RECIPROCITY FORMULAS FOR \( p \)-ADIC DEDEKIND SUMS

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Dedicated to Professor Katsumi Shiratani on his 62nd birthday

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1. Introduction

In this paper, we consider the Carlitz's reciprocity formulas \( p \)-adically, using \( p \)-adic Dedekind sums.

For positive integers \( h, k \) and \( m \), the higher-order Dedekind sums are defined by

\[
S_{m+1}^{(r)}(h, k) = \sum_{a=0}^{k-1} \frac{B_{m+1-r} \left( \frac{a}{k} \right) B_r \left( \frac{ha}{k} \right)}{B_r \left( \frac{a}{k} \right)}, \quad 0 \leq r \leq m + 1,
\]

where \( B_n(x), n \geq 0, \) is \( n \)-th periodic Bernoulli function. For \( m \) even, we have \( S_{m+1}^{(r)}(h, k) = 0 \). Hence, in the sequel, we assume that \( m \) is odd positive integer. For \( r = 0 \) and \( r = m + 1 \), \( S_{m+1}^{(0)}(h, k) = k^{-m}B_{m+1} \) and \( S_{m+1}^{(m+1)}(h, k) = d^{m+1}k^{-m}B_{m+1}, \) \( d = (h, k) \), are essentially the Bernoulli numbers. Carlitz's reciprocity formula is written as

\[
S_{m+1}^{(r)}(h, k) = \sum_{a=1}^{m+1-r} \left( \begin{array}{c} m+1-r \\ r \end{array} \right) k^a S_{m+1}^{(m+1-r-a)}(h, k)
\]

provided \( (h, k) = 1 \) and \( 0 \leq r \leq m \). For \( r = 0 \), this formula reduces to Apostol's reciprocity theorem. And for \( r \geq 1 \), this derives several interesting formulas \([2]\).

2. \( p \)-adic Dedekind sums

Let \( p \) be a prime number. Let \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \) be the ring of rational \( p \)-adic integers and the field of rational \( p \)-adic numbers, respectively. The \( p \)-adic Dedekind sum is a \( \mathbb{Q}_p \)-valued analytic function on \( \mathbb{Z}_p \) which interpolates the numbers \( k^mS_{m+1}^{(r)}(h, k) \) with respect to \( m \). It was first constructed by Rosen and Snyder \([7]\) and generalized by the author.

Put \( e = p - 1 \) or 2 according as \( p > 2 \) or \( p = 2 \). Let \( \alpha \) be an even integer such that
0 < α ≤ e and let r, h, k be positive integers. Then, we obtained in [6] the following

**Theorem.** There exists a p-adic analytic function \( S_{p, α}(s; r, h, k) \) of variable \( s \in \mathbb{Z}_p \) which satisfies

\[
S_{p, α}(m; r, h, k) = k^m S^{(r)}_{m+1}(h, k) - p^{m-r} k^m S^{(r)}_{m+1}(ph, k),
\]

for all integers \( m \) such that \( m ≥ r \) and \( m + 1 ≡ α \pmod{e} \).

Let \( d \) be a positive integer. Then, from the formula

\[
S^{(r)}_{m+1}(dh, dk) = d^{-m} S^{(r)}_{m+1}(h, k),
\]

we see that

\[
S_{p, α}(s; r, dh, dk) = d^α S_{p, α}(s; r, h, k),
\]

for all \( s \in \mathbb{Z}_p \) and that, if \( p \mid k \),

\[
S_{p, α}(m; r, h, k) = k^m S^{(r)}_{m+1}(h, k) - k^m S^{(r)}_{m+1}(h, k/p),
\]

for \( m + 1 ≡ α \pmod{e} \), \( m ≥ r \). When \( (h, k) = 1 \), we denote by \( h^*_k \) a positive integer satisfying \( h^*_k h ≡ 1 \pmod{k} \). Then, since

\[
S^{(r)}_{m+1}(h, k) = S_{m+1}^{(m+1-r)}(h^*_k, k),
\]

it follows that, if \( (h, k) = (p, k) = 1 \),

\[
S_{p, α}(s; 1, h^*_k, k) = k^m s_m(h, k) - p^{m-1} k^m s_m(p^h, k),
\]

for all positive integers \( m, m + 1 ≡ α \pmod{e} \). Here,

\[
s_m(h, k) = \sum_{a=1}^{k-1} B_m \left( \frac{ha}{k} \right) = S_{m+1}^{(m)}(h, k), \quad (h, k) = 1,
\]

mean Apostol’s higher-order Dedekind sums. Hence, Rosen-Snyder’s functions \( S_p(s; h, k) \) are given by \( S_{p, α}(s; 1, h^*_k, k) \) when \( (h, k) = (p, k) = 1 \).

Now, we define \( S_{p, α}(s; r, h, k) \) for \( r = 0 \). Let \( L_p(s, ω^a) \) be Kubota-Leopoldt’s \( p \)-adic \( L \)-function for the character \( ω^a \), where \( ω \) is Teichmüller character on the group \( \mathbb{Z}_p^* \) of units in \( \mathbb{Z}_p \). We put

\[
S_{p, α}(s; 0, h, k) = -(s + 1)L_p(-s, ω^a),
\]

for any positive integers \( h \) and \( k \). Then, we have

\[
S_{p, α}(m; 0, h, k) = (1 - p^m) B_{m+1}
\]

\[
= k^m S^0_{m+1}(h, k) - p^m k^m S^0_{m+1}(ph, k),
\]

for \( m + 1 ≡ α \pmod{e} \), \( m > 0 \).

### 3. Reciprocity formulas

We assume that \( p \mid hk, (h, k) = 1 \). First, let \( p \mid k \). Let \( r ≥ 0, m + 1 ≡ α \pmod{e} \) and \( m ≥ r + 1 \). Then we have

\[
S_{p, α}(m; r + 1, h^*_k, k) = k^m S^{(r+1)}_{m+1}(h^*_k, k) - k^m S^{(r+1)}_{m+1}(h^*_k, k/p)
\]

\[
= k^m S^{(m-r)}_{m+1}(h, k) - k^m S^{(m-r)}_{m+1}(h, k/p).
\]

Similarly we have
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\[
S_{p,a}(m; r, h_k^*, k) = k^m S_{m+1}^{(m+1-r)}(h, k) - k^m S_{m+1}^{(m+1-r)}(h, k/p),
\]
and

\[
S_{p,a}(m; j, h_j^*, h) = h^m S_{m+1}^{(j)}(k_j^*, h) - p^{m-j} h^m S_{m+1}^{(j)}(p k_j^*, h)
= h^m S_{m+1}^{(m+1-j)}(k, h) - p^{m-j} h^m S_{m+1}^{(m+1-j)}(k/p, h),
\]
for \( 0 \leq j \leq r + 1 \). Hence

\[
\left( m + 1 \right) S_{p,a}(m; r + 1, h_k^*, k) + \left( m + 1 \right) S_{p,a}(m; r, h_k^*, k)
- \left( m + 1 \right) h^{r+1} \sum_{j=0}^{r+1} \left( r + 1 \right) (-k)^j S_{p,a}(m; j, h_k^*, h)
= \left( m + 1 \right) h^m S_{m+1}^{(m-r)}(h, k) - \left( m + 1 \right) k^m S_{m+1}^{(m+1-r)}(h, k)
- \left( m + 1 \right) h^{r+1} \sum_{j=0}^{r+1} \left( r + 1 \right) (-k)^j S_{m+1}^{(m+1-j)}(k, h)
- p^m \left( \left( m + 1 \right) h(k/p)^m S_{m+1}^{(m-r)}(h, k/p) + \left( m + 1 \right) (k/p)^m S_{m+1}^{(m+1-r)}(h, k/p) \right)
- \left( m + 1 \right) h^r \sum_{j=0}^{r} \left( m + 1 - r \right) \left( 1 - p^{m-j} h^m \right) S_{m+1}^{(m+1-r-j)}(-k) B_{m+1-j} B_j,
\]
for \( r \geq 0 \) and \( m \geq r + 1 \) such that \( m + 1 \equiv \alpha \pmod{e} \). The right hand side is written formally as

\[
\left( m + 1 \right) \{ B' (Bh - B'k)^{m+1-r} - p^m B' (Bh - B'k/p)^{m+1-r} \}.
\]

Next, let \( p \mid h \). Then, for \( m + 1 \equiv \alpha \pmod{e} \), \( m \geq r + 1 \), \( r \geq 0 \),

\[
S_{p,a}(m; r + 1, h_k^*, k) = k^m S_{m+1}^{(r+1)}(h_k^*, k) - p^{m-r-1} k^m S_{m+1}^{(r+1)}(p h_k^*, k)
= k^m S_{m+1}^{(m-r)}(h, k) - p^{r-1} k^m S_{m+1}^{(m-r)}(h/p, k),
\]

\[
S_{p,a}(m; r, h_k^*, k) = k^m S_{m+1}^{(m+1-r)}(h, k) - p^{m-r} k^m S_{m+1}^{(m+1-r)}(h/p, k)
\]
and, for \( 0 \leq j \leq r + 1 \),

\[
S_{p,a}(m; j, h_j^*, h) = h^m S_{m+1}^{(j)}(k_j^*, h) - p^{m-j} h^m S_{m+1}^{(j)}(p k_j^*, h)
= h^m S_{m+1}^{(m+1-j)}(k, h) - h^m S_{m+1}^{(m+1-j)}(k/h, p).
\]
Hence, in this case, the left hand side of (12) is equal to
Therefore we obtain by (2)

**Theorem 2.** If \((h, k)\) and \(p | h\), then

\[
(m + 1)S_{p, \alpha}(m; r + 1, h^*, k) + (m + 1)S_{p, \alpha}(m; r, h^*, k)
\]

\[
- \sum_{j=0}^{r+1} \binom{r+1}{j} (-k)^j S_{m+1-r}^{(m+1-r)}(h, h)
\]

\[
= \sum_{j=0}^{m+1-r} \binom{m+1-r}{j} (1 - p^{-j}) h^{m+1-r-j} B_{m+1-j}.
\]

For all integers \(m \geq r + 1\) such that \(m + 1 \equiv \alpha \pmod{e}\). The right hand side of the above is written formally as

\[
(m + 1)B'(Bh - B'k)^{m+1-r} - p^{m-r}B'(Bh/p - B'k)^{m+1-r}.
\]

For \(r = 0\), the formula (13) becomes

\[
(m + 1)S_{p, \alpha}(m; 1, h^*, k) + S_{p, \alpha}(m; 0, h^*, k)
\]

\[
- \sum_{j=0}^{m+1} \binom{m+1}{j} (1 - p^{-j}) h^{m+1-j} (-k)^j B_{m+1-j}.
\]

Thus we get

\[
hS_{p, \alpha}(m; 1, h^*, k) + kS_{p, \alpha}(m; 1, h^*, h)
\]

\[
= (1 - p^m) m B_{m+1} + \frac{1}{m+1} \{(Bh - B'k)^{m+1} - p^m(Bh - B'k/p)^{m+1}\},
\]

for all \(m \geq 1\) such that \(m + 1 \equiv \alpha \pmod{e}\). This is a \(p\)-adic interpolation of Apostol's reciprocity law in the case of \(p | hk\).

If \(r = 1\), we see from (13) that

\[
(m + 1)S_{p, \alpha}(m; 2, h^*, k) + 2S_{p, \alpha}(m; 1, h^*, k)
\]

\[
- \sum_{j=0}^{m} \binom{m}{j} (1 - p^{-j}) h^{m+1-j} k B_{m+1-j}.
\]

for \(m \geq 3, m + 1 \equiv \alpha \pmod{e}\). We exchange \(h\) and \(k\) in (15), then we get
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(19) \[ mh^2S_{p, m}(m; 2, k_h^*, h) + 2kS_{p, m}(m; 1, k_h^*, h) \]
\[ = \sum_{j=0}^{m} \left( \frac{m}{j} \right)(1 - p^{j-1})k^{m+1-j}B_{m+1-j}B_j. \]

By subtracting, we obtain

(20) \[ m(h^2S_{p, m}(m; 2, k_h^*, h) - k^2S_{p, m}(m; 2, k_h^*, h)) \]
\[ = (m - 1)\{hS_{p, m}(m; 1, k_h^*, k) - kS_{p, m}(m; 1, k_h^*, h)\} \]
\[ + (B_h - B'h)(B_h + B'h)^m - p^n(B_h - B'h/p)(B_h + B'h/p)^m, \]
for $m \geq 3$, $m + 1 \equiv \alpha \pmod{p}$, where $p \nmid k$ and $(h, k) = 1$.

Finally, let $p \nmid hk$, $(h, k) = 1$. In this case we assume further that $p \equiv 1 \pmod{hk}$. Then we have

(21) \[ hS_{p, m}(m; 1, k_h^*, k) + kS_{p, m}(m; 1, k_h^*, h) \]
\[ = (1 - p^{m-1})\left( mB_{m+1} + \frac{1}{m + 1} \sum_{j=0}^{m-1} \left( \frac{m + 1}{j} \right)(-1)^jB_jB_{m+1-j}h^k \right), \]
for all positive integers $m$ such that $m + 1 \equiv \alpha \pmod{e}$. This is a direct extension of the result of Rosen-Snyder [7].

References


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