) and \( b \in \mathbb{Z} \). We put

\[
S(q, a, b) = \{[(qn + b)/a] : n \in \mathbb{Z}\}
\]

where \([x]\) denotes the greatest integer \( \leq x \).

Our theme is the disjointness properties of Beatty sequences. We treat the following three problems.

Problem A. Let \( q_i, a_i \in \mathbb{N} \) for \( i = 1, \ldots, k \) be given. Under what condition the system of sequences \( S(q_i, a_i, b_i) \) \((1 \leq i \leq k)\) can be made disjoint by taking suitable \( b_i \) \((1 \leq i \leq k)\)?

Problem B. Assume that the condition of Problem A is satisfied. List up all the set \( b_i \) \((1 \leq i \leq k)\) for which the sequences \( S(q_i, a_i, b_i) \) \((1 \leq i \leq k)\) are mutually disjoint.

Problem A and B are twin ones. The third one is of somewhat different type.

Problem C. Let \( q_i, a_i \in \mathbb{N} \) \((i = 1, 2)\) be given. Obtain a criterion to decide whether \( e_1 + e_2 \) sequences such as

\[
S(q_i, a_i, b_i^{(1)}) \leq i \leq e_1 \text{ and } S(q_j, a_j, b_j^{(2)}) 1 \leq j \leq e_2
\]

can be made disjoint by taking suitable \( b \)'s.

For the understanding of the background of the problems, we refer to [2], especially pp. 19-20. First we take up Problems A and B. We gave in I a complete solution of the problems for \( k = 2 \). Namely we obtained the following two Theorems.

Let \((a, b)\) mean the greatest common divisor of \( a \) and \( b \). We take \( q_i, a_i \in \mathbb{N} \) \((i = 1, 2)\) for which \((q_i, a_i) = 1\). And we put \((q_1, q_2) = q_1(a_1, a_2) = a \) and \( a_i = a_1 \) \((i = 1, 2)\).

**Theorem A.** Notations being as above, consider \( q_1, a_i \) \((i = 1, 2)\) are given. Two sequences \( S(q_1, a_1, b_i) \) \(and S(q_2, a_2, b_i)\) are disjoint with suitable two integers \( b_i \) \(and b_i\) if and
only if
(2) \( xu_1 + yu_2 = q - 2u_1u_2(a - 1) \)

holds with some \((x, y) \in \mathbb{N}^2\).

In case this condition is satisfied, we take a solution \((x_0, y_0)\) of (2) such that \(1 \leq y_0 \leq u_1\).

**Theorem B.** Assume that \(q_i, a_i (i = 1, 2)\) satisfy the condition of Theorem A. Then \(S(q_i, a_i, b_i) (i = 1, 2)\) are disjoint if and only if

\( u_i b_2 - u_2 b_1 \equiv (E_1 \cup E_2) \pmod{q} \)

where

\( E_1 = \{ u_1 x + u_2 y + u_1 u_2 (a - 1) : 0 \leq x \leq x_0 - 1, 1 \leq y \leq y_0 \} \)

\( E_2 = \{ u_1 x + u_2 y + u_1 u_2 (a - 1) : 0 \leq x \leq x_0 - u_2 - 1, y_0 + 1 \leq y \leq u_1 \} \).

(In case \( x_0 \leq u_2 \), we define \( E_2 = \emptyset \).)

In considering Problems A, B for \( k \geq 3 \), we treat only the most interesting case. Namely we assume

\((a_i, a_j) = 1\) for \(1 \leq i \neq j \leq k\).

For \( k = 3 \), we gave in I a partial result for Problem A. And a complete solution is given in IV. We quote here Theorem 4 of I, since we need it in the discussion of this paper. We take \(q_i, a_i \in \mathbb{N} (1 \leq i \leq 3)\) for which \((q_i, a_i) = (a_i, a_j) = 1\) for \(1 \leq i \neq j \leq 3\). And furthermore we assume

\((q_i, q_j) = q\) for \(1 \leq i \neq j \leq 3\).

If \(S(q_i, a_i, b_i) (1 \leq i \leq 3)\) are disjoint, we obtain by Theorem A the following three relations.

\[
\begin{pmatrix}
  x_1 & y_1 & 0 \\
  0 & x_2 & y_2 \\
  y_3 & 0 & x_3
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3
\end{pmatrix}
= \begin{pmatrix}
  q \\
  q \\
  q
\end{pmatrix}
\]

with \(x_i, y_i \in \mathbb{N} (1 \leq i \leq 3)\).

We define \(f\) by

\( x_1 x_2 x_3 + y_1 y_2 y_3 = qf. \)

Here we see easily the fact \(f \in \mathbb{N}\).

**Theorem C.** Notations being as above, the sequences \(S(q_i, a_i, b_i) (1 \leq i \leq 3)\) are disjoint with suitable \(b\)'s if and only if \(f \geq 2\) with a suitable solution system \((x_i, y_i) (1 \leq i \leq 3)\) of (3).

We remark that Theorem C lies on the line of the famous Minkowski Theory, which asserts the close relation between the existence of a diophantine solution and the volume of some set.

In V, we gave a solution of Problem B for \( k = 3\). As noted there, Problem B seems to be insignificant at a casual glance. But in many cases, Problem B becomes indispensable.
to solve other problems. And the result of \( V \) is essential to study the theory of disjoint quadruples of rational Beatty sequences, which is the theme of this paper. Hence we discuss the result in \( \S 2 \).

Before going to the theme of this paper, we give some remarks about our results concerning Problem C. Our investigation of the problem are in II and III. We solved there the problem for some cases. And for some cases, we proposed only a plausible conjecture. Thus our answer to Problem C is an incomplete one. But our present view is as follows. The most interesting result given in II and III is not the answer to Problem C itself, but the solution of two problems (\( \beta \)) and (\( \tau \)) given there. (We are going to discuss full detail of it in another place.) For more precise information, we refer to II, III or [3].

Now we explain the contents of this paper. Our theme is Problem A for \( k = 4 \). We initiated this study in IV. But the result given there is an inadequate one. And it turns out that there are new phenomena which did not appear in case \( k = 2, 3 \). Thus Problem A for \( k = 4 \) does not allow a simple answer. Recently we obtained a fairly satisfactory method to treat the problem, and also rather an unexpected byproduct. To report them is the aim of this paper.

In this paper we treat the problem under some restrictions. We believe that the interesting part of the problem is not spoiled by that. The first restriction is

\[ (q_i, a_i) = 1, (a_i, a_j) = 1, (q_i, q_j) = q \text{ for all } 1 \leq i \neq j \leq 4.\]

Under this condition, we see by Theorem A that our problem is equivalent to seek the criterion for disjointness of \( S(q, a, b) \) (\( 1 \leq i \leq 4 \)). Thus we denote \( S(q, a, b) \) simply by \( S \).

In case \( S_i(1 \leq i \leq 4) \) are mutually disjoint, we call \( \{ S_i \} \) as a disjoint system.

By Theorem A, we see that the relation \( S_i \cap S_j = \emptyset (1 \leq i \neq j \leq 4) \) implies the existence of the relations of the following type.

\[
\begin{pmatrix}
x_2 & y_1 & 0 & 0 \\
x_3 & 0 & z_1 & 0 \\
0 & y_3 & z_2 & 0 \\
x_4 & 0 & 0 & w_1 \\
0 & y_4 & 0 & w_2 \\
0 & 0 & z_4 & w_3
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
= \begin{pmatrix}
q \\
q \\
q \\
q
\end{pmatrix}
\]

with \( x_i, y_i, z_i, w_i \in \mathbb{N} \) (\( 1 \leq i \leq 4 \)).

In (6), \( x_i \) is the coefficient of \( a_i \) in the relation which combines \( a_i \) with \( a_i \). A similar rule is applied for \( y_i \) etc. We use \( x \) as a generic name for \( x \)'s, \( y \)'s, \( z \)'s and \( w \)'s.

Now we impose two conditions. The first one is

(\( F \)) Two \( x \)'s which lie in the same row of the matrix of (6) are relatively prime.

This condition means the relations in (6) cannot be reduced.

The second condition is the following (\( \# \)).

(\( \# \)) The relation (6) is the unique solution system for given \( q, a_i \) (\( 1 \leq i \leq 4 \)).

This condition allows us to consider only \( E_i \) of Theorem B. Namely by (\( \# \)), we are saved from troublesome classification of cases.
We state here our problem more precisely;

Let \( q, a_i \in \mathbb{N} \) \((1 \leq i \leq 4)\) be given, which satisfy (5), (6), (F) and (#). Decide whether \( S(q, a_i, b_i) \) \((1 \leq i \leq 4)\) can be made disjoint by taking suitable \( b_i \)'s, by using only the information included in (6).

We give here a rough sketch of our results. The quadruple \( \{S_i\} \) contains the following four triples as its subsets. We put

\[
T_1 = \{S_2, S_3, S_4\}, T_2 = \{S_3, S_1, S_4\}, T_3 = \{S_1, S_2, S_4\}, T_4 = \{S_1, S_2, S_3\}.
\]

Here the number \( i \) is put to \( T_i \) which does not contain \( S_i \). We consider in the following the order of \( S_i \) in each triple to be fixed. We define \( f_i \) \((1 \leq i \leq 4)\) as in (4) for each triple. Note that by (#), they are uniquely determined. Our final condition is

\[
(*) \quad a_i \geq 4, 16f_i < q, (q, f_i) = 1 \text{ for } 1 \leq i \leq 4.
\]

The former two conditions are not heavy. The third one is rather essential.

Now by Theorem C, we obtain a necessary condition for \( \{S_j\} \) to be a disjoint system. That is

\[
(7) \quad f_i \geq 2 \quad (1 \leq i \leq 4).
\]

Needless to say, (7) is not a sufficient condition. By studying some numerical examples, we are suggested the following two phenomena (cf. Remark (i), (ii) of IV, §4);

(H.1) For \( S(q, a_i, b_i) \) \((1 \leq i \leq 4)\) to be a disjoint system with large \( q \), some of \( f_i \)'s have to be also large.

(H.2) Conversely, if \( f_i \geq K \) \((1 \leq i \leq 4)\) holds with some \( K \), \( S(q, a_i, b_i) \) \((1 \leq i \leq 4)\) can be made a disjoint system.

As noted already in IV, (H.1) allows an exceptional type. In this paper, we call it a zero-system. We give a complete characterization of this type in Theorem 1, §4. We believed, for the time being, there is no other exceptional ones. But it turns out that there is another exceptional type. We name it a keen system. And excluding these two types, a proposition of (H.1) type holds (cf. Theorem 2, §6).

Note that in Theorems 1 and 2, we assert the possibility of exceptional ones. But it is a different problem to ascertain the existence of infinitely many exceptional ones, or to have a method to construct them. Inquiring this problem, we come upon rather unexpectedly a method to construct disjoint systems. By the method, starting from a disjoint system, we obtain new systems (including four parameters!) of a similar nature. If we note the difficulty in finding \( q, a_i \) \((1 \leq i \leq 4)\) which satisfies (6) and (#), we think the method to be rather astonishing. And we think the significance of it lies apart from the theory of Beatty sequences. Thus we discuss its details in another place. We explain in §8 the basic device of the method by treating an example.

Returning to our main concerns, we seek a proposition of (H.2) type. We tried several plans to obtain some theoretical results. In the present state, however, we need some numerical calculations to decide the existence of a disjoint system from (6). Thus we
confine ourselves to state comments about the plans we adopted, in the closing part of this paper.

2. Disjoint triples. As noted in §1, our theory of disjoint quadruples of Beatty sequences is founded on that of triples. Thus we need a precise theory of disjoint triples, including an answer to Problem B. Here we revisit the results of V, and add some more to them.

We take up $T_1 = \{S_2, S_3, S_3\}$. Namely we have

$$\begin{pmatrix} y_3 & z_2 & 0 \\ 0 & z_4 & w_3 \\ y_4 & 0 & w_2 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} q \\ q \\ q \end{pmatrix}. \tag{8}$$

Note that the condition (#) and (*) are imposed. We take $a_i \in \mathbb{N}$ for which $a_i \equiv 1 \pmod{q}$ ($1 \leq i \leq 4$), and consider them to be fixed. We first study the condition for $T_1$ to be disjoint. By translating $S_i$'s simultaneously, we may assume

$$b_i = -1. \tag{9}$$

Now by Theorem B and (#), we see that $S_2 \cap S_4 = S_3 \cap S_4 = \emptyset$ if and only if $b_2$ and $b_3$ are of the following form.

$$\begin{cases} b_2 \equiv m_2 + \hat{a}a_2 n_2 \pmod{q}, & 0 \leq m_2 < w_2, 0 \leq n_2 < y_4, \\ b_3 \equiv m_3 + \hat{a}a_3 n_3 \pmod{q}, & 0 \leq m_3 < w_3, 0 \leq n_3 < z_4. \end{cases} \tag{10}$$

We take $m, n \in \mathbb{Z}$ so that

$$w_3 - m_3 \leq \hat{m} < y_3 + w_3 - m_3 \quad \text{and} \quad \hat{n} = z_4 + n_2 - n_3. \tag{11}$$

We start our discussion with the following Lemma.

**Lemma 1.** Let $\hat{m}, \hat{n}, m_1$ and $n_1 (i = 2, 3)$ be as above. $T_1$ is a disjoint triple if and only if there exists $\chi \in \mathbb{Z}$ which satisfies the following two relations.

$$\begin{cases} \hat{a}_2 \chi + \hat{a}_3 \hat{m} + \hat{a}_4 \hat{n} \equiv 0 \pmod{q}, \\ m_2 < \chi \leq m_2 + z_2. \end{cases} \tag{12}$$

In case this condition is satisfied, the triple $(\chi, \hat{m}, \hat{n})$ is uniquely determined by $b_2$ and $b_3$.

**Proof.** The first assertion is Lemma 3 of V. Thus we consider the latter assertion. As stated above, we define the numbers by two steps.

$$b_2, b_3 \longrightarrow m_i, n_i (i = 2, 3) \longrightarrow (\chi, \hat{m}, \hat{n}). \tag{13}$$

By (#), we see the former map of (13) is injective. As for the latter mapping, $\hat{n}$ is uniquely determined by (11). Thus by (12), the value of $\hat{a}_2 \chi + \hat{a}_3 \hat{m} \pmod{q}$ is determined. Now again by (#), we see the second map is injective.

We denote the injection given in Lemma 1 by

$$\sigma(b_2, b_3, -1) = (\chi, \hat{m}, \hat{n}).$$

In the above discussion, we assumed (9). By that, our result lacks the symmetry with respect to $S_i (2 \leq i \leq 4)$. Here we try to regain the symmetry. We explain the notations
used. Let \( \{S(q, a_2, b_2), S(q, a_3, b_3), S(q, a_4, -1)\} \) be a disjoint triple. Assume that, by translating simultaneously the values of \( S_i \) (\( 1 \leq i \leq 4 \)), we obtain the triple \( \{S(q, a_2, -1), S(q, a_3, b'_3), S(q, a_4, b'_4)\} \). Let \( (\chi, \hat{m}, \hat{n}) \) and \( m_i, n_i \) (\( i = 2, 3 \)) be as above.

**Lemma 2.** Notations being as above, we have

\[
\begin{align*}
\hat{b}_3 &\equiv (\hat{m} + m_3 - w_3) + \hat{a}_3 a_3 (\chi - m_2 - 1) \pmod{q}, \\
\hat{b}_4 &\equiv (y_4 - n_2 - 1) + \hat{a}_2 a_4 (w_2 - m_2 - 1) \pmod{q}.
\end{align*}
\]

*Proof.* First we note the relation \( S(q, a_2, -1) = S(q, a_3, b_2) - a_2 (m_2 + a_4 a_2 n_2 + 1) \), which follows from (1) and (10). Thus we have \( b'_3 \equiv -a_3 a_2 (m_2 + a_4 a_2 n_2 + 1) + b_3 \equiv m_3 + a_3 \hat{a}_4 (n_3 - n_2) - a_3 a_2 (m_2 + 1) \pmod{q} \). Note that \( a_3 a_4 (n_3 - n_2) \equiv a_3 a_4 (z_4 - \hat{n}) \pmod{q} \). Thus by (12), we have the expression for \( b'_3 \). As for \( b'_4 \), we have \( b'_4 \equiv -a_4 a_2 (m_2 + a_4 a_2 n_2 + 1) - 1 \pmod{q} \). By noting \( y_4 + \hat{a}_2 a_4 w_2 \equiv 0 \pmod{q} \), we obtain (10).

**Lemma 3.** We start from \( \{S(q, a_3, b'_3), S(q, a_4, b'_4), S(q, a_2, -1)\} \) and carry the same discussion as above. Then we have \( \sigma(b'_3, b'_4 - 1) = (\hat{m}, \hat{n}, \chi) \). And if we start from \( \{S(q, a_3, b_3), S(q, a_2, b_2), S(q, a_4, -1)\} \), we have \( \sigma(b_3, b_2 - 1) = (y_3 + w_3 - \hat{m}, z_2 + w_2 - \chi, z_4 + y_4 - \hat{n}) \).

*Proof.* The second relation is easy to see. Thus we treat the first relation. We note that the coefficients of (14) satisfy the corresponding required inequalities of (10). And note the coefficients of (12) are of type \( \hat{a} \). Thus by the uniqueness theory of Lemma 1, and by following the process (13), we obtain the conclusion of Lemma.

Lemma 3 shows that the numbers \( \chi, \hat{m} \) and \( \hat{n} \), which are of different birth, have a symmetric nature.

Now we answer to Problem B for \( k = 3 \).

**Lemma 4.** Notations being as above, \( T_1 \) can be made a disjoint triple with suitable \( b \)'s if and only if there exists \( (\chi, \hat{m}, \hat{n}) \in \mathbb{Z}^3 \) which satisfies the following relations.

\[
\begin{align*}
\hat{a}_2 \chi + \hat{a}_3 \hat{m} + \hat{a}_4 \hat{n} &\equiv 0 \pmod{q}, \\
0 < \chi < x_2 + z_2, 0 < \hat{m} < y_3 + w_3, 0 < \hat{n} < y_4 + z_4.
\end{align*}
\]

And we obtain all the set \( \{b_2, b_3\} \) from \( (\chi, \hat{m}, \hat{n}) \), by reversing the process (13), of which disjoint triples are made with \( b_4 = -1 \).

*Proof.* From a disjoint triple \( T_1 \), we obtain \( (\chi, \hat{m}, \hat{n}) \) by following the above discussion. And it is easy to see that \( (\chi, \hat{m}, \hat{n}) \) satisfies (15). Conversely, we start from \( (\chi, \hat{m}, \hat{n}) \) which satisfies (15). Then as shown in Lemma 4 of \( V \), we obtain \( b_2 \) and \( b_3 \) which make a disjoint triple with \( b_4 = -1 \).

Thus Problem B for a triple is reduced to list up all \( (\chi, \hat{m}, \hat{n}) \) which satisfies (15). But Lemma 4 has a defect. Namely the essential quantity \( \delta \) does not appear in it. As a remedy
Disjoint sequences generated by the bracket function $\nu$ for that, we define a new triple $(a, \beta, \gamma)$ as follows.

\[ x = z_{\nu} + \beta z_4, \hat{\nu} = y_{\nu} + \gamma y_4, \hat{\nu} = y_4 + \gamma y_4. \]

And we define the set $W$ by

\[ W = \{(a, \beta, \gamma) : (f_{\nu}, f_{\beta}, f_{\gamma}) \in \mathbb{Z}_1, \text{with (Q) and (C)}, \] where

(Q) $0 < a z_{\nu} + \beta z_4 < z_2 + z_2, 0 < a y_{\nu} + \gamma y_4 < y_3 + y_3, 0 < \beta y_4 < \gamma y_4 < y_4 + z_4$.

(C) $a z_{\nu} + \beta z_4 \equiv a y_{\nu} + \gamma y_4 \equiv \beta y_4 + \gamma y_4 \equiv 0 \pmod{Z}$.

(Note that (Q) is satisfied by taking $(a, \beta, \gamma)$ so that all elements are contained in $(0, 1)$, or at most one of them is 0 or 1. We say in the case that $(a, \beta, \gamma)$ satisfies $(0, 1)$ condition.

Lemma 5. The relation (16) defines a bijection between $W$ and the set of $(x, \hat{\nu}, \hat{\nu})$ which satisfies (15).

Proof. This is the main result of $V$. Thus we refer to it.

3. Criterion for a disjoint system. In search of the criterion, we proceed first the same way adopted in IV. We start putting $b_4 = -1$. Then by Theorem B and (#), we see $S_1 \cap S_4 = \phi$ if and only if $b_i$'s are of the following form.

\[ b_i = m_1 + a_i \hat{a}_{i1} (\mod q), \quad 0 \leq m_1 < w_i, \quad 0 \leq n_1 < x_i, \]

\[ b_2 = m_2 + a_2 \hat{a}_{i2} (\mod q), \quad 0 \leq m_2 < w_2, \quad 0 \leq n_2 < y_i, \]

\[ b_3 = m_3 + a_3 \hat{a}_{i3} (\mod q), \quad 0 \leq m_3 < w_3, \quad 0 \leq n_3 < z_i. \]

Now again by Theorem B and (#), we see that $S_1 \cap S_2 = \phi$ if and only if $a_1 b_2 - a_2 b_1 \in \{a_i[0, x_2 - 1] + a_2[1, y_2]\}$ (mod q).

(We denote $[g, g + h] = \{g, g + 1, \ldots, g + h\}$, for $g \in \mathbb{N}$ and $h \in \mathbb{N} \cup \{0\}$.)

Thus substituting ($\nu$) and noting $\hat{a} y_{\nu} + \hat{a} z_4 \equiv 0 \pmod{q}$, we have

\[ \{-a_i \hat{a}_3[w_3 - m_3, y_3 + w_3 - m_3 - 1] - a_i \hat{a}_4(y_4 + n_1 - n_2)\} \cap [m_3 + 1, m_3 + y_3] \neq 0 \pmod{q}. \]

By a similar process, we see that $S_2 \cap S_3 = \phi$ if and only if

\[ \{-a_i \hat{a}_3[w_3 - m_3, y_3 + w_3 - m_3 - 1] - a_i \hat{a}_4(z_4 + n_2 - n_3)\} \cap [m_3 + 1, m_3 + z_3] \neq 0 \pmod{q}. \]

And we see that $S_3 \cap S_1 = \phi$ if and only if

\[ \{-a_i \hat{a}_3[w_3 - m_3, z_3 + w_3 - m_3 - 1] - a_i \hat{a}_4(x_4 + n_2 - n_3)\} \cap [m_3 + 1, m_3 + x_3] \neq 0 \pmod{q}. \]

Now we define $(\chi, \hat{\mu}, \hat{\nu})$ for each $T_i$ $(1 \leq i \leq 3)$ as in §2. Namely

\[ \sigma(b_{i+1}, b_{i+2}, -1) = (\chi, \hat{\mu}, \hat{\nu}) (1 \leq i \leq 3). \]

(The suffix $i+1, i+2$ are considered modulo 3.)

Note the relation.

\[ \hat{a}_3 \chi_1 + \hat{a}_3 \hat{\mu}_1 + \hat{a}_3 \hat{\nu}_1 = \hat{a}_3 \chi_2 + \hat{a}_3 \hat{\mu}_2 + \hat{a}_3 \hat{\nu}_2 = \hat{a}_3 \chi_3 + \hat{a}_3 \hat{\mu}_3 + \hat{a}_3 \hat{\nu}_3 = 0 \pmod{q}. \]

Lemma 6. $S(q, a_i, b_i)$ $1 \leq i \leq 4$ is a disjoint system if and only if $(\chi, \hat{\mu}, \hat{\nu})$ $(1 \leq i \leq 3)$ satisfy the following relations.

\[ m_2 < \chi_1 \leq m_2 + z_2, w_2 - m_2 \leq \hat{\mu}_1 < y_3 + w_3 - m_3, \hat{\nu}_1 = z_4 + n_2 - n_3, \]

\[ m_3 < \chi_2 \leq m_3 + x_3, w_1 - m_1 \leq \hat{\mu}_2 < z_1 + w_1 - m_1, \hat{\nu}_2 = x_4 + n_3 - n_1, \]
\( m_1 < \chi_3 \leq m_1 + y_1, w_2 - m_2 \leq \hat{m}_3 < x_4 + w_2 - m_2, \hat{n}_3 = z_4 + n_1 - n_2. \)

**Proof.** Note that the first term of \((19)\) is \(-a_1 \hat{m}_3 - a_2 \hat{n}_3 \equiv \chi_3 \pmod{q}\). Thus \((19)\) means that \(\chi_3 \in [m_1 + 1, m_1 + y_1]\) \(\pmod{q}\). As shown in \(V\), the condition \(\pmod{q}\) can be eliminated under \((*)\).

Now we give a tentative answer to Problem A for \(k = 4\).

**Lemma 7.** Let \(q, a_i \in \mathbb{N} \ (1 \leq i \leq 4)\) with \((#), (*)\) be given. Then \(S_i \ (1 \leq i \leq 4)\) can be made disjoint by suitable \(b's\) if and only if there exist triples \((\chi_i, \hat{m}_i, \hat{n}_i) \ (1 \leq i \leq 3)\) which satisfy \((20)\) and the following relations.

\[
\begin{align*}
(A) & \quad 0 < \chi_1 < w_2 + z_2, 0 < \hat{m}_1 < y_3 + w_3, 0 < \hat{n}_1 < z_4 + y_4, \\
& \quad 0 < \chi_2 < w_3 + x_3, 0 < \hat{m}_2 < z_1 + w_1, 0 < \hat{n}_2 < x_4 + y_4, \\
& \quad 0 < \chi_3 < w_1 + y_1, 0 < \hat{m}_3 < x_2 + w_2, 0 < \hat{n}_3 < y_4 + x_4.
\end{align*}
\]

\[
(\mathbb{N}) \quad \hat{n}_1 + \hat{n}_2 + \hat{n}_3 = x_4 + y_4 + z_4.
\]

\[
(J) \quad \begin{align*}
0 < \hat{m}_1 + \chi_2 & < z_1 + y_1 + w_1, \\
0 < \hat{m}_2 + \chi_1 & < x_2 + z_2 + w_2, \\
0 < \hat{m}_3 + \chi_3 & < y_3 + x_3 + w_3.
\end{align*}
\]

**Proof.** We obtain \((A), (\mathbb{N})\) and \((J)\) from \((21)\). Thus our problem is to obtain \(m_i, n_i \ (1 \leq i \leq 3)\) which satisfy \((18)\) and \((21)\) from \((\chi_i, \hat{m}_i, \hat{n}_i) \ (1 \leq i \leq 3)\) which satisfy \((A), (\mathbb{N})\) and \((J)\). We first consider \(m_i\). By taking all inequalities in \((21)\) that contains \(m_i\), we have

\[
0 < w_1 - \hat{m}_2 \leq m_1 < z_1 + w_1 - \hat{m}_1 \quad \text{and} \quad \chi_2 - y_1 \leq m_1 < \chi_3.
\]

Considering also the inequalities in \((18)\), we see that the inequality which has to be satisfied by \(m_1\) is

\[
(22) \quad \text{Max} \left(0, w_1 - \hat{m}_2, \chi_2 - y_1\right) \leq m_1 < \text{Min} \left(w_1, \chi_2, z_1 + w_1 - \hat{m}_1\right).
\]

To ascertain the existence of such \(m_1\), we check taking three terms of the left-hand side of \((22)\) and the right in turn. It is an easy task. A similar reasoning works for \(m_2\) and \(m_3\).

Now we take up \(n_i \ (1 \leq i \leq 3)\). We see the inequality to be satisfied by \(n_1\) is

\[
\text{Max} \left(0, \hat{n}_3 - y_4, x_4 - \hat{n}_2\right) \leq n_1 < \text{Min} \left(x_4, \hat{n}_3, x_4 + z_4 - \hat{n}_2\right).
\]

The existence of such \(n_1\) is easily checked. From this \(n_1\), we define \(n_2\) by \(n_2 = y_4 + n_1 - \hat{n}_3\) and \(n_3 = \hat{n}_2 - x_4 + n_1\). We can easily ascertain that these \(n_i \ (1 \leq i \leq 3)\) satisfy the desired properties.

Note that Lemma 7 lacks the symmetry with respect to \(S_i \ (1 \leq i \leq 4)\). In particular, \(T_4\) is concealed. To regain the symmetry is our next task. For the purpose, we take \(T_4 = \{S_1, S_2, S_3\}\) and define \((\chi_4, \hat{m}_4, \hat{n}_4)\) in a somewhat modified form. Namely we put

\[
(23) \quad \sigma(b_1, b_2, -1) = (y_1 + z_1 - \chi_4, z_2 + x_2 - \hat{m}_4, x_3 + y_3 - \hat{n}_4).
\]

We note that, by this modified definition, the relation satisfied by them is not altered. Namely we have

\[
(24) \quad a_1 \chi_4 + a_2 \hat{m}_4 + a_3 \hat{n}_4 = 0 \pmod{q}.
\]
Lemma 8. Let $q, a_i \in \mathbb{N} (1 \leq i \leq 4)$ be given as above. Then $S(q, a_i, b_i) (1 \leq i \leq 4)$ can be made disjoint by taking suitable $b$'s if and only if there exist $(x_{u_i}, m_i, n_i) (1 \leq i \leq 4)$ which satisfy (20), (24) and the following (A) and (N).

\[
\begin{align*}
(A) & \quad 0 < x_1 < w_2 + z_3, 0 < \hat{m}_1 < y_3 + w_3, 0 < \hat{n}_1 < z_4 + y_4, \\
& \quad 0 < x_2 < w_3 + x_3, 0 < \hat{m}_2 < z_1 + w_1, 0 < \hat{n}_2 < x_4 + z_4, \\
& \quad 0 < x_3 < w_1 + y_1, 0 < \hat{m}_3 < x_2 + w_2, 0 < \hat{n}_3 < y_4 + x_4, \\
& \quad 0 < x_4 < y_1 + z_1, 0 < \hat{m}_4 < z_2 + x_2, 0 < \hat{n}_4 < x_3 + y_3. \\
(N) & \quad \hat{m}_2 + x_3 + x_4 = y_1 + z_1 + w_1, \\
& \quad \hat{m}_1 + \hat{m}_3 + \hat{m}_4 = z_2 + w_2 + x_2, \\
& \quad \hat{m}_1 + \hat{m}_2 + \hat{n}_4 = w_3 + x_3 + y_3, \\
& \quad \hat{n}_1 + \hat{n}_2 + \hat{n}_3 = x_4 + y_4 + z_4.
\end{align*}
\]

Proof. We get (A), (N) and (J) of Lemma 7 easily from (A) and (N). Thus by Lemma 7, we have a disjoint system. Conversely, assume that we are given a disjoint system $(S_i)$. Then $T_i (1 \leq i \leq 4)$ are all disjoint triples. Thus we have (A). As mentioned above, we have the last relation of (N) by translating $S_i (1 \leq i \leq 4)$ simultaneously so that $b_4 = -1$. By using Lemma 3, we obtain the relations of (N) by letting $b_1 = -1, b_2 = -1$ and $b_3 = -1$ respectively.

Here by the same reason mentioned in §2, we reform the conditions of Lemma 8 as follows. We define the numbers $a_i, \beta_i, \gamma_i (1 \leq i \leq 4)$.

\[
\begin{align*}
(25) & \quad \begin{cases}
\chi_1 = a_1 z_2 + \beta_1 w_2, \hat{m}_1 = a_1 y_3 + \gamma_1 w_3, \hat{n}_1 = \beta_1 y_4 + \gamma_1 z_4, \\
\chi_2 = a_2 x_3 + \beta_2 w_3, \hat{m}_2 = a_2 z_1 + \gamma_2 w_1, \hat{n}_2 = \beta_2 z_4 + \gamma_2 x_4, \\
\chi_3 = a_3 y_1 + \beta_3 w_1, \hat{m}_3 = a_3 x_2 + \gamma_3 w_2, \hat{n}_3 = \beta_3 x_4 + \gamma_3 y_4, \\
\chi_4 = a_4 y_1 + \beta_4 x_2 + \gamma_4 x_3, \hat{m}_4 = a_4 z_2 + \gamma_4 z_2, \hat{n}_4 = \beta_4 x_3 + \gamma_4 y_3.
\end{cases}
\end{align*}
\]

(We shall use $a$ as a generic name for $a$'s, $\beta$'s and $\gamma$'s.) By this reformulation, (A), (20) and (24) turns to conditions (Q) and (C) for each triple. About the reformulation of (N), we put

\[
(26) & \quad \begin{cases}
\mu_1 = \gamma_2 + \beta_3 - 1, \mu_2 = \gamma_3 + \beta_1 - 1, \mu_3 = \gamma_1 + \beta_2 - 1, \\
\mu_4 = a_4 + \alpha_1 - 1, \mu_5 = \beta_4 + \alpha_2 - 1, \mu_6 = \gamma_4 + \alpha_1 - 1.
\end{cases}
\]

Then as easily seen (N) becomes the following relation (§).

\[
(§) & \quad \begin{bmatrix}
w_1 & 0 & 0 & y_1 & z_1 \\
0 & w_2 & 0 & x_2 & 0 & z_2 \\
0 & 0 & w_3 & 0 & x_3 & y_3 \\
x_4 & y_4 & z_4 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4 \\
\mu_5 \\
\mu_6
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

In the following, (§) is the main concern of our investigation. To aid our study, we use the diagram (E) shown in Figure 1. And we call two $\alpha$'s which compose $\mu_i (1 \leq i \leq 6)$ a coupled pair.
4. zero system. Note that if all $\mu_i = 0$ $(1 \leq i \leq 6)$, then (§) is satisfied trivially. We call a disjoint system of this type a zero system. For the case, the sum of a coupled pair is 1. This implies the following property (S). (We put 0 = 0/1.) (S): The reduced denominators of coupled $\alpha$'s are equal. We seek in the following in what situation (S) can happen? For the purpose, we use (D)-diagram, which is made from (E) by putting the reduced denominator of $\alpha$ at each vertex.

We first study the properties of the reduced denominators of $\alpha$'s which belong to a disjoint triple. We take $T_1$ and $W$ as in (7). From $(\alpha, \beta, \gamma) \in W$, we take out the reduced denominator of each element, and make a new triple $(s, t, u) \in \mathbb{N}^3$. We denote this process by $r(\alpha, \beta, \gamma) = (s, t, u)$.

We use the following notations and terminology in this section and in Lemma 14 of §5. Let $p$ be a prime. For $a \in \mathbb{Z}$, $v(a)$ denotes the order of $p$-factor of $a$. (We put $v(0) = \infty$).

For $(s, t, u) \in \mathbb{N}^3$ and $m \in \mathbb{Z} - \{0\}$, we put

$$m \otimes (s, t, u) = (s/(s, m), t/(t, m), u/(u, m)).$$

**Lemma 9.** We take $T_1$. Let $W$ be the set given in (7). Then the set $r(W)$ is given by

$$\{m \otimes (f/w_2, w_3, f/(z_2, z_4), f/(y_3, y_4)) : 1 \leq m \leq f-1\}.$$

**Proof.** We note that (Q) holds if $(\alpha, \beta, \gamma)$ satisfies the (0, 1)-condition. Thus (C) is the essential condition for our problem. We put $a = a/f$, $\beta = b/f$, $\gamma = c/f$. Then $a, b, c \in \mathbb{Z}$. And (C) is equivalent to the following condition;

$$az_2 + bw_2 - ay_3 + cw_3 - by_4 + cz_4 = 0 \pmod{p^k}$$

holds for all prime $p$ for which $v(f) = k > 0$.

By (F), either $v(w_2)$ or $v(y_4)$ is 0. We assume $v(w_2) = 0$, and take $\hat{w}_2$ so that $\hat{w}_2w_2 \equiv 1 \pmod{p^k}$. From the first relation of (28), we have $b \equiv -\hat{w}_2z_2a \pmod{p^k}$. We show the relation

$$c \equiv \hat{w}_2(y_4 - y_3)a \pmod{p^k}.$$

From (7), we obtain
Disjoint sequences generated by the bracket function VI

\[
\begin{pmatrix}
a_2 \\
a_3 \\
a_4
\end{pmatrix} =
\begin{pmatrix}
z_4w_2 + z_2w_3 - z_2w_2 \\
y_3w_2 + y_4w_3 - y_3w_3 \\
y_3z_4 + y_4z_2 - y_4z_4
\end{pmatrix}
\]

Since \( a_i \in \mathbb{N} \), we have

\[
z_4w_2 + z_2w_3 - z_2w_2 \equiv y_3w_2 + y_4w_3 - y_3w_3 \\
\equiv y_3z_4 + y_4z_2 - y_4z_4 \equiv 0 \pmod{f}.
\]

Using (30), we see that \( c \) taken as (29) satisfies the latter two relations of (28). Conversely we see, by considering \((w_3, z_4) = 1\), that (28) determines uniquely the class of \( c \pmod{p^k} \).

Now again by (30), we have \( v(z_4) \geq \min (v(z_2), k) \). And by \((\ast)\), we have \( \min (v(z_2), v(z_4)) \leq k \). Thus by putting \( a = 1 \), we obtain the triple of (28) for \( m = 1 \). A similar reasoning works for the case \( v(y_4) = 0 \). As easily seen, \((a, \beta, \gamma) \in \mathbb{Z}^3 \). Thus we obtain all possible triples letting \( m \) (or \( a \)) run through 1 to \( f-1 \).

By virtue of the discussion above, we use in the following the matrix (31) to aid the study of (C).

\[
\begin{pmatrix}
w_2 & -z_2 & y_4 - y_3 \\
w_2 - w_3 & -z_4 & y_4 \\
& & z_4 - z_2 & -y_3
\end{pmatrix}
\]

We read (31) as follows: For \( p \) with \( v(f) = k > 0 \), we take an element in (31) which is relatively prime to \( p \). Then the row of (31) which contains that element shows the ratio \( \pmod{p^k} \) of \((f_\alpha, f_\beta, f_\gamma) \pmod{p^k} \). If there exist plural such elements in (31), their consistency follows from (30).

We say the triple given by putting \( m = 1 \) in (27) to be \emph{generic}.

\textbf{Lemma 10.} Take \((a, \beta, \gamma) \in \mathbb{W} \). We put \( r(a, \beta, \gamma) = (s, t, u) \). We assume \( s \leq t \leq u \).

Then \((s, t, u)\) is one of the following four types (0)-(3).

\((0)\) \((s, s, s)\) with \( s > 1 \).

\((1)\) \((s, t, t)\) where \( s \neq t \) \text{ and } \( s/t \).

\((2)\) \((s, t, u)\) where \( s, t, u \) all differ. \( s \mid u \) and \( t \mid u \). \( s/u, s/t \) = 1.

\((3)\) \((s, t, u)\), for which we put \( r = L.C.M\) \((s, t, u)\).

Here, \( r, s, t, u \) all differ. \( r/s, r/t, r/u \) are mutually disjoint.

\textit{Proof.} First we consider the generic triple. By Lemma 9, it is of type (i), depending the cardinalty \( j \) of the elements which are not \( f \). For general \( m \), we treat separating the cases by the type of the generic one.

(a) Let \((f, f, f)\) be the generic triple. Then we have \( m \otimes (f, f, f) = (f/ (m, f), f/ (m, f), f/ (m, f)) \).

(b) Let \((s, f, f)\) be the generic triple. For \( m \in \mathbb{Z} \), we put \( m' = m/ (m, s) \). Then it is easy to see that \( m \otimes (s, f, f) \) is of type (0) if \( f/s \) divides \( m' \), and of type (1) in other cases.

(c) Let \((s, t, f)\) of type (2) be the generic triple. We put \( s' = f/s \) and \( t' = f/t \). For \( m \)
If \( m \in \mathbb{Z} \), we put \( m' = m/ (m, s, t) \), and define \( j \) by

\[ j = \text{the cardinality of the relations satisfied between } s' \not\in m' \text{ and } t' \not\in m'. \]

Then it is easy to see that \( m \otimes (s, t, f) \) is of type \((i)\).

(d) Let \( (s, t, u) \) of type \((3)\) be the generic triple. Then by \( (F) \) and \( (27) \), we see \( r = f \). We put \( s' = f/s \), \( t' = f/t \) and \( u' = f/u \). And for \( m \in \mathbb{Z} \), we put \( m' = m/ (m, s, t, u) \). We define \( j \) as follows.

\[ j = \text{the cardinality of the relations satisfied among } s' \not\in m', t' \not\in m' \text{ and } u' \not\in m'. \]

Now a similar reasoning used above works, and we see that \( m \otimes (s, t, u) \) is of type \((i)\).

Note that our aim is to decide, for given \( q, a_i \) \( (1 \leq i \leq 4) \), the existence of a disjoint system and not to list up all of them. Thus in case there are plural choices of \( b \)'s, we satisfy with having a simple one. We say a zero system to be minimal, if the cardinality of the different prime numbers which appear in \((D)\)-diagram of the system is minimal (by choosing \( b \)'s).

**Lemma 11.** Let \( q, a \in \mathbb{N} \) \( (1 \leq i \leq 4) \) with \((F)\), \((#)\) and \((*)\) be given. Assume that there exists a zero system \( S (q, a, b) \) \( (1 \leq i \leq 4) \) by taking suitable \( b \)'s. Then up to the equivalence of the order of \( a \), we can choose a zero system, by taking suitable \( b \)'s, whose \((D)\)-diagram is one of the following three types.

![Figure 2](image-url)

**Proof.** As noted above, our interest is focussed on the minimal one. We note the following two facts.

(i) Lemma 10 implies that at most one element of \( (a, \beta, \gamma) \) is in \( \mathbb{Z} \).

(ii) We obtain a simple \((D)\)-diagram by multiplying a factor of \( f_i \) \( (1 \leq i \leq 4) \) for all triples simultaneously, unless the operation contradicts (i).

We consider the cases separating by the cardinality of 1-1 couples which appear in a minimal \((D)\)-diagram.

We treat first a \((D)\)-diagram which has no 1-1 couples. By considering (i) and (ii), we see that such \((D)\)-diagram contains a triple \((p, p, p)\) with a prime \( p \). Assume \( T \) to be of the
Disjoint sequences generated by the bracket function VI

type. We determine other denominators using (S) and Lemma 10. Then we obtain (A) of Figure 2.

Next we assume that exactly one 1-1 couple appears in a minimal (D)-diagram. Let this couple make $\mu_0$. By Lemma 10, we see that $T_1$ and $T_2$ of the diagram have to be of type (1). We put them as $(1, t, t)$ and $(1, u, u)$. We determine other denominators using (S) and Lemma 10. Then we obtain (B) of Figure 2 in case $(t, u) > 1$, and (C) in case $(t, u) = 1$.

Finally we show that there can not appear two 1-1 couples in a (D)-diagram. By (i) noted above, such (D) diagram has to be as shown in Figure 3 (up to the equivalence).

Now we note $t$ of Figure 3. Then by Lemma 9, we have $t | (y_3, y_4)$ and $(w_1, w_2)$. It contradicts (F).

Summing the discussions of this section, we give the following

**Theorem 1.** Let $q, a_i \in \mathbb{N}$ ($1 \leq i \leq 4$) be given. Assume that they satisfy (5), (6), (F), (#) and (*). Then $S(q, a_i, b_i)$ ($1 \leq i \leq 4$) can be made a zero system with suitable $b_i$'s if and only if one of the following two conditions is satisfied.

(i) There exist a prime $p$ and $k \in \mathbb{N}$ which satisfy, by rearranging the orders of $a_i$ (if necessary), $p | f_i$, $p^k | f_i$ $(2 \leq i \leq 4)$, and they allow either (A) or (B) of Figure 2 as a (D)-diagram.

(ii) There exist two primes $p$ and $r$ which satisfy, by suitable ordering of $a_i$ ($1 \leq i \leq 4$), the following (a) and (b).

(a) $p \nmid f_i$, $r \nmid f_i$, $pr \nmid f_3$ and $f_4$.

(b) They allow (C) of Figure 2 as a (D)-diagram.

**Proof.** The only if part of Theorem follows easily from Lemma 10. Thus we try to construct a zero system under the conditions given above. First we assume (i) of Theorem with (A) to be satisfied. We put $y_2 = 1/p^k$ and $\beta_3 = (p^k - 1)/p^k$. From these values, we determine $\alpha_2$, $\beta_2$ and $\alpha_3$, $\gamma_3$ so that they satisfy (0,1)-condition and (C). Next we put $\beta_4 = 1 - \gamma_3$, $\gamma_1 = 1 - \beta_2$, $\alpha_1 = 1 - \alpha_3$ and $\beta_1 = 1 - \alpha_2$. Finally we obtain $\alpha_i$ and $\gamma_i$ by (C) and (0,1)-condition.

To see that these value system satisfy (C) in $T_1$ and $T_4$, we use (31) and the following three matrices.

\[
(31) \begin{pmatrix}
W_3 & -X_3 & Z_4 - Z_1 \\
W_3 - W_1 & -X_4 & Z_4 \\
W_1 & X_4 - X_3 & -Z_1
\end{pmatrix}
\begin{pmatrix}
W_1 & -Y_1 & X_4 - X_2 \\
W_1 - W_2 & -Y_4 & X_4 \\
W_2 & Y_4 - Y_1 & -X_2
\end{pmatrix}
\begin{pmatrix}
Z_1 & -Y_1 & X_3 - X_2 \\
Z_1 - Z_2 & -Y_3 & X_3 \\
Z_2 & Y_3 - Y_1 & -X_2
\end{pmatrix}
\]

These three matrices has the same meaning of (31) for $T_1$ (for $T_2-T_4$ respectively). By separating the cases by the distribution of elements of (31) which are prime to $p$, and considering (30) and the definitions of $f_i$, we see that the value set given above are consistent.
Thus we have a zero system. For other cases, a similar reasoning works by starting the couple \((a_0, a_0)\).

We give here three numerical examples.

**Example 1.** \(q = 4200013, a_1 = 7079, a_2 = 3207, a_3 = 6673, a_4 = 20948\). Then we have \(x_2 = 413, x_3 = 171, x_4 = 543, y_1 = 398, y_2 = 546, y_4 = 1179, z_1 = 448, z_2 = 367, z_4 = 485, w_1 = 17, w_2 = 20, w_3 = 46\). From these values, we have \(f_1 = 6, f_2 = f_3 = 3, f_4 = 30\). We obtain a zero system of type (i), (B) of Theorem 1.

**Example 2.** \(q = 30011, a_1 = 521, a_2 = 193, a_3 = 107, a_4 = 607\). We have \(x_2 = 74, x_3 = 142, x_4 = 58, y_1 = 147, y_3 = 261, y_4 = 150, z_1 = 5, z_2 = 4, z_4 = 11, w_1 = 31, w_2 = 23, w_3 = 40\). From these values, we have \(f_1 = 3, f_2 = 2, f_3 = 6, f_4 = 18\). We obtain a zero system of type (ii) of Theorem with \(p = 3\) and \(r = 2\).

**Example 3.** \(q = 4200013, a_1 = 6673, a_2 = 7079, a_3 = 5237, a_4 = 5116\). We have \(x_2 = 448, x_3 = 485, x_4 = 13, y_1 = 171, y_3 = 543, y_4 = 383, z_1 = 184, z_4 = 68, z_4 = 125, w_1 = 804, w_2 = 291, w_3 = 693\). And \(f_1 = 9, f_2 = 12, f_3 = 33, f_4 = 12\). Thus \(f\)'s have a common divisor 3. But there exist no (D)-diagrams which correspond to zero systems.

5. **Further reformulation of \((\$)\).** To investigate disjoint systems of general type, we need further reformulation of \((\$)\). We introduce first the following numbers.

\[
\begin{align*}
G_1 &= (x_2y_3z_1w_3-x_3y_1z_4w_2)/q, \\
G_2 &= (x_2y_3z_4w_1-x_4y_1z_2w_3)/q, \\
G_3 &= (x_4y_3z_1w_2-x_3y_4z_2w_1)/q.
\end{align*}
\]

**Lemma 12.** \(G_i \in \mathbb{Z} (1 \leq i \leq 3)\). And they satisfy the following relations \((\Gamma)\).

\[
\begin{align*}
G_1 &= (x_2z_if_1-w_2z_if_4)/z_2 = (z_1w_3f_3-y_1w_2f_2)/w_1, \\
&= (y_4w_3f_4-x_3y_1f_1)/y_3 = (x_2y_4f_3-x_3z_4f_2)/x_3, \\
G_2 &= (x_2y_3f_2-x_4w_3f_4)/x_3 = (x_2w_1f_1-z_2w_3f_3)/w_2, \\
&= (y_3w_4f_4-x_3y_1f_1)/y_4 = (w_1z_4f_3-y_3z_2f_2)/z_1, \\
G_3 &= (x_4z_if_1-y_4z_if_2)/z_4 = (y_3z_if_3-y_4w_if_4)/y_1, \\
&= (w_2y_4f_4-x_3z_3f_3)/x_2 = (y_3w_2f_2-x_3w_1f_1)/w_3.
\end{align*}
\]

**Proof.** For the first assumption, we take up \(G_1\). We put \(K = x_2y_4z_1w_3-x_3y_1z_4w_2\). We take the following minor matrix of \((6)\).

\[
\begin{pmatrix}
1 & x_2 & y_1 & 0 & 0 \\
0 & x_3 & 0 & z_1 & 0 \\
0 & 0 & z_4 & w_3 & 0 \\
0 & y_4 & 0 & w_2 & 0
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
= 
\begin{pmatrix}
q \\
q \\
q \\
q
\end{pmatrix}
\]

The determinant of this matrix is \(K\). If \(K = 0\), there is no problem. Assume \(K \neq 0\). Then
by solving (34), we have a relation of the following type.
\[ a_i = \frac{q}{k} \text{ (polynomial of } x\text{'s) } (1 \leq i \leq 4). \]
Since \( a_i \)'s are relatively prime integers, we see \( q | K \). A similar reasoning works for \( G_2 \) and \( G_3 \).

For the latter assertion, we obtain it by (33) and the definitions of \( f_i \) (1 \( \leq i \leq 4 \)).

We give here an example of some \( G_t = 0 \).

**Example 4.** \( q = 503, a_1 = 17, a_2 = 35, a_3 = 24, a_4 = 43 \). Then we have \( x_2 = 9, x_3 = 7, x_4 = 22, y_1 = 10, y_3 = 13, y_4 = 7, z_1 = 16, z_2 = 2, z_4 = 12, w_1 = 3, w_2 = 6, w_5 = 5 \). Thus \( G_1 = 0, G_2 = 4, G_3 = 54 \).

**Lemma 13.** The following six vectors are solutions of (8). All solutions are generated by independent two of them.

\[
\begin{bmatrix}
0 & -z_4 f_4 & y_4 f_4 & z_3 f_1 & -y_1 f_3 & -G_1 \\
-z_4 f_4 & 0 & -x_4 f_4 & -z_2 f_2 & -G_2 & x_2 f_2 \\
y_4 f_4 & x_4 f_4 & 0 & -G_3 & y_3 f_3 & -x_3 f_3 \\
-z_1 f_1 & z_2 f_2 & G_3 & 0 & w_1 f_1 & -w_2 f_2 \\
y_1 f_1 & G_2 & -y_3 f_3 & -w_1 f_3 & 0 & w_3 f_3 \\
G_1 & -x_2 f_2 & x_3 f_3 & w_2 f_2 & -w_3 f_3 & 0
\end{bmatrix}
\]

**Proof.** Easy calculations.

Since \( \mu \)'s are obtained from \( \alpha \)'s by (30), they are rational numbers whose denominator are closely related to \( f \)'s. We introduce several sets of numbers: (Here \( \{a, b\} \) means the least common multiple of \( a \) and \( b \)).

\[
\tau_1 = \{f_2/(z_1, z_4), f_3/(y_1, y_4)\}, \quad \tau_2 = \{f_1/(z_2, z_4), f_3/(x_2, x_4)\}
\]

\[
\tau_3 = \{f_2/(x_3, x_4), f_1/(y_3, y_4)\}, \quad \tau_4 = \{f_3/(w_1, w_2), f_4/(z_1, z_2)\}
\]

\[
\tau_5 = \{f_2/(w_1, w_3), f_4/(y_1, y_3)\}, \quad \tau_6 = \{f_4/(x_2, x_3), f_1/(w_2, w_3)\}
\]

We put

\[
\varphi_i = \tau_i \mu_i \quad (1 \leq i \leq 6).
\]

Then we see \( \varphi_i \in Z \) (1 \( \leq i \leq 6 \)) by Lemma 9. We define \( R_i \) (1 \( \leq i \leq 6 \)) by

\[
\begin{align*}
R_1 \tau_1 &= f_2 f_3, \quad R_2 \tau_2 &= f_1 f_3, \quad R_3 \tau_3 &= f_1 f_2, \\
R_4 \tau_4 &= f_3 f_4, \quad R_5 \tau_5 &= f_2 f_4, \quad R_6 \tau_6 &= f_1 f_4, \\
D &= \text{G.C.M. of } R_i \quad (1 \leq i \leq 6).
\end{align*}
\]

And finally we define \( F_{ij} \) (1 \( \leq i + j \leq 6 \)) as follows.

\[
\begin{align*}
F_{12} &= Dz_4 f_3 / R_1 R_2, \quad F_{13} = -Dy_4 f_2 / R_1 R_3, \quad F_{14} = -Dz_2 f_2 / R_1 R_4, \\
F_{15} &= Dy_4 f_2 / R_1 R_5, \quad F_{16} = DG_1 / R_1 R_6, \quad F_{23} = Dx_4 f_1 / R_2 R_3, \\
F_{24} &= Dz_2 f_3 / R_2 R_4, \quad F_{25} = DG_2 / R_2 R_5, \quad F_{26} = -Dx_4 f_1 / R_2 R_6, \\
F_{34} &= DG_3 / R_3 R_4, \quad F_{35} = -Dy_3 f_2 / R_3 R_5, \quad F_{36} = Dx_4 f_1 / R_3 R_6, \\
F_{45} &= -Dw_1 f_4 / R_4 R_5, \quad F_{46} = Dw_2 f_4 / R_4 R_6, \quad F_{56} = -Dw_3 f_4 / R_5 R_6.
\end{align*}
\]
And $F_{ij} = -F_{ji}$ ($1 \leq i \neq j \leq 6$).

We introduce here the following terminology which we use in Lemma 14. For $a, b, c \in \mathbb{N} \cup \{0, \infty\}$, we say $(a, b, c)$ is an $L$-triple if the smaller two of them are equal. And in case all elements are not 0, we say the $L$-triple to be positive.

**Lemma 14.** Let $p$ be a prime. Then $v(F_{ij})$ ($1 \leq i \neq j \leq 6$) satisfy the following three properties.

(i) $v(F_{ij}) \geq 0$. Namely we have $F_{ij} \in \mathbb{Z}$.

(ii) Let $1 \leq i < j < k \leq 6$. Then $(v(F_{ij}), v(F_{jk}), v(F_{ki}))$ is an $L$-triple.

(iii) For any $i$ with $1 \leq i \leq 6$, $v(F_{ij}) = 0$ holds with some $j$.

**Proof.** As we cannot see through the true reason of this properties, we prove this Lemma by wholly elementary reasonings. Thus it becomes lengthy, and we state it by separating to nine steps.

**(Step 1)** We start by giving the value set $v(x)$ for all $x$ in (6). By (F) and the equivalence among the orderings of $S_i$, we have the following four possible types.

(A) $v(x_2) = a$, $v(y_3) = b$, $v(x_4) = c$, $v(y_4) = d$, $v(z_4) = f$.

(B) $v(x_2) = a$, $v(y_3) = b$, $v(x_4) = c$, $v(x_4) = d$, $v(y_4) = e$, $v(z_4) = f$.

(C) $v(x_2) = a$, $v(y_3) = b$, $v(z_4) = c$, $v(x_4) = d$, $v(y_4) = e$, $v(z_4) = f$.

(D) $v(x_2) = a$, $v(y_3) = b$, $v(z_4) = c$, $v(x_4) = d$, $v(y_4) = e$, $v(w_3) = f$.

Here $a$ to $f$ are non-negative integers. And other $v(x) = 0$.

**(Step 2)** Next we determine $v(f_i)$ and $v(G_i)$. These are easy calculations, and we record here the result of Case A only. (Throughout this proof, $\{t, u, \cdots, v\}$ denotes the minimum value of them.)

$$v(f_1) = \{b+f, e\}+t_1, v(f_2) = \{c+f, d\}+t_2,$$

$$v(f_3) = \{a+e, d\}+t_3, v(f_4) = \{a+b, c\}+t_4,$$

$$V(G_1) = \{a+e, c+f\}+g_1, V(G_2) = \{a+b+f, d\}+g_2,$$

$$V(G_3) = \{b+d, c+e\}+g_3.$$ 

Here $t$'s and $g$'s are non-negative integers. ($g$'s can be $\infty$.)

**(Step 3)** We first treat the case $t$'s and $g$'s are all 0. We calculate $v(R_i)$ ($1 \leq i \leq 6$) using the above values. We record here only the result of Case A. They are

$$v(R_1) = \{d, a+e, c+f\}, v(R_2) = \{a+b+f, d, a+e\},$$

$$v(R_3) = \{b+d, c+e, d+e, b+c+f\}, v(R_4) = \{c, d, a+b, a+e\},$$

$$v(R_5) = \{c, d, a+b\}, v(R_6) = \{a+b, c, a+e\}.$$ 

From these values, we obtain $v(D) = v(R_4)$.

**(Step 4)** We calculate $v(F_{ij})$ using the above values. And ascertain the properties stated in Lemma, by running through all combinations of $i$, $j$, and $k$. It is a long and tiresome work, but can be done using wholly elementary reasonings.

**(Step 5)** Next we assume some $t_i > 0$. To treat the case, we use (G) and the following simple fact (L).
Let $s, t, u \in \mathbb{Z}$. Assume that $s+t+u = 0$, then $(v(s), v(t), v(u))$ is an L-triple.

We explain the reasoning used by taking a typical one. We take up Case A, and assume $t_1 > 0$. Then we have $b+f=e$. Using this relation, we eliminate $e$ from (39). Next we apply (L) for the relation in (G) which contains $f_i$. Then we see that $(t_1, t_4, g_1), (t_1, t_3, g_2)$ and $(t_1, t_2, g_3)$ are all L-triples. Since $t_1 > 0$, these triples are either positive, or the remained two elements in it are 0. If some $t_i > 0$ for $2 \leq i \leq 4$, we can develop a similar discussion with it.

Note that the following six triples are obtained by the above process from (G).

- $(t_1, t_4, g_1), (t_1, t_3, g_2), (t_1, t_2, g_3)$,
- $(t_2, t_3, g_1), (t_2, t_4, g_2), (t_3, t_4, g_3)$.

As a conclusion of the above discussions, we have only three possible cases.

(a) All six triples of (40) are positive.
(b) Exactly one triple of (40) is positive.
(c) Exactly one of $t$'s is positive. Namely other $t$'s and $g$'s are all zero.

And we consider in the following these cases separately.

(Step 6) (In case all $t$'s and $g$'s are positive.) Here we take up Case A. From the assumption, we have the following seven relations from (39).

- $b+f=e, c+f=d, a+e=d, a+b=c, a+e=c+f, a+b+f=d, b+d=e+c$.

Note that only three of (41) are independent. Thus we take $a, b, d$ and $f$ as a base system, and express other $c, d$, and $e$ with them. Substituting these values in (39), we obtain

- $v(R_1) = a+b+f+(t_1, t_4), v(R_2) = a+b+f+(t_1, t_3), v(R_3) = a+2b+f+(t_1, t_2),
- v(R_4) = a+b+f+(t_1, t_4), v(R_5) = a+b+f+(t_1, t_2), v(R_6) = a+b+f+(t_1, t_3)$.

From these values, we obtain $v(D) = a+b+f+(t_1, t_2, t_3, t_4)$. Using these values, we calculate $v(F_{ij})$, and ascertain (i)-(iii) of Lemma, by running through all combinations of $i$, $j$, and $k$. A similar reasoning works for the other cases.

(Step 7) (The case where only one triple of (40) is positive.) As a sample, we take Case A, with positive L-triple $(t_1, t_4, g_1)$. Then we have three relations; $b+f=e, a+b=c$ and $a+e=c+f$. We take $a, b, d$ and $f$ as a base system, and eliminate $c$ and $e$ from (39). And we carry the same process mentioned above.

(Step 8) (The case exactly one of t's is positive.) We take up Case A with $t_i > 0$. Then we have $b+f=e$. Thus we eliminate $e$ from (39), and obtain the following values.

- $v(f_1) = b+f+t_i, v(f_2) = c+f+g, v(f_3) = a+b+f, v(f_4) = a+b, c, v(G_1) = a+b+f, c, f$, $v(G_2) = a+b+f, d, v(G_3) = b+d, b+c+f$.

From these values, we calculate $v(R_i) (1 \leq i \leq 6)$ and $v(D)$. And we prove (i)-(iii) of Lemma, by running through all combinations of $i$, $j$, and $k$.

(Step 9) (The case t's are all 0 and some g's are positive.) In this case the reasoning
used is same with that of Step 8. But we note here the following three facts, which simplify our discussion.

(i) Since \( t \)'s are all 0, the values \( v(t_i), v(R_i) \) and \( v(D) \) are same with that give in Step 3.

(ii) New phenomena arise only for \( v(F_{16}), v(F_{25}) \) and \( v(F_{34}) \). And property (i) of Lemma holds obviously.

(iii) Since the set \( \{1, 6\}, \{2, 5\} \) and \( \{3, 4\} \) are disjoint, we may consider the effect of \( g_1, g_2 \) and \( g_3 \) to be mutually independent.

After these long and tiresome steps, we conclude the proof.

**Lemma 15.** For \( 1 \leq i < j < k \leq 6 \), we have

\[ F_{ij}^k + F_{jk}^i + F_{ki}^j = 0. \]

*Proof.* Easy calculations, using Lemmas 12 and 13.

As noted above, \( G_i \) can be 0. But excluding \( F_{16}, F_{25} \) and \( F_{34} \), \( F_i \)'s are \( \neq 0 \). We develop our discussion by taking \( F_{12} \). We obtain from (42) the following relations.

\[ \varphi_k = (-F_{2k}^i F_{ik}^2) / F_{12} (3 \leq k \leq 6). \]

**Lemma 16.** There exists unique \( h \) which satisfies the following relations.

\[ F_{12}^h \equiv F_{2k} \pmod{F_{12}} \text{ for all } 3 \leq k \leq 6, \]

\[ 1 \leq h \leq F_{12}, (h, F_{12}) = 1. \]

*Proof.* We take \( p \) for which \( v(F_{12}) = a > 0 \). By (ii) and (iii) of Lemma 14, there exists \( k \) in \( 3 \leq k \leq 6 \) such that \( v(F_{1k}) = v(F_{2k}) = 0 \). Hence the congruence class of \( h \pmod{p^a} \) is determined by (42) of such \( k \). Accumulating these relations for all prime factors of \( F_{12} \), we get the unique \( h \) in \([1, F_{12}]\). And we have \( (h, F_{12}) = 1 \).

Next we show that this \( h \) satisfies all relations of (44). Let \( p \) be as above. We eliminate the common factors of \( F_{1k}, F_{2k} \) and \( F_{12} \) from (44). Here again by (ii) of Lemma 15, we see that \( p \)-factor remains at most one of them. Now we note the following relation (43), which is an easy conclusion of Lemma 13.

\[ F_{12} F_{uv} - F_{1k} F_{2v} + F_{1v} F_{2u} = 0 (3 \leq u \neq v \leq 6). \]

Now we apply (ii) of Lemma 15 for \( F_{1u}, F_{1v} \) and \( F_{uv} \) (\( i = 1, 2 \)), and we see that (45) assures the consistency of (44).

By virtue of Lemma 16, we may put

\[ \varphi_1 = a \text{ and } \varphi_2 = ha - F_{12} m \text{ with } (a, m) \in \mathbb{Z}^2. \]

By substituting these values in (43), we obtain

\[ \varphi_k = h_{ik} a - F_{1k} m \text{ where } h_{ik} = (F_{1k} h - F_{2k}) / F_{12} \in \mathbb{Z}. \]

Our investigation of \( \varphi_k (1 \leq k \leq 6) \) in the following depend on the expression (46) and (47).
We call them \((a, m)\)-expression of \(\varphi_k\) over \(F_{12}\). We give here some remarks.

(i) By (47), we see that \(\varphi_k \equiv h_{ik}\varphi_i \pmod{|F_{ik}|}\) is satisfied. This relation corresponds that of (44). But in some cases, the corresponding inequality does not hold.

(ii) There is \((a, m)\)-expression of \(\varphi\)'s over another \(F_{sk}\) \((\neq 0)\). It has a form of type

\[
\varphi_i = h_{0j}A - F_{0j}M, \quad \varphi_s = A.
\]

Here \(h_{0s}\) satisfies the congruence relation of above type.

(iii) In particular we have the following relation.

\[
(48) \quad h_{0i}h_{0i} \equiv 1 \text{ or } -1 \pmod{|F_{1i}|}.
\]

6. Estimates of \(\varphi\). Note that \(\varphi\)'s are made of \(\alpha\)'s, which satisfy (Q). Thus the value of \(\mu\) must remain in some limits. In this section, we estimate the limit. We start, as usual, with \(T_1\) and \(W_1\).

**Lemma 17.** We take \(T_1\). Let \((\alpha, \beta, \gamma) \in W\). Then the following inequalities hold.

\[
|\alpha| < w_2s_5(y_4+z_4)/q_f+1/2, \quad |\beta| < s_5z_4(y_3+w_3)/q_f+1/2, \quad |\gamma| < y_3y_4(z_2+w_2)/q_f+1/2.
\]

**Proof.** Easy deduction from (Q).

We define \(c_i - c_6\) as follows.

\[
\begin{align*}
    c_1 &= f_2f_3(y_1y_4(x_2+w_2)/q_f+z_1z_4(x_3+w_3)/q_f+1)/R_1, \\
    c_2 &= f_1f_3(x_2x_4(y_1+w_1)/q_f+z_2z_4(y_3+w_3)/q_f+1)/R_2, \\
    c_3 &= f_2(f_3(y_3y_4(z_2+w_2)/q_f+x_3x_4(z_1+w_1)/q_f+1)/R_3, \\
    c_4 &= f_2f_3(z_1z_2(x_3+y_3)/q_f+w_1w_3(x_4+y_4)/q_f+1)/R_4, \\
    c_5 &= f_2f_3(y_1y_3(x_2+z_2)/q_f+w_1w_3(x_4+z_4)/q_f+1)/R_5, \\
    c_6 &= f_1f_4(z_1z_2(y_3+y_3)/q_f+w_2w_3(y_4+y_4)/q_f+1)/R_6.
\end{align*}
\]

Now by Lemma 17 and the definitions of \(\varphi\)'s, we obtain

\[
|\varphi_i| < c_i \quad (1 \leq i \leq 6).
\]

(In usual cases, \(c_i\) is of size \(Kf_{ih}\) with \(1 < K < 5\).)

7. Keen systems. We combine (50) with (46).

Then we obtain the following inequality of diophantine type.

\[
|ha-Fm| < C \quad \text{for } |a| \leq B \text{ with } (a, m) \in \mathbb{Z}^2,
\]

where \(h, F \in \mathbb{Z}\), \(|h| \leq |F|\) and \((h, F) = 1\).

We say (51) to be a strong inequality in case the following relation holds.

\[
|F| > 2BC.
\]

To study the inequality, we quote some notations and results from the theory of continued fraction.

Let \(h_0, h_1, \ldots, h_6 \in \mathbb{N}\) be given. We put \([\varphi] = 1, [h_0] = h_0, [h_0, h_1, \ldots, h_i] = h_i, [h_0, \ldots, h_{i-1}]

\[
\begin{align*}
    & \in [h_0, \ldots, h_{i-2}] \text{ for } i \geq 1. \quad \text{For } [u] = [h_0, \ldots, h_s], \text{ we denote } [-u] = [h_s, \ldots, h_0], \text{ and } [u-] = \\
    & [h_0, \ldots, h_{s-1}] \text{ and } [-u-] = [h_s, \ldots, h_0]. \quad \text{Let } [v] \text{ be another continued fraction, and } b \in \mathbb{N}.
\end{align*}
\]
As easily seen, we have the following relations.

\[
\begin{align*}
[u-][u][u][u-] &= (-1)^s, \\
[u][v]+[u-][v] &= u, \\
[uv][v][u][u-v] &= v.
\end{align*}
\]

Lemma 18. Assume that \( h, F \) of (51) satisfy \( 1 \leq h < F \). We consider solutions of (51) such that \( (a, m) \in \mathbb{N}^2 \). And if (51) is a strong inequality, there exist at most one solution. And it is given as follows. Let \([h_0, ..., h_n] \) be the continued fraction of \( F/h \). We take \( t \) so that \([h_0, ..., h_{t-1}] \leq h < [h_0, ..., h_t] \). Then by putting \( a = [h_0, ..., h_{t-1}] \) and \( m = [h_{t-1}, ..., h_1] \), we have \( |a-Fm| < \). Thus to prove Lemma, it is sufficient to show the inequality \( [h_t, ..., h_n] > C \), where \( t \) is the number mentioned in Lemma. Again by (53), we have \( F - [h_0, ..., h_n] = [h_0, ..., h_{t-1}] [h_0, ..., h_n] + [h_0, ..., h_{t-2}] [h_{t-1}, ..., h_n] < 2[h_0, ..., h_n] [h_0, ..., h_{t-1}] \). Since \([h_0, ..., h_{t-1}] \leq B \), we have the desired inequality.

Let \( q, a_i \in \mathbb{N} \) (1 \( \leq i \leq 4 \)) with (6) and (#) be given. And \( F_{1j} \) and \( c_i \) (1 \( \leq i \neq j \leq 6 \)) be as above. We say \( q, a_i \) (1 \( \leq i \leq 4 \)) allows a keen system if there exist \((t, u) \) (1 \( \leq t < u \leq 6 \)) and \( q_i \) (1 \( \leq i \leq 6 \)) which satisfy the following relation and (50).

\[
|F_{1u}| > 2c_{c_0}.
\]

We note here two remarks.

(i) For a keen system with \((t, u)\), we consider the \((a, m)\)-expression over \( F_{1u} \). Then by taking \( \varphi_i = a > 0 \), we see by Lemma 18 that there exists only one system of \( \varphi \)'s.

(ii) In case there is a \((t, u)\) which satisfies (53), it is usual to appear plural such \((t, u)'s\). And in the case, the continued fractions of such \( F_{1u}/h_{1u} \) are of similar type. (cf. Example 5, stated later.)

Summing the above discussions, we obtain

Theorem 2. Let \( q, a_i \) (1 \( \leq i \leq 4 \)) be given with (5), (6), (F), (#) and (*). Let \( F_{1j} \) (1 \( \leq i \neq j \leq 6 \)) and \( c_i \) (1 \( \leq i \leq 6 \)) be given as above. Assume that there exists \((t, u)\) such that \( |F_{1u}| > 2c_{c_0} \) and \( 1 \leq t < u \leq 6 \). Then the quadruple \( \{S(q, a_i, b_i)\} \) allows no disjoint system, excluding the possibilities of either zero systems or that which corresponds \( \varphi \)'s of the keen system.

8. Construction of over systems. As stated in §1, we have a method to construct disjoint systems from a given disjoint system. We call them over systems of the original one. We explain here the basic device of the method by taking an example. We separate
the process to seven steps. We take the following system as an origin.

**Example 5.** $q=4200013$, $a_1=6673$, $a_2=7079$, $a_3=5237$, $a_4=15348$. Then the solution system is as follows. $x_2=448$, $x_3=485$, $x_4=13$, $y_1=171$, $y_3=543$, $y_4=383$, $z_1=184$, $z_2=68$, $z_4=125$, $w_1=268$, $w_2=97$, $w_3=231$. From these values, we have $f_1=3$, $f_2=4$, $f_3=11$, $f_4=12$. $R_i=R_2=R_3=1$, $R_4=4$, $R_5=12$, $R_6=3$, $D=1$.

(Step 1) We calculate the $(a, m)$-expression, and the continued fractions of $F/h$ as follows.

$$
\varphi_1 = a \\
\varphi_2 = 823a - 1375m \\
\varphi_3 = -917a + 1532m \\
\varphi_4 = -303a + 506m \\
\varphi_5 = 34a - 57m \\
\varphi_6 = 299a - 499m
$$

By taking $a=[1, 1, 2]=5$ and $m=[1, 2]=3$, we obtain $\varphi_1=5$, $\varphi_2=-10$, $\varphi_3=11$, $\varphi_4=3$, $\varphi_5=-1$, $\varphi_6=-2$.

(Step 2) We take the middle terms of the above continued fractions as parameters. Namely we put

$$
\varphi_2 = 823a - 1375m \\
\varphi_3 = -917a + 1532m \\
\varphi_4 = -303a + 506m \\
\varphi_5 = 34a - 57m \\
\varphi_6 = 299a - 499m
$$

(Step 3) By considering (48), we define other $(a, m)$-expressions of $\varphi_1$ by taking the reverse continued fractions.

$$
\varphi_2 = (30b+13)a-(50b+25)m \\
\varphi_3 = -(33c+26)a+(55c+25)m \\
\varphi_4 = -(9d+6)a+(15d+11)m \\
\varphi_5 = (3e+1)a-(5e+2)m \\
\varphi_6 = (6f+5)a-(10f+9)m
$$

(Step 4) By comparing the results of Step 2 and those of 3, we have all $(a, m)$-expressions for $\varphi$'s. We take $f$'s and $R$'s from the original one. Since the coefficients of $M$ is $F_{ij}$, we obtain the values of $x$'s. They are

$$
x_2 = 20(f_1-b)+8, x_3 = 22(f_1-c)+1, x_4 = (110(c-b)+39)/3, y_1 = 15e+6, \\
y_3 = 33(c-e)+15, y_4 = (55c+47)/4, z_1 = (60d+44)/11, \\
z_2 = (120(d-b)+28)/11, z_4 = (50b+25)/11, w_1 = 12(d-e)+4,
$$
\( w_2 = 6(f-d)+1, \ w_3 = 6(f-e)+3. \)

(Step 5) We rewrite the value system by taking \( y_i, y_4, z_i, z_4, w_2 \) as a base. Then other \( x's \) are represented homogeneously by them. Namely we have

\[
\begin{align*}
x_2 &= (20z_1+50w_2-66z_4)/15, \\
x_3 &= (121z_1+110w_2-48y_4)/30, \\
x_4 &= (40y_4-121z_4)/15, \\
y_3 &= (12y_4-11y_1)/5, \\
z_2 &= (11z_1-4y_1)/5, \\
w_1 &= (11z_1-4y_1+10w_2)/10.
\end{align*}
\]

(Step 6) From these \( x's \), we calculate \( q, a_i \ (1 \leq i \leq 4) \). Then we see that the following relation (55) is necessary, in order that they make a consistent system.

\[
10w_2 = -8y_1+8y_4+11z_1-22z_4.
\]

Substituting these relation in the values obtained in Step 5, we have

\[
\begin{align*}
x_2 &= (-40y_1+40y_4+110z_1-176z_4)/15, \\
x_3 &= (-44y_1+20y_4+121z_1-121z_4)/15, \\
x_4 &= (40y_4-121z_4)/15, \\
y_3 &= (-12y_4+12y_1)/5, \\
z_2 &= (10z_1-12z_4)/5, \\
w_1 &= (11z_1-11z_4-6y+4y_4)/5,
\end{align*}
\]

From these value set, we have

\[
\begin{align*}
q &= (88y_1^2z_4-120y_1y_4z_4-40y_4z_4^2-121y_1z_4^2+242y_1z_4^2) \\
&+80y_1^2z_4+220y_4z_4^2-352y_4z_4^2)/150, \\
a_i &= (-2y_1z_4+2y_4z_4-y_1z_4)/10, \\
a_2 &= (-40y_1z_4+8y_4z_4+40y_4z_4+110z_4^2-77z_4z_4-110z_4^2)/150, \\
a_3 &= (-44y_1^2+12y_1y_4+121y_1z_4+40y_1^2-22y_1z_4-110y_4z_4)/150, \\
a_4 &= (10y_4z_4-11y_1z_4-5y_4z_4)/15.
\end{align*}
\]

It is easy to see that these value set satisfies (6).

(Step 7) To obtain a disjoint system, there are some additional conditions. Namely \( q, a_i \ (1 \leq i \leq 4) \) are relatively prime natural numbers, and \( x's \) are in \( \mathbb{N} \). To consider these condition, we go back to the expression with \( a-f \). Note that (55) turns to the following (56).

\[
-10b+11c+12d-12e-6f+3 = 0.
\]

We remark the following three facts.

(i) As easily seen the following conditions have to be satisfied for \( x's \) to be in \( \mathbb{Z} \).

\[
b \equiv 5 \pmod{11}, \ c \equiv 3 \pmod{4}, \ b \equiv c \equiv 0 \pmod{3}, \ d \equiv 0 \pmod{11}.
\]

(ii) To satisfy the condition \( q, a_i \in \mathbb{N} \), we take \( x's \) so that their size are proportional to the original one.

(iii) If we can take \( q \) to be prime or to have only large prime factors, the condition (5) is satisfied.

By considering these facts, we obtain the following numerical example. We take \( b=258, \ c=279, \ d=330, \ e=113, \ f=516 \). Then we have \( q=4850146493=293.1229.134649 \) and \( a_1=681763, a_2=780009, a_3=717712, a_4=1655033 \). The solution set is \( x_2=5168, x_3=5215, x_4=783, x_1=1701, y_3=5493, y_4=3848, z_1=1804, z_2=788, z_4=1175, w_1=2608, w_2=1117, w_3=2421 \).

Under some calculations, we decompose \( \varphi's \) to \( \alpha's \) so that \( \alpha_1=\beta_1=1/3, \ \gamma_1=5/3, \ \alpha_2=\beta_2=1/4, \ \gamma_2=3/4, \ \alpha_3=1/11, \ \beta_3=\gamma_3=4/11, \ \alpha_4=1, \ \beta_4=\gamma_4=1/2 \). These value set satisfies (Q).
and (C) in each triple. Thus we obtain a disjoint set.

9. $\varphi$ to $\alpha$. To obtain a disjoint system from $\varphi$'s, we have to decompose them into $\alpha$'s which satisfy (Q) and (C) in each triple. As noted in §1, we failed to get some tolerable theory about it. Thus we explain here the barrier which we cannot overcome.

(i) About (Q), there is no problem if we can take $\alpha$'s which satisfy the $(0, 1)$-condition. This problem is closely related with the following problem: Let $a, b \in \mathbb{N}$. For $m \in [1, 2ab - 1]$, decide whether it can be expressible as $m = ax + by$ with $1 \leq x < b$ and $1 \leq y < a$. It is clear that the probability is $1/2$, and those $m$ which lie near the center have good chance. But to obtain a decisive criterion seems to be difficult.

(ii) About (C), we think the following proposition is plausible: Assume $D = 1$. Let a system of $\varphi$ be given. Then it can be decomposed to $\alpha$'s uniquely (up to modulo $\mathbb{Z}$) so that they satisfy (C) in each triple. And if $D > 1$, there are plural (D?) possibilities. We may adopt the reasoning used in the proof of Theorem 1, but the situation seems to be much more complicated.

References

