Fundamental Study on Free Vibration Problems of Plates with Non-homogeneity

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Chapter 1

General Introduction
1.1 The General Review

A plate is one of the fundamental composition components used for various structures. In the recent years, with the development of light-weight, high strength and high efficiency materials, such as composite materials or functionally graded materials, the plates made of non-homogeneous materials are increasingly used in aeronautical, mechanical and civil structures. Composite material is an anisotropic material that its characteristics are varying in different directions according to the combination of fibers. Composite materials have gained popularity in high-performance products such as aerospace components (tails, wings, fuselages, propellers), boat and scull hulls, and racing car bodies. More mundane uses include fishing rods and storage tanks. Carbon composite is a key material in today’s launch vehicles and spacecraft. It is widely used in solar panel substrates, antenna reflectors, payload adapters and heat shields. Functionally graded material (FGM) is a non-homogeneous material which compositions and/or functions are varying continuously or step-wisely from one side to other side. FGM is used to produce light-weight, strong and durable materials and is applicable to a broad range of fields such as structural material, energy conversion material. Especially, due to its light-weight, strong and durable property, FGM is very important for rocket and space station construction. FGM is also applicable to an outer wall of space plane and parts of rocket engine.

By adding non-homogeneity and anisotropy to materials, these composite materials or FGMs give advanced performance and advanced function to structure components, and make structure components with high efficiency. With the development of the light weight, high intensity and high efficient materials, the thin and weight saving of structures progress further, while plates with non-homogeneity, such as rectangular plates with variable thickness, non-rectangular plates, plates with a hole and plates with point supports, are used as structure components.

Due to the practical importance, the free vibration analysis of plates with non-homogeneity has received considerable attention. Free vibration of plates has been analyzed by many researchers. Usually two kinds of plate theories are used to analyze the plate problems. One is the classical plate theory (Kirchhoff theory). The other is thick plate theory such as Mindlin plate theory. The classical plate theory neglects the transverse shear
deformation. It is successfully used to thin plates. But for thick plates, it will make some errors because the shear deformation on thick plates is very important. Mindlin plate theory takes into account shear deformation. So it provides a more realistic alternative to Kirchhoff theory. Some of the important assumptions used in the development of Mindlin plate theory are: (1) the deflection of the mid-surface of the plate is small compared with the plate thickness, (2) transverse normal stresses are negligible, and (3) the normal to the mid-surface of the plates remains straight after deformation but not necessarily remains normal.

Based on the above two kinds of plate theories, lots of researchers have been done on free vibration of plates with non-homogeneous.

1.1.1 Research on free vibration of plates with point supports and resting on elastic foundations

Plates on elastic supports is a common model for several types of engineering structures and real life applications can be found for plate structures on different types of elastic supports. For example, plates on elastic foundation models are always used in the analysis of foundations of buildings, reinforced concrete pavements of highways, airport runways, and in the buckling of face metal sheet (in-plane) of sandwich panels. Plates on column supports find applications in column supported slabs, highway bridge decks, panels in ships and aircraft, and in the selection of optimum hold-down positions of solar panels.

According to the positions of the point supports, two kinds of the plates with point supports have been studied. One is the plate with point supports along the edges. Another is the plate with interior point supports. Saliba [1] used the superposition method to analyze the free vibration of rectangular cantilever isotropic plates with symmetrically distributed point supports along the free edges. After 4 years, Saliba [2] used the same method to analyze the free vibration of rectangular cantilever isotropic plates with symmetrically distributed interior point support. Bapat et al. investigated the free vibration of isotropic plates with symmetrical point supports along the edges [3] and with asymmetrical point supports along the edges [4]. The flexibility function approach and the impulse function approach were used to simulate the point supports. A comparison of these
two methods was also given and the advantage of the flexibility function method was shown. By using the same methods, Bapat and Suryanarayan [5] also studied the free vibration of isotropic plates with interior point supports. By dividing the plate into two sub-plates and satisfying the continuity conditions along the partition line and the compatibility condition of zero deflection at the point support, a set of equations were obtained. By utilizing the set of equations and the equivalent equations, the characteristic equation was obtained. Huang and Thambiratnam [6] used the finite strip element method to study the free vibration analysis of isotropic plates on elastic intermediate supports. The spring system was employed to simulate elastic intermediate supports. Results indicated that the spring system could simulate elastic intermediate supports such as point supports, line supports, local uniformly distributed supports and mixed edge supports. From the results, it was also evident that support stiffness and support areas had influence on the free vibration response of plates on line supports and local uniformly distributed supports. Kim and Dickinson [7] employed the Lagrangian multiplier method to analyze the free vibration of isotropic plates with arbitrarily located point supports. All of the above studies are limited to the isotropic plates with uniform thickness. Recently, Zhao, Wei and Xiang [8] used discrete singular convolution method (DSC) to solve isotropic plate vibration with irregular internal supports. Case studies were considered to the combination of a few different boundary conditions and irregular internal supports. The topologies of irregular internal supports were generated using an image processing algorithm. Completely independent verifications were conducted by using the established pb-2 Ritz method, which was available for two relatively simpler support patterns. The morphology of the first few eigen-modes is found to be localized to largest support-free spatial regions.

The fundamental frequency of vibration of circular and regular polygonal plates on a non-homogeneous foundation was obtained by Laura and Guri’rrez [9]. The Rayleigh-Ritz method was used. By using the same method, they [10] analyzed the transverse vibration of rectangular isotropic plates on non-homogeneous foundations. The boundary conditions were elastically restrained. The fundamental frequency coefficients were given for various aspect ratios and the modulus of the foundations. Based on Gâteaux differential, a mixed finite element formulation was derived and used to analyze
the static and dynamic problems of thin plate on elastic foundation by Omurtag et al. [11]. The numerical results were obtained for clamped and simply supported isotropic plates with variable thickness on Winkler or Pasternak foundations. By the same method, Omurtag and Kadioğlu [12] studied the free vibration of isotropic plates resting on Pasternak foundation and presented some numerical results for plates with simply supported boundary conditions. Matsunaga [13] used the method of power series expansion of the displacement components to investigate the vibration and stability of thick plates on elastic foundation. Based on the higher-order theory of thick plate, the natural frequency and the buckling stress were given for a simply supported square plate on a two-parameter elastic foundation and subjected to in-plane stress. Huang and Thambiratnam [14] used the finite strip method to analyze the static and dynamic responses of isotropic plates resting on elastic supports or elastic foundations. A spring system was used to simulate different elastic supports, such as elastic foundation, line and point elastic supports, and mixed boundary conditions. A three-span simply supported plate and a plate resting on Winkler elastic foundation were discussed. Ju, Lee and Lee [15] analyzed the free vibration of rectangular and circular plates with stepped thickness resting on non-homogeneous elastic foundations by using the finite element method. Based on Mindlin plate theory, the model included transverse shear deformation as well as bending extension coupling in cases of plates with stepped sections eccentrically located with respect to the mid-plane. The section of elastic foundation under a plate element was treated as a separate foundation element. The transverse deformation of these foundation elements was made to be consistent with the deflection of plate elements being supported, resulting in a consistent stiffness matrix for the elastic foundation. Natural frequency parameters and mode shapes of these isotropic plates were presented.

1.1.2 Research on free vibration of plates with variable thickness and with a hole

Varying thickness plates are frequently used in order to economize on the plate materials or to lighten the plates, especially when used in wings for high-speed, high-performance aircrafts. By carefully designing the thickness distribution, a substantial increase in stiffness, buckling and vibration capacities of the plate may be obtained over its uniform
thickness counterpart. Plates with varying thickness are also extensively used in modern structures due to their unique functions. For example, stepped plates possess a number of attractive features, such as material saving, weight reduction, stiffness enhancing, designated strengthening, fundamental vibration frequency increasing, etc.

Plates with holes are extensively used in aeronautical, mechanical and civil structures to lighten the structure and to obtain the convenient connection of structural members.

Bhat et al. [16] used four kinds of methods to obtain natural frequencies of isotropic rectangular plates with linearly variable thickness in one direction. A comparison of the results of the four methods was given. Roy and Ganesan [17] investigated the dynamic response of an isotropic square plate with linear or parabolic thickness variation in one direction. The effects of thickness variation on natural frequencies, dynamic displacements and stresses were considered. The Rayleigh–Ritz method was used to study the free vibration of isotropic rectangular plates with variable thickness in two directions by Singh and Saxena [18]. Liew and Lim [19] analyzed the free vibration of isotropic trapezoidal plates with variable thickness. They also analyzed the free vibration of isotropic doubly-tapered rectangular plates [20]. The Rayleigh–Ritz method was employed. The first eight frequencies were presented for plates with six kinds of boundary conditions and various aspect ratios. Liew et al. [21] presented a semi-analytical method to analyze the free vibration of isotropic rectangular plates with abrupt thickness variation in the central part. The frequency parameters and mode shapes were given for three boundary conditions. Sakata [22] utilized the double trigonometric series to obtain the characteristic equation of a clamped orthotropic rectangular plate with linearly varying thickness in one direction. The effects of aspect ratios and flexural rigidity on fundamental frequency were evaluated. Malhotra et al. [23] investigated the vibrations of orthotropic square plates with parabolic thickness variation in one direction. Rayleigh–Ritz method was used to obtain fundamental frequencies for four boundary conditions. Bambill et al. [24] used the Rayleigh–Ritz method and the finite element method to analyze the transverse vibration of an orthotropic rectangular plate with linearly varying thickness in one direction. Fundamental frequencies were presented for plates with a free edge. Bert and Malik [25] adopted a semi-analytical approach in the differential quadrature method to investigate free vibration of isotropic and
orthotropic rectangular plates with linearly varying thickness in one direction. They realized the information published on tapered orthotropic plates was very scant and they presented a number of numerical results for plates with two opposite edges simply supported. Ashour [26] studied the flexural vibration of orthotropic plates with variable thickness in one direction by employing the finite strip transition matrix technique. The frequencies were obtained for plates with two opposite edges having the same boundary conditions and the same thickness. But the boundary conditions of the two opposite edges were no longer restricted to simply supported conditions. Although the results obtained by Ashour were accurate enough for some boundary conditions, for the other boundary conditions, some of the frequency parameters, even the fundamental frequency parameter, seemed to be lost. Fan [27] studied the bending of the plate with variable thickness in one direction. The concentrated load and the uniform load were considered. Malhotra, Ganesan and Veluswami [23] investigated the vibrations of orthotropic square plates with parabolic thickness variation in one direction. Rayleigh-Ritz method was used to obtain fundamental frequencies for four kinds of boundary conditions. There are very few information about the free vibration problem of the orthotropic plates with variable thickness, especially for the orthotropic plates with general boundary conditions and variable thickness in two directions.

Plates with holes characteristics have been studied for many years, early studies were focused on isotropic plates. Paramasivam [28] extended a grid framework model to determine the effects of the holes on the fundamental frequencies. Numerical solutions were presented for isotropic square plates with square holes. Hegarty and Ariman [29] used a least-squares point-matching method to investigate the free vibration of the isotropic rectangular plates with a central circular hole. Clamped and simply supported plates were considered. The Rayleigh method was used to analyze the dynamic characteristics of plates with holes by Ali and Atwal [30]. Frequencies were shown for the isotropic simply supported square plates with square and rectangular holes. Aksu and Ali [31] proposed a definite difference formulation for the prediction of dynamic behavior of isotropic rectangular plates with holes. Experimental and theoretical frequencies were given for the plates with single hole or double holes. Compared with the study of isotropic plates with
holes, the study of orthotropic plates with holes is rather limited. Reddy [32] studied the large amplitude free vibration of layered composite plates with rectangular cutouts by finite element method. Frequencies corresponding to linear and non-linear situations were presented for thin and thick orthotropic and laminated composite plates. Avalos, Larrondo and Laura [33] obtained the frequency parameters for orthotropic rectangular plates with free-edge holes by using the Rayleigh-Ritz method. The effects of aspect ratio, hole side to plate side ratio and the position of the hole on the frequency properties were investigated. However, in these studies the effect of the variation of the thickness on frequency properties was not considered. Most previous investigations have been confined to plates with circular holes, and square or rectangular holes. Further, in these studies the positions of holes were limited to the central part of the plates.

1.2 The Purpose and the Outline of the Thesis

1.2.1 The purpose and present research work

From above section 1.1, the problem of free vibration of plates with non-homogeneity is very important since these plates are extensively used in aircraft structures, ship structures, traffic, buildings and other structures. Lots of researches have been done on the plates with non-homogeneity. Generally, the fundamental differential equations of the problems of plates with non-homogeneity are simultaneous partial differential equations with variable coefficients. These coefficients include the flexural rigidity of the plate, the different axial and shear modulus, the thickness of the plate and etc. It is almost impossible to obtain the analytical solutions. So many numerical methods such as Rayleigh-Ritz method, the finite element method, the supposition method, the finite strip element method and etc. are used to analyze the problems. These methods have many advantages in solving elastic bending and vibration problems of the plates. However, it is found that almost all the methods need prior assumption of the deflection. For some cases, they may encounter some difficulties such as the phenomenon of shearing locking, the corner problems and etc. In order to solve these problems, a special treatment is needed. On the other hand, the problems of plates with non-homogeneity are generally more difficult to be described easily and accurately. All these make the calculation more complex and need more computer memory space and
time. Moreover, there is very few information about the free vibration problem of the orthotropic plates with variable thickness in two directions. So in order to understand the dynamic behavior of such structure components with non-homogeneity which is different from homogeneous isotropic structure component, much more precise and more efficient method is needed even for the analytical method of a thin plate. It means in order to design and manufacture these new types of structure components safely and economically, it is indispensable to develop a more precise and efficient method for analyzing those structural mechanics characteristics.

In this thesis, under the concept of considering that the plate with non-homogeneity as a kind of rectangular plates with variable thickness, a new discrete method is developed to solve the structural mechanics actions, and the practicality is investigated also. The fundamental equations replacing the assumption of deflection are used as starting point. Throughout two steps, the discrete solution is directly obtained. The two steps are as follows.

- Transforming the differential equations into integral equations.
- Applying trapezoidal rule of numerical integration.

The solutions are obtained based on the discrete points. By increasing the number of crosswise line divided equally, the governing differential equations and boundary conditions are satisfied to any desired degree of accuracy in theory. So in fact, the discrete method is a semi-analytical method. The shear forces, the bending moments, the twisting moments, the slopes and the deflection of all the intersection of crosswise line can be obtained.

Using above discrete method, the free vibration problems of plates with non-homogeneity can be solved by using the Green function. The discrete-form solution for deflection of the plate with a concentrated load gives the discrete-type Green function of the plate. The characteristic equation of free vibration of rectangular plate with non-homogeneity can be obtained.

With the development of the computers, it is possible to obtain much more exact
solution than before. The quantities at any intersection point of crosswise line can be finally expressed by the quantities of the boundary points. So the computer time is saved and even not very large size digital computer can be used to solve the problem of plates with non-homogeneity by using the discrete method.

By using the discrete method, the following works have been carried out. The convergence and accuracy of the numerical solutions have been investigated.

1. Free vibration analysis of rectangular plates with variable thickness and point supports [34], [35]
2. Free vibration analysis of square plates resting on non-homogeneous elastic foundations [36], [37]
3. A discrete method for bending and free vibration analysis of orthotropic plates with non-uniform thickness [38], [39]
4. Free vibration analysis of orthotropic rectangular plates with variable thickness and general boundary conditions [40], [41]
5. Free vibration analysis of simple supported orthotropic square plates with a square hole [42]
6. Experimental research on free vibration of plates with variable thickness and a hole defect. [43], [44]

1.2.2 Outline of the thesis

This thesis is composed of six chapters which are as follows:

In chapter 1, general review of background of researches on free vibration problems of plates with non-homogeneity and the necessary of simple and high accuracy method for solving the problems are described. At the end, the outline of the thesis has been concluded.

In chapter 2, the basic theory is given out. At first, in order to obtain discrete solution equations of orthotropic plate with variable thickness and distributed load combined with a concentrated load, the fundamental differential equations of the orthotropic plates with variable thickness are converted into integral equations. By applying numerical integration, the discrete solutions based on the Mindlin plate theory are obtained. Secondly, the discrete
method is extended to analyze the free vibration of orthotropic plates with variable thickness by using the Green function. The discrete form solution for deflection of the orthotropic plate with a concentrated load gives the discrete type Green function of the plate. Thirdly, integral constants and boundary conditions of some rectangular plates are shown in figures. Fourthly, Comparison of FEM and proposed method is made. Some numerical results are analyzed for SSSS isotropic plate and CCCC isotropic plate under uniform load and a concentrated load. It is concluded that proposed method has the advantage of using less unknown quantities to obtain satisfactory accuracy and proposed method is closer to the exact solution than FEM. At last, the equivalent rectangular plate is obtained by proper transformation of the irregular-shaped and variable thickness plate.

In chapter 3, in order to verify the proposed discrete method described in Chapter 2, the experimental research is done with a laser holographic interferometry. Aluminum alloy plates are used as specimen. Plates with uniform thickness, with non-uniform thickness and with a hole defect are investigated. The experimental results are compared with analytical results and reference results. The convergence and accuracy of proposed discrete method are investigated.

In chapter 4, applying the basic theory described in chapter 2, the discrete method is used for analyzing the free vibration problems of isotropic plates with non-homogeneity. Firstly, the discrete method is used for analyzing free vibration of isotropic rectangular plates with multiple point supports. No prior assumption of shape of deflection, such as shape functions used in the finite element method, is employed. The efficiency and accuracy of the proposed method for the isotropic rectangular plates with uniform thickness and point supports by comparing the present results with those previously published and the effect of the point support on the frequency parameter of isotropic plate with variable thickness are investigated. Secondly, the same discrete method is extended for analyzing the free vibration problem of isotropic square plates with stepped thickness on non-homogeneous elastic foundations. The spring system is used to simulate the foundations. The characteristic equation of the free vibration is gotten by using the Green function. The convergence and accuracy are investigated.

In chapter 5, use the discrete method described in Chapter 2 to analyze the free vibration
problems of orthotropic plates with non-homogeneity. Some numerical analyses are carried out for the free vibration problems of orthotropic plates with non-uniform thickness. The efficiency, convergence and accuracy of the numerical solutions are investigated. Secondly, the discrete method is developed for analyzing the free vibration problems of orthotropic rectangular plates with general boundary conditions. The efficiency and accuracy for the free vibration problem of tapered orthotropic rectangular plates with general boundary conditions are investigated. The effects of the boundary conditions, the aspect ratio and variable thickness on the frequency parameter are discussed. Some new data and mode shapes for the orthotropic plates with general boundary condition and variable thickness in one or two directions are given. At last, the method is extended for analyzing the free vibration problem of simply supported orthotropic square plate with a square hole. In this section, a square plate with a square hole is transformed into an equivalent square plate with non-uniform thickness by considering the hole as an extremely thin part of the equivalent orthotropic plate. Therefore, the dynamic characteristics of the plate with a hole can be obtained by analyzing the equivalent plate. The effects of the side to thickness ratio, hole side to plate side ratio and variation of the thickness on the frequency properties are considered. Some numerical analyses are carried out for the simply supported orthotropic square plate with a square hole. The efficiency and accuracy of the numerical solutions are investigated.

In chapter 6, the conclusions of this work are summarized.
Chapter 2

Basic Theory
2.1 Discrete Method for Bending Problems of Orthotropic Plates

2.1.1 Fundamental differential equation of orthotropic plates with variable thickness

In order to obtain the fundamental differential equation of a rectangular orthotropic plate with variable thickness, an $x - y - z$ coordinate system (shown in Figure 2.1) is used with its $x - y$ plate contained in middle plate of a rectangular plate and the $z$-axis perpendicular to the middle plane of the plate. The thickness, the length and the width of the orthotropic plate are $h$, $a$ and $b$, respectively. The principle material axes of the orthotropic plate in the longitudinal, transverse and normal directions are designated as 1, 2 and 3.

In this chapter, an uniform load $q = q(x, y)$ and a concentrated load $P$ at point $(x_q, y_r)$ are considered for the analysis of the orthotropic rectangular plate.

The displacements $u$, $v$ and $w$ in the $x$, $y$ and $z$ directions are assumed to be

$$u = z \theta_z(x, y), \quad v = z \theta_y(x, y), \quad w = w(x, y),$$

(2.1)

where $\theta_z(x, y)$ and $\theta_y(x, y)$ are the rotations in the $x - z$ and $y - z$ planes.
For small displacements, the strain-displacement relations of elasticity yield

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \theta_x}{\partial x} \\
\frac{\partial \theta_y}{\partial y} \\
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}
\end{bmatrix} + \begin{bmatrix}
\theta_x \\
\theta_y \\
\theta_z
\end{bmatrix}
\]

(2.2)

For orthotropic plates, the stress-strain relations can be expressed as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

(2.3)

where \(\overline{Q}_{11} = E_1(1-\nu_{12}\nu_{21})\), \(\overline{Q}_{12} = \nu_{12}E_2(1-\nu_{12}\nu_{21})\), \(\overline{Q}_{16} = 0\), \(\overline{Q}_{22} = E_2(1-\nu_{12}\nu_{21})\), \(\overline{Q}_{26} = 0\), \(\overline{Q}_{66} = G_{12}\), \(\overline{Q}_{44} = G_{23}\), \(\overline{Q}_{45} = 0\), \(\overline{Q}_{55} = G_{13}\), \(E_1\) is the axial modulus in the 1-direction, \(E_2\) is the axial modulus in the 2-direction, \(\nu_{12}\) is the Poisson’s ratio associated with loading in the 1-direction and strain in the 2-direction, \(\nu_{21}\) is the Poisson’s ratio associated with loading in the 2-direction and strain in the 1-direction, \(G_{23}\), \(G_{13}\) and \(G_{12}\) are the shear modulus in 2-3, 1-3 and 1-2 planes.

The moments and the shear forces can be given by

\[
M_y = \int_{-h/2}^{h/2} \sigma_y z dz, \quad M_x = \int_{-h/2}^{h/2} \sigma_x z dz, \quad M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz, \quad Q_y = \kappa \int_{-h/2}^{h/2} \tau_{yx} dz,
\]

(2.4)

where, \(\kappa = 5/6\) is the shear correction factor.

By using the equations (2.2) ~ (2.4), the relations of the moment-displacement and the shear force-displacement can be obtained as
where the extensional stiffness $A_{ij} = \bar{Q}_{ij} h$ ($i, j = 4, 5$), and the bending stiffness $D_{ij} = \bar{Q}_{ij} h^3 / 12$ ($i, j = 1, 2, 6$).

From the element analysis (see Figure 2.2), with the shear force equilibrium in the $z$-direction and moment equilibrium in the $x$- and $y$- directions, Eqs. (2.6-1)-(2.6-3) can be obtained. Combining with Eq. (2.5), the eight fundamental differential equations of orthotropic plates with variable thickness are written as follows:
\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{q} + \bar{P} \delta(x - x_q) \delta(y - y_r) = 0 \tag{2.6-1}
\]

\[
\frac{\partial M_{sy}}{\partial x} + \frac{\partial M_{sx}}{\partial y} - Q_y = 0 \tag{2.6-2}
\]

\[
\frac{\partial M_{sx}}{\partial x} + \frac{\partial M_{sy}}{\partial y} - Q_x = 0 \tag{2.6-3}
\]

\[
M_x = D_{11} \frac{\partial^2 \theta_x}{\partial x^2} + D_{12} \frac{\partial^2 \theta_y}{\partial y^2} + D_{16} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \tag{2.6-4}
\]

\[
M_y = D_{12} \frac{\partial^2 \theta_x}{\partial x^2} + D_{22} \frac{\partial^2 \theta_y}{\partial y^2} + D_{26} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \tag{2.6-5}
\]

\[
M_{xy} = D_{16} \frac{\partial^2 \theta_x}{\partial x^2} + D_{26} \frac{\partial^2 \theta_y}{\partial y^2} + D_{66} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \tag{2.6-6}
\]

\[
Q_y = kA_{44} \left( \frac{\partial w}{\partial y} + \theta_y \right) + kA_{45} \left( \frac{\partial w}{\partial x} + \theta_x \right) \tag{2.6-7}
\]

\[
Q_x = kA_{45} \left( \frac{\partial w}{\partial y} + \theta_y \right) + kA_{55} \left( \frac{\partial w}{\partial x} + \theta_x \right) \tag{2.6-8}
\]

Where \( \theta_x, \theta_y \) are slopes, \( w \) is the deflection, \( \delta(x - x_q) \) and \( \delta(y - y_r) \) are the Dirac’s delta functions.

By using the non-dimensional expressions,

\[
\begin{align*}
[X_1, X_2] &= \frac{a^2}{D_0 (1 - \nu_{12} \nu_{21})} [Q_y, Q_x], \\
[X_3, X_4, X_5] &= \frac{a}{D_0 (1 - \nu_{12} \nu_{21})} [M_{sy}, M_y, M_x], \\
[X_6, X_7, X_8] &= [\theta_y, \theta_x, \frac{w}{a}], [\eta, \xi, \zeta] = \left[ \frac{x}{a}, \frac{y}{b}, \frac{z}{h} \right]
\end{align*}
\]

the differential equations of the orthotropic plate with uniform load \( \bar{q} \), and a concentrated load \( \bar{P} \) at point \((x_q, y_r)\) are establish as follows:
\[
\frac{\partial^2 X_2}{\partial \eta^2} + \frac{\partial X_1}{\partial \zeta} + q + P \delta(\eta - \eta_0) \delta(\zeta - \zeta_0) = 0
\]
\[
\mu \frac{\partial X_1}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} - \mu X_1 = 0
\]
\[
\mu \frac{\partial X_1}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} - \mu X_1 = 0
\]
\[
\mu \frac{\partial X_1}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} - \mu X_1 = 0
\]
\[
D_{11} \mu \frac{\partial X_1}{\partial \eta} + D_{12} \frac{\partial X_6}{\partial \zeta} + D_{16} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \vec{D} X_5 = 0
\]
\[
D_{12} \mu \frac{\partial X_7}{\partial \eta} + D_{22} \frac{\partial X_6}{\partial \zeta} + D_{26} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \vec{D} X_4 = 0
\]
\[
D_{16} \mu \frac{\partial X_7}{\partial \eta} + D_{66} \frac{\partial X_6}{\partial \zeta} + D_{66} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \vec{D} X_3 = 0
\]
\[
k \vec{A}_{44} \frac{\partial X_6}{\partial \zeta} + \mu X_6 + k \mu \vec{A}_{45} \left( \frac{\partial X_6}{\partial \eta} + X_7 \right) - \mu \vec{D} \vec{T} X_1 = 0
\]
\[
k \vec{A}_{45} \frac{\partial X_6}{\partial \eta} + \mu X_6 + k \mu \vec{A}_{55} \left( \frac{\partial X_6}{\partial \eta} + X_7 \right) - \mu \vec{D} \vec{T} X_2 = 0
\]

(2.7)

the Eq.(2.7) can be written as following simple form:

\[
\sum_{s=1}^{8} \left\{ F_{ls} \frac{\partial X_s}{\partial \zeta} + F_{2ls} \frac{\partial X_s}{\partial \eta} + F_{3ls} X_s \right\} + q \delta_{ls} + P \delta(\eta - \eta_0) \delta(\zeta - \zeta_0) \delta_{ls} = 0
\]

(2.8)

where \( \delta_{ls} \) is Kronecker’s delta, \( F_{111} = F_{123} = F_{134} = 1, \quad F_{146} = D_{12}, \quad F_{156} = D_{22}, \quad F_{167} = D_{66}, \quad F_{288} = \mu k \vec{A}_{55}, \quad F_{212} = F_{235} = F_{223} = \mu, \quad F_{247} = \mu \vec{D}_{11}, \quad F_{257} = \mu \vec{D}_{12}, \quad F_{266} = \mu \vec{D}_{66}, \quad F_{178} = k \vec{A}_{44}, \quad F_{332} = F_{321} = -\mu, \quad F_{345} = F_{354} = F_{363} = -\mu \vec{D}, \quad F_{371} = F_{382} = -\mu \vec{D} \vec{T}, \quad F_{376} = \mu k \vec{A}_{44}, \quad F_{387} = \mu k \vec{A}_{55}, \quad F_{147} = \vec{D}_{16} = 0, \quad F_{157} = F_{166} = \vec{D}_{26} = 0, \quad F_{188} = F_{278} = F_{377} = F_{386} = \mu k \vec{A}_{45} = 0, \quad F_{246} = F_{267} = \mu \vec{D}_{16} = 0, \quad F_{256} = -\mu \vec{D}_{26} = 0, \quad q = \mu (\bar{q} / q_0) q_0 a^3 / (D_0 (1 - \nu_{12} \nu_{21})), \quad \vec{D} = (h_0 / h)^3, \quad P = \bar{P} a / (D_0 (1 - \nu_{12} \nu_{21})), \quad \vec{D}_{\theta} = \vec{D} / (2 E_2), \quad \vec{A}_{\theta} = 12 (a / h_0)^2 (\bar{Q}_{\theta} / E_2), \quad \vec{D} \vec{T} = h_0 / h, \quad \mu = b / a, \quad D_0 = E_2 h_0^3 / (12 (1 - \nu_{12} \nu_{21})), \quad \bar{Q}_{\theta} = (E_2 / E_1)^2 (\bar{Q} / E_1), \quad \bar{D} = (h_0 / h)^3, \quad \bar{E}_1 = 12 (a / h_0)^2 (\bar{Q} / E_1). \]

The standard bending rigidity \( h_0 \) is the standard thickness of the plate.

2.1.2 The discrete solution of fundamental differential equations

With an orthotropic rectangular plate divided vertically into \( m \) equal-length parts and horizontally into \( n \) equal-length parts as shown in Figure 2.3, the plate can be considered as a
group of discrete points which are the intersections of the \((n+1)\)-horizontal and \((m+1)\)-vertical dividing lines. In this chapter, the rectangular area, \(0 \leq \eta \leq \eta_i, \ 0 \leq \zeta \leq \zeta_j\), corresponding to the arbitrary intersection \((i, j)\) is denoted as the area \([i, j]\), the intersection \((i, j)\) denoted by \(\bigcirc\) is called the main point of the area \([i, j]\), the intersections denoted by \(\circ\) are called the inner dependent points of the area, and the intersection denoted by \(\bullet\) are called the boundary dependent points of the area.

![Figure 2.3 Discrete points on an orthotropic rectangular plate](image)

By integrating the equations (2.8) over the area \([i, j]\), the integral equations are obtained as follows:

\[
\sum_{s=1}^{8} \left\{ F_{1s} \int_0^{\eta_i} \left[ X_s(\eta, \zeta) - X_s(\eta, 0) \right] d\eta + F_{2s} \int_0^{\zeta} \left[ X_s(\eta, \zeta) - X_s(0, \zeta) \right] d\zeta \\
+ F_{3s} \int_0^{\zeta} X_s(\eta, \zeta) d\eta d\zeta \right\} + \int_0^{\eta_i} \int_0^{\zeta} q(\eta, \zeta) \delta_{ij} d\eta d\zeta + Pu(\eta - \eta_i)u(\zeta - \zeta_j) \delta_{ij} = 0 \quad (2.9)
\]

Next, by applying the numerical integration method, the simultaneous equation for the unknown quantities \(X_{sij} = X_s(\eta_i, \zeta_j)\) at the main point \((i, j)\) is obtained as follows:
\[
\sum_{s=1}^{9} \left\{ F_{1s} \sum_{k=0}^{j} \beta_{ik} (X_{s,k} - X_{s,0}) + F_{2n} \sum_{l=0}^{j} \beta_{jl} (X_{s,l} - X_{s,0}) + F_{3s} \sum_{k=0}^{l} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} X_{s,k} \right\} \\
+ \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} q_{kl} \delta_{it} + Pu_{iq} u_{jr} \delta_{it} = 0, 
\]

where \( \beta_{ik} = \alpha_{ik} / m, \beta_{jl} = \alpha_{jl} / n, \alpha_{ik} = 1 - (\delta_{0k} + \delta_{ik}) / 2, \alpha_{jl} = 1 - (\delta_{0l} + \delta_{jl}) / 2, t = 1 \sim 8, \)

\( i = 1 \sim m, j = 1 \sim n, u_{iq} = u(\eta_i - \eta_q), u_{jr} = u(\zeta_j - \zeta_r). \)

\[
\begin{align*}
\alpha_{ik} &= \begin{cases} 0.5 & \text{if } k = 0, i \\ 1 & \text{if } k \neq 0, i \end{cases} ; \\
\alpha_{jl} &= \begin{cases} 0.5 & \text{if } l = 0, j \\ 1 & \text{if } l \neq 0, j \end{cases} ;
\end{align*}
\]

By retaining the quantities at main point \((i, j)\) on the left hand side of the equation and putting other quantities on the right hand side, and using the matrix transition, the solution \(X_{p_{ij}}\) of the above equation (2.10) is obtained as follows.

\[
\begin{align*}
\beta_{ii} X_{11} + \mu \beta_{jj} X_{22} &= \sum_{k=0}^{i} \beta_{ik} \left[ X_{1k,0} - X_{1k} (1-\delta_{ik}) \right] \\
+ \mu \sum_{l=0}^{j} \beta_{jl} \left[ X_{2l,0} - X_{2l} (1-\delta_{jl}) \right] - \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} A_{p_l q_{kl}} - A_{p_l} P u_{iq} u_{jr} \\
\beta_{ii} X_{33} + \mu \beta_{jj} X_{55} - \mu \beta_{ii} \beta_{jj} X_{22} &= \sum_{k=0}^{i} \beta_{ik} \left[ X_{3k,0} - X_{3k} (1-\delta_{ik}) \right] \\
+ \mu \sum_{l=0}^{j} \beta_{jl} \left[ X_{5l,0} - X_{5l} (1-\delta_{jl}) \right] + \mu \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} X_{2l,0} (1-\delta_{ik} \delta_{jl}) \\
\beta_{ii} X_{44} + \mu \beta_{jj} X_{33} - \mu \beta_{ii} \beta_{jj} X_{11} &= \sum_{k=0}^{i} \beta_{ik} \left[ X_{4k,0} - X_{4k} (1-\delta_{ik}) \right] \\
+ \mu \sum_{l=0}^{j} \beta_{jl} \left[ X_{3l,0} - X_{3l} (1-\delta_{jl}) \right] + \mu \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} X_{1l,0} (1-\delta_{ik} \delta_{jl}) 
\end{align*}
\]
\[
\overline{D}_{16} \beta_{ji} X_{7ij} + \overline{D}_{12} \beta_{ji} X_{6ij} + \mu \overline{D}_{16} \beta_{ji} X_{6ij} + \mu \overline{D}_{11} \beta_{ji} X_{7ij} - \mu \overline{D}_{ji} \beta_{ji} X_{5ij}
\]
\[
= \sum_{k=0}^{i} \overline{D}_{12} \beta_{ik} [X_{6ik} - X_{6ij} (1 - \delta_{ki})] + \sum_{k=0}^{i} \overline{D}_{16} \beta_{ik} [X_{7k0} - X_{7ij} (1 - \delta_{ki})] \\
+ \sum_{l=0}^{j} \overline{D}_{16} \beta_{jl} [X_{6il} - X_{6ij} (1 - \delta_{jl})] + \sum_{l=0}^{j} \overline{D}_{11} \beta_{jl} [X_{7il} - X_{7ij} (1 - \delta_{jl})] \\
+ \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} \mu \overline{D}_{kl} X_{5kl} (1 - \delta_{ki} \delta_{jl})
\]

\[
\overline{D}_{22} \beta_{ji} X_{6ij} + \overline{D}_{26} \beta_{ji} X_{7ij} + \mu \overline{D}_{26} \beta_{ji} X_{6ij} + \mu \overline{D}_{12} \beta_{ji} X_{7ij} - \mu \overline{D}_{ji} \beta_{ji} X_{4ij}
\]
\[
= \sum_{k=0}^{i} \overline{D}_{22} \beta_{ik} [X_{6ik} - X_{6ij} (1 - \delta_{ki})] + \sum_{k=0}^{i} \overline{D}_{26} \beta_{ik} [X_{7k0} - X_{7ij} (1 - \delta_{ki})] \\
+ \sum_{l=0}^{j} \overline{D}_{26} \beta_{jl} [X_{6il} - X_{6ij} (1 - \delta_{jl})] + \sum_{l=0}^{j} \overline{D}_{12} \beta_{jl} [X_{7il} - X_{7ij} (1 - \delta_{jl})] \\
+ \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} \mu \overline{D}_{kl} X_{4kl} (1 - \delta_{ki} \delta_{jl})
\]

\[
\overline{D}_{26} \beta_{ij} X_{6ij} + \overline{D}_{66} \beta_{ij} X_{7ij} + \mu \overline{D}_{66} \beta_{ij} X_{6ij} + \mu \overline{D}_{16} \beta_{ij} X_{7ij} - \mu \overline{D}_{ij} \beta_{ij} X_{3ij}
\]
\[
= \sum_{k=0}^{i} \overline{D}_{26} \beta_{ik} [X_{6ik} - X_{6ij} (1 - \delta_{ki})] + \sum_{k=0}^{i} \overline{D}_{66} \beta_{ik} [X_{7k0} - X_{7ij} (1 - \delta_{ki})] \\
+ \sum_{l=0}^{j} \overline{D}_{66} \beta_{jl} [X_{6il} - X_{6ij} (1 - \delta_{jl})] + \sum_{l=0}^{j} \overline{D}_{16} \beta_{jl} [X_{7il} - X_{7ij} (1 - \delta_{jl})] \\
+ \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} \mu \overline{D}_{kl} X_{3kl} (1 - \delta_{ki} \delta_{jl})
\]

\[
k \overline{A}_{44} \beta_{ij} X_{8ij} + \mu k \overline{A}_{45} \beta_{ij} X_{8ij} + \mu k \overline{A}_{44} \beta_{ij} X_{7ij} + \mu k \overline{A}_{44} \beta_{ij} X_{6ij} - \mu \overline{D}_{ij} \beta_{ij} X_{1ij}
\]
\[
= \sum_{k=0}^{i} k \overline{A}_{44} \beta_{ik} [X_{8ik} - X_{8ij} (1 - \delta_{ki})] + \sum_{l=0}^{j} k \mu \overline{A}_{45} \beta_{jl} [X_{8lj} - X_{8ij} (1 - \delta_{jl})] \\
+ \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} [\mu \overline{D}_{kl} X_{1kl} - \mu k \overline{A}_{45} X_{7kl} - \mu k \overline{A}_{44} X_{6kl}] (1 - \delta_{ki} \delta_{jl})
\]
$k \overline{A}_{ss} \beta_{ij} X_{8ij} + \mu k \overline{A}_{ss} \beta_{ij} X_{8ij} + \mu k \overline{A}_{ss} \beta_{ij} X_{7ij} + \mu k \overline{A}_{ss} \beta_{ij} X_{6ij} - \mu \overline{D} \overline{T}_{ij} \beta_{ij} X_{2ij} = \sum_{k=0}^{i} k \overline{A}_{ss} \beta_{ik} \left[ X_{8k0} - X_{8kj} (1 - \delta_{kj}) \right] + \sum_{j=0}^{j} \mu k \overline{A}_{ss} \beta_{ij} \left[ X_{8ij} - X_{8kj} (1 - \delta_{kj}) \right] + \sum_{k=0}^{i} \sum_{k=0}^{j} \beta_{il} \beta_{lj} \left[ \mu \overline{D} \overline{T}_{il} X_{2kj} - \mu k \overline{A}_{ss} X_{7il} - \mu k \overline{A}_{ss} X_{6il} \right] (1 - \delta_{ik}, \delta_{kj}) \tag{2.11}
$

where

$\beta_{ij} = \beta_{ji}, \quad \delta_{ik} = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}, \quad \delta_{ij} = \begin{cases} 0 & \text{if } l \neq j \\ 1 & \text{if } l = j \end{cases}$

Rewriting the equation (2.11) in matrix form yields:

$[\rho_{pt}] [X_{pjq}] = \{A_i\} \quad (p=1 \sim 8, \ t=1 \sim 8) \tag{2.12}$

Where $\{A_i\}$ is the right-hand term of equation (2.11), $\{X_{pjq}\}$ is the quantity at the main point $(i, j)$, the coefficient matrix $[\rho_{pt}] \ (8 \times 8)$ is as follow:

$[\rho_{pt}] = \begin{bmatrix}
\beta_{ii} & \mu \beta_{jj} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mu \beta_{jj} & \beta_{ii} & 0 & \mu \beta_{jj} & 0 & 0 & 0 \\
-\mu \beta_{ij} & 0 & \beta_{ii} & -\mu \beta_{ij} & \beta_{ii} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\mu \beta_{ij} D_{ij} & 0 & 0 \\
0 & 0 & 0 & -\mu \beta_{ij} D_{ij} & 0 & 0 & -\mu \beta_{ij} D_{ij} & 0 \\
-\mu \beta_{ij} DT_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mu \beta_{ij} DT_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$

(2.13)

Using the inverse of matrix, the equation (2.13) can be written as

$\{X_{pjq}\} = [\rho_{pt}]^{-1} \{A_i\} = [\gamma_{pt}] \{A_i\} \tag{2.14}$

The Eq.(2.14) can be written as the following simple form:

$X_{pjq} = \sum_{t=1}^{8} \left[ \sum_{k=0}^{i} \beta_{ik} A_{pt} \left[ X_{8ko} - X_{8kj} (1 - \delta_{kj}) \right] \right] + \sum_{j=0}^{j} \beta_{lj} B_{pt} \left[ X_{8ij} - X_{8il} (1 - \delta_{ij}) \right]$
\begin{equation}
\begin{aligned}
\sum_{k=0}^{l} \sum_{l=0}^{j} \beta_{il} \beta_{jl} C_{ptkl} X_{ui} + (1 - \delta_{il} \delta_{jl}) \sum_{k=0}^{l} \sum_{l=0}^{j} \beta_{il} \beta_{jl} A_{pl} q_{il} - A_{pj} P u_{iq} u_{jr},
\end{aligned}
\end{equation}

where \( p = 1 \sim 8; A_{pl}, B_{pl}, C_{ptkl} \) are given in Appendix A.

In the equation (2.15), the quantity \( X_{pij} \) at the main point \((i, j)\) of the area \([i, j]\) is related to the quantities \( X_{tk0} \) and \( X_{t0l} \) at the boundary dependent points of the area and quantities \( X_{tkj}, X_{tlk} \) at the inner dependent points of the area. With the spreading of the area \([i, j]\) according to the regular order as \([1, 1], [1, 2], \ldots, [1, n]; [2, 1], [2, 2], \ldots, [2, n]; \ldots, [m, 1], [m, 2], \ldots, [m, n]\), a main point of a smaller area becomes one of the inner dependent points of the following large areas. Whenever the quantity \( X_{pij} \) at the main point \((i, j)\) is obtained by using the equation (2.15) in above mentioned order, the quantities \( X_{tkj}, X_{tlk} \) at the inner independent points of the following large area can be eliminated by substituting the obtained results into the corresponding terms of the right side of equation (2.15). By repeating this process, the quantity \( X_{pij} \) at the main point is only related to the quantities \( X_{rk0} (r=1, 3, 4, 6, 7, 8) \) and \( X_{s0l} (s=2, 3, 5, 6, 7, 8) \) which are six independent quantities at each boundary dependent point along the horizontal axis and the vertical axis in Figure 2.3, respectively.

\begin{equation}
X_{pij} = \sum_{d=1}^{6} \left\{ \sum_{f=0}^{l} a_{pijf} X_{r0f} + \sum_{g=0}^{j} b_{pijg} X_{s0g} \right\} + \bar{q}_{pij} P
\end{equation}

where \( a_{pijf}, b_{pijg}, \bar{q}_{pij} \) and \( \bar{q}_{pij} \) are given in Appendix B.

The equation (2.16) gives the discrete solution of the fundamental differential equation (2.6) of the bending problem of an orthotropic plate under uniform load and a concentrated load.

By using the boundary conditions, the quantities at any point of the orthotropic plate can be obtained from equation (2.16). The above coefficients \( a_{251003}, a_{14003}, \ldots, a_{221001}, a_{12005}, a_{12006}, \alpha_{l}, \beta_{lk}, \gamma_{ik} \) and etc. are derived in Appendix C.

The equation (2.16) is the approximate solution of the equation (2.6). In this equation, the integral constants \((Q_{r})=X_{r}, (Q_{s})=X_{s}, (M_{xy})=X_{3}, (M_{y})=X_{4}, (M_{x})=X_{5}, (\theta_{y})=X_{6}, (\theta_{x})=X_{7}, (w)=X_{8}\) can be determined from the boundary conditions.
For an isotropic plate, which is considered as a special case of orthotropic plates with the axial modulus $E_1 = E_2 = E$ and the shear modulus $G_{12} = G_{13} = G_{23} = G = E / (2(1 + \nu))$, the fundamental differential equations of the isotropic plate having a concentrated load $\bar{P}$ at a point $(x_q, y_q)$ and distributed load $\bar{q} = \bar{q}(x, y)$ are obtained from Equation (2.6) as follows:

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{q} + \bar{P} \delta(x - x_q) \delta(y - y_q) = 0
\]

\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_x}{\partial y} - Q_y = 0
\]

\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0
\]

\[
\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} = \frac{M_x}{D},
\]

\[
\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} = \frac{M_y}{D},
\]

\[
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2}{(1 - \nu)} \frac{M_{xy}}{D},
\]

\[
\frac{\partial w}{\partial x} + \theta_x = \frac{Q_x}{Gt},
\]

\[
\frac{\partial w}{\partial y} + \theta_y = \frac{Q_y}{Gt}.
\]

where $t = h / 1.2$.

2.2 Discrete Method for Vibration Problems of Orthotropic Plates

In the section 2.1, a discrete method for analyzing elastic bending problems of orthotropic plates with variable thickness has been proposed. In this section, the discrete method is extended to analyze the free vibration of orthotropic rectangular plates with variable thickness by using the Green function. The discrete form solution for deflection of the orthotropic plate with a concentrated load gives the discrete type Green function of the plate.

2.2.1 Discrete Green function of an orthotropic plate with variable thickness

The Green function of an orthotropic plate bending problem is given by the displacement function of the plate with a unit concentrated load, so the Green function $G(x, y, x_q, y_q) = w(x, y, x_q, y_q) / \bar{P}$ of an orthotropic plate with variable thickness can be obtained from the fundamental
differential equations of the orthotropic plate with a concentrated load $\bar{P}$ at a point $(x_q, y_r)$ as shown in Figure 2.4, which is given by the following equations:

$$
\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} + \bar{P} \delta(x - x_q) \delta(y - y_r) = 0
$$

$$
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_x = 0
$$

$$
\frac{\partial M_y}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0
$$

$$
M_x = D_{11} \frac{\partial^2 \theta_x}{\partial x^2} + D_{12} \frac{\partial^2 \theta_y}{\partial y^2} + D_{16} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right)
$$

$$
M_y = D_{12} \frac{\partial^2 \theta_x}{\partial x^2} + D_{22} \frac{\partial^2 \theta_y}{\partial y^2} + D_{26} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right)
$$

$$
M_{xy} = D_{16} \frac{\partial^2 \theta_x}{\partial x^2} + D_{26} \frac{\partial^2 \theta_y}{\partial y^2} + D_{66} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right)
$$

$$
Q_x = kA_{44} \left( \frac{\partial \hat{w}}{\partial y} + \theta_y \right) + kA_{45} \left( \frac{\partial \hat{w}}{\partial x} + \theta_x \right)
$$

$$
Q_y = kA_{45} \left( \frac{\partial \hat{w}}{\partial y} + \theta_y \right) + kA_{55} \left( \frac{\partial \hat{w}}{\partial x} + \theta_x \right)
$$

(2.18)

With the same method described in section 2.1, the discrete solution of fundamental differential equation (2.18) of the bending problem of orthotropic plate under a concentrated load, and the discrete Green function is chosen as $X_{\delta y} / [\bar{P}a / D_y (1 - \nu_{12} \nu_{21})]$.

2.2.2 Characteristic equation of free vibration of orthotropic plates with variable thickness

During the free vibration, there is inertial force $-\rho \partial^2 \hat{w}(x, y) / \partial t^2 \, dx \, dy$ at every point $(x, y)$ of the orthotropic plate, in which $\rho$ is the mass density of the plate material and $\hat{w}(x, y)$ is the displacement at point $(x, y)$. Supposing the displacement $\hat{w}(x, y)$ can be expressed in terms of sin and cos functions, that is, $\hat{w}(x, y) = A \sin(\omega t) + B \cos(\omega t)$, then the
inertial force $-\rho h^2 \ddot{w}(x,y)/\partial t^2 dx dy$ can be written as $\rho h \omega^2 \hat{w}(x,y) dx dy$. Considering the inertial force $\rho h \omega^2 \hat{w}(x,y) dx dy$ as a concentrated load at point $(x, y)$ (refer to Figure 2.4), it can be concluded that the free vibration problem of an orthotropic plate will correspond to be bending problem of an orthotropic plate with a concentrated load $\rho h \omega^2 \hat{w}(x,y) dx dy$ at any point $(x, y)$. By applying the Green function $w(x_0,y_0,x,y)/P$, which is the displacement at point $(x_0, y_0)$ of an orthotropic plate with a unit concentrated load at point $(x, y)$, and using the method above mentioned, the displacement at point $(x_0, y_0)$ of the orthotropic plate with a concentrated load $\rho h \omega^2 \hat{w}(x,y) dx dy$ at every point $(x, y)$ will be obtained as follow:

$$\hat{w}(x_0,y_0) = \int_0^\pi \int_0^\pi \rho h \omega^2 \hat{w}(x,y)[w(x_0,y_0,x,y)/P] dx dy$$

(2.19)

Figure 2.4 An orthotropic plate with a concentrated load $\rho h \omega^2 \hat{w}(x,y) dx dy$

The above equation (2.19) gives the displacement amplitude $\hat{w}(x_0,y_0)$ at the point $(x_0, y_0)$ of the orthotropic plate during the free vibration.

By using the following non-dimensional expressions,

$$\lambda^4 = \frac{\rho_0 h \omega^2 a^4}{D_0(1 - v_{12} v_{21})}, \quad H(\eta, \zeta) = \frac{\rho(x,y)}{\rho_0} \frac{h(x,y)}{h_0},$$

$$W(\eta, \zeta) = \frac{\hat{w}(x,y)}{a}, \quad G(\eta_0, \zeta_0, \eta, \zeta) = \frac{w(x_0,y_0,x,y)}{a} \frac{D_0(1 - v_{12} v_{21})}{P a}$$

($\rho_0$: the standard mass density)

the integral equation (2.19) can be rewritten as follow:
\[ W(\eta_0, \zeta_0) = \int_0^1 \int_0^1 \mu \chi^4 H(\eta, \zeta) G(\eta_0, \zeta_0, \eta, \zeta) W(\eta, \zeta) d\eta d\zeta \] (2.20)

By applying the numerical integration method mentioned in section 2.1, equation (2.20) is discretely expressed as follow:

\[ \Lambda W_{ij} = \sum_{j=0}^{m} \sum_{l=0}^{n} \beta_{ml} \beta_{nj} H_{jl} G_{klj} W_{lj}, \quad \Lambda = 1/(\mu \chi^4) \] (2.21)

From equation (2.21) homogeneous linear equations in \((m+1) \times (n+1)\) unknown quantities \(W_{00}, W_{01}, \ldots, W_{0n}, W_{10}, W_{11}, \ldots, W_{1n}, \ldots, W_{m0}, W_{m1}, \ldots, W_{mn}\) are obtained as follow:

\[ \sum_{j=0}^{m} \sum_{l=0}^{n} (\beta_{ml} \beta_{nj} H_{jl} G_{klj} - \Lambda \delta_{ik} \delta_{jl}) W_{lj} = 0, \quad (k = 0, 1, \ldots, m; \quad l = 0, 1, \ldots, n) \] (2.22)

The characteristic equation of the free vibration of an orthotropic plate with variable thickness is obtained from the equation (2.22):

\[
\begin{bmatrix}
K_{00} & K_{01} & K_{02} & \cdots & K_{0n} \\
K_{10} & K_{11} & K_{12} & \cdots & K_{1m} \\
K_{20} & K_{21} & K_{22} & \cdots & K_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_{m0} & K_{m1} & K_{m2} & \cdots & K_{mm}
\end{bmatrix}
= 0
\] (2.23)

Where

\[
K_{ij} = \beta_{mj} \begin{bmatrix}
\beta_{n0} H_{j0} G_{i0j0} - \Lambda \delta_{ij} & \beta_{n1} H_{j1} G_{i0j1} & \beta_{n2} H_{j2} G_{i0j2} & \cdots & \beta_{nm} H_{jn} G_{i0jn} \\
\beta_{n0} H_{j0} G_{i1j0} & \beta_{n1} H_{j1} G_{i1j1} - \Lambda \delta_{ij} & \beta_{n2} H_{j2} G_{i1j2} & \cdots & \beta_{nm} H_{jn} G_{i1jn} \\
\beta_{n0} H_{j0} G_{i2j0} & \beta_{n1} H_{j1} G_{i2j1} & \beta_{n2} H_{j2} G_{i2j2} - \Lambda \delta_{ij} & \cdots & \beta_{nm} H_{jn} G_{i2jn} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n0} H_{j0} G_{i(n-1)j0} & \beta_{n1} H_{j1} G_{i(n-1)j1} & \beta_{n2} H_{j2} G_{i(n-1)j2} & \cdots & \beta_{nm} H_{jn} G_{i(n-1)jn} \\
\beta_{n0} H_{j0} G_{i(n-1)j0} & \beta_{n1} H_{j1} G_{i(n-1)j1} & \beta_{n2} H_{j2} G_{i(n-1)j2} & \cdots & \beta_{nm} H_{jn} G_{i(n-1)jn} - \Lambda \delta_{ij}
\end{bmatrix}
\]

For isotropic plates, the Green function \(G(x, y, x_q, y_r) = w(x, y, x_q, y_r)/P\) of an isotropic plate with variable thickness can be obtained from the fundamental differential equations of the isotropic plate with a concentrated load \(P\) at a point \((x_q, y_r)\). The fundamental differential equations of the isotropic plate with a concentrated load \(P\) at a point \((x_q, y_r)\) are shown as:
\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P\delta(x-x’)\delta(y-y’) = 0
\]
\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0,
\]
\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0,
\]
\[
\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} = \frac{M_x}{D},
\]
\[
\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} = \frac{M_y}{D},
\]
\[
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2}{(1-\nu)} \frac{M_{xy}}{D},
\]
\[
\frac{\partial w}{\partial x} + \theta_x = \frac{Q_x}{Gt},
\]
\[
\frac{\partial w}{\partial y} + \theta_y = \frac{Q_y}{Gt},
\]

(2.24)

2.3 Integral Constants and Boundary Conditions

The integral constants are all quantities at the discrete points along the edges \(y=0\) (\(\zeta = 0\)) and \(x=0\) (\(\eta = 0\)) of the rectangular plate. There are six integral constants at every discrete point except the point at corners, and half of them are self-evident according to the boundary conditions along the edges \(y=0\) and \(x=0\). The remaining three integral constants can be determined by the boundary conditions along the edges \(y=b\) and \(x=a\).

2.3.1 Integral constants and boundary conditions for some plates

The integral constants and the boundary conditions of rectangular plates with four edges simple supported (SSSS), four edges clamped (CCCC), four edges free (FFFF), three edges free and one edge clamped (FFCF), two adjacent edges simple supported and the other edges clamped (SSCC) or free (SSFF) are shown in Figures 2.5~2.10, respectively. In this thesis, the symbolism used for the boundary conditions will identify a rectangular plate with the edges \(x=0, y=0, x=a,\) and \(y=b\). For example, a plate having boundary SSFF has a simple supported boundary condition at \(x=0\) \((\eta = 0)\), a simple supported boundary condition at \(y=0\) \((\zeta = 0)\), a free boundary condition at \(x=a\), a free boundary condition at \(y=b\). For the plates with symmetrical axes, 1/2 or 1/4 parts cut out of the plates along the horizontal or vertical
symmetrical axis or both are designated by (b), (c), or (d) and also shown in these figures. The integral constants and the boundary conditions at the corners of the plates are shown in the boxes. The simple supported, clamped and free edges are shown by solid line, thick line and dotted line, respectively.

Figure 2.5 Integral constants and boundary conditions of a plate with four simple supported edges (SSSS)
Figure 2.6 Integral constants and boundary conditions of a plate with four clamped edges (CCCC)
Figure 2.7 Integral constants and boundary conditions of a plate with four free edges (FFFF)
Figure 2.8 Integral constants and boundary conditions of a cantilever plate (FFCF)
Figure 2.9 Integral constants and boundary conditions of a plate with simple support edges and clamped edges (SSCC)

Figure 2.10 Integral constants and boundary conditions of a plate with simple support edges and free edges (SSFF)
2.3.2 Integral constants and boundary conditions at the corners

There are eight integral constants at each corner. There are \( Q_y, Q_x, M_{xy}, M_y, M_x, \theta_x, \theta_y \), and \( w \). Some of them can be determined to be zero by using boundary conditions at the corners. For example, in Figure 2.5(a), the isotropic rectangular plate is simply supported. The boundary conditions along the side \( x=0 \) and the side \( y=0 \) are

\[
M_x = 0, \quad \theta_y = 0, \quad w = 0, \quad \text{for } x = 0
\]
\[
M_y = 0, \quad \theta_x = 0, \quad w = 0, \quad \text{for } y = 0
\]

So the boundary conditions at the corner \((0,0)\) are \( M_y = M_x = \theta_y = \theta_x = w = 0 \). In order to satisfy these conditions, the five integral constants \( M_y, M_x, \theta_x, \theta_y \), and \( w \) at the corner \((0,0)\) must be zero.

Observing that

\[
\frac{\partial w}{\partial x} = 0 \quad \text{when } w = 0 \text{ on the side } y = 0
\]

and using isotropic plates equation (2.17), the following equation can be obtained:

\[
\frac{\partial w}{\partial x} = \frac{Q_x}{Gt} - \theta_x = 0
\]

from which

\[
\frac{Q_x}{Gt} = \theta_x
\]

considering \( \theta_x = 0 \) on the side \( y = 0 \), then \( Q_x = 0 \)

observing that

\[
\frac{\partial w}{\partial y} = 0 \quad \text{when } w = 0 \text{ on the side } x = 0
\]

and using isotropic plates equation (2.17), \( Q_y = 0 \) can be also obtained. Therefore, at last, there is only one integral constant \( M_{xy} \) at the corner \((0,0)\).

In the similar manner, the boundary conditions at the corner \((a,b)\) can be obtained. The boundary conditions along the side \( x=a \) and the side \( y=b \) are

\[
M_x = 0, \quad \theta_y = 0, \quad w = 0, \quad \text{for } x = a
\]
\[
M_y = 0, \quad \theta_x = 0, \quad w = 0, \quad \text{for } y = b
\]
Hence, the boundary conditions at corner \((a,b)\) are \(M_x = M_y = \theta_x = \theta_y = w = 0\). But not all of these boundary conditions are independent.

By using isotropic plate equation (2.17), the following equations can be obtained.

\[
\frac{\partial \theta_x}{\partial x} = \frac{1}{(1 - v^2)D}(M_x - vM_y) \\
\frac{\partial \theta_y}{\partial y} = \frac{1}{(1 - v^2)D}(M_y - vM_x)
\]

Observing that

\[
\frac{\partial \theta_x}{\partial x} = 0 \quad \text{when} \quad \theta_x = 0 \quad \text{on the side} \quad y = b
\]

\[
\frac{\partial \theta_y}{\partial y} = 0 \quad \text{when} \quad \theta_y = 0 \quad \text{on the side} \quad x = a
\]

from which \(M_x = M_y = 0\).

Thus, there are only three independent boundary conditions at the corner \((a, b)\), namely,

\[
\theta_x = 0, \quad \theta_y = 0, \quad w = 0.
\]

### 2.4 Equivalent Rectangular Plate with Variable Thickness and Irregular Shape

Irregular shaped plates such as plate with various shaped holes usually more complicated than uniform rectangular plates, but they can be considered as a kind of rectangular plate by translating them into the equivalent rectangular plate with non-uniform thickness. An opening in the plate can be considered as an extremely thin part of the equivalent rectangular plate. Furthermore, a non-rectangular plate can be translated into an equivalent rectangular plate which additional parts are extremely thin or thick according to the boundary condition of the original plate.

A typical translation from an original irregular-shaped plate into its equivalent rectangular plate is shown in Figure 2.11. The edge 1-2-3-4 is clamped, the edge 4-5-6-1 is simply supported, the edge 7, which is the periphery of the hole, is free. The thickness of the irregular-shaped plate is denoted by \(h_0\). By adding three parts \(A_1, A_2, A_3\), the irregular-shaped plate is translated into a rectangular plate as shown in Fig. 2.11(b). The parts \(A_1, A_2, A_3\), are enclosed by the edges 2-3-8-2, the edge 7 and edge 4-5-6-9-4, respectively. The thickness of
these additional parts is denoted by $h$. The thickness of the rectangular plate along the curved lines 2-3, 7 and 4-5-6 is taken as $(h_0 + h)/2$. The thickness of the part $A_1$ is chosen to be extremely thick ($h \gg h_0$) and the edge 3-8-2 of the part $A_1$ are chosen to be clamped because the edge 2-3 of the irregular-shaped plate is clamped. In this case, there will be almost no deflection and slope along the curved line 2-3 when rectangular plate is subjected to the bending load. So the curved line 2-3 of the rectangular plate can be approximately considered as clamped. The thickness of the part $A_2$ is chosen to be extremely thin ($h \ll h_0$) because the edge 7 of the irregular-shaped plate is free. In this case, the part $A_2$ is similar to a hole and there are almost no forces along its edge 7 when the rectangular plate is subjected to the bending load. So the edge 7 of the rectangular plate can be approximately considered as free. The part $A_3$ is chosen to be an extremely thin overhang ($h \ll h_0$) and there are many point supports along its edge 4-5-6, because the edge 4-5-6 of the original irregular-shaped plate is simply supported. The values of reactions at the each point support of the rectangular plate are adjusted to satisfy the three restrained conditions, $M_n=0$, $\theta_r=0$, $w=0$. The first condition $M_n=0$ means that the bending moment $M_n$ around the tangential axis of the
curved line of point supports is zero. The second condition $\theta_t=0$ means that the slope $\theta_t$ around the normal axis of the curved line of point supports is zero. And the third condition $w=0$ means that the deflection $w$ is zero. So the curved line 4-5-6 can be considered as simply supported when the rectangular plate is subjected to the bending load. By above translation of an original irregular-shaped plate, an equivalent rectangular plate is finally obtained. The boundary conditions of this equivalent rectangular plate is clamped along the edges 1-2-8 and 8-3-4, free along the edges 4-9 and 9-6, and simply supported along the edge 6-1. This equivalent plate has many point supports along the curved line 4-5-6 and the thickness of the plate is variable. After the equivalent rectangular plate is obtained, all analyses are carried out in the equivalent rectangular plate. A general method is obtained for analyzing the bending and free vibration problems of plates.

In order to express the conditions $M_n=0$, $\theta_t=0$, a relationship between the two coordinate axes, $n-t$ axis and $x-y$ axis, is needed.

In Figure 2.12, point $A$ is any point at the curved line 4-5-6 of point supports. The $t$ and $n$ axes are the coordinate axes in the direction of the tangent and the normal at the point $A$, respectively. The angle between $n$ axis and $x$ axis is $\alpha$. $A$ is the point at $(x_0, y_0)$ in the $x-y$ axes. $A'$ is the point at $(x, y)$ in $x-y$ axes and at $(u, v)$ in $n-t$ axes. Then the following relationship can be obtained as follows:

$$x = x_0 + u \sin \alpha + v \cos \alpha$$

$$y = y_0 + v \sin \alpha - u \cos \alpha$$

From which

$$\frac{\partial x}{\partial u} = \sin \alpha, \quad \frac{\partial x}{\partial v} = \cos \alpha, \quad \frac{\partial y}{\partial u} = -\cos \alpha, \quad \frac{\partial y}{\partial v} = \sin \alpha$$

By using the above expression, the slopes of $n$ and $t$ directions are derived as follows:

$$\theta_n = -\frac{\partial w}{\partial v} = -\left(\frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v}\right) = -\left(\frac{\partial w}{\partial x} \cos \alpha + \frac{\partial w}{\partial y} \sin \alpha\right) = \theta_x \cos \alpha + \theta_y \sin \alpha \quad (2.25)$$

$$\theta_t = -\frac{\partial w}{\partial u} = -\left(\frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u}\right) = \left[\frac{\partial w}{\partial x} \sin \alpha + \frac{\partial w}{\partial y} (-\cos \alpha)\right] = \theta_x \sin \alpha - \theta_y \cos \alpha \quad (2.26)$$
Figure 2.12 Transformation of coordinates

The element $abc$ through the point $A$ is shown in Figure 2.13. The normal stress $\sigma_n$ and the shear stress $\tau_{nt}$ on the side $ac$ can be obtained by projecting the forces acting on the element $acb$ onto the $n$ and $t$ directions, respectively, which give the following equations:

\[
\begin{align*}
\sigma_n ds - (\sigma_x dy) \cos \alpha - (\tau_{xy} dy) \sin \alpha - (\sigma_y dx) \sin \alpha - (\tau_{xy} dx) \cos \alpha &= 0 \\
\tau_{nt} ds - (\sigma_x dy) \sin \alpha + (\tau_{xy} dy) \cos \alpha + (\sigma_y dx) \cos \alpha - (\tau_{xy} dx) \sin \alpha &= 0
\end{align*}
\]

because of $\quad dx = ds \sin \alpha, \quad dy = ds \cos \alpha$

the following equations can be obtained:

\[
\begin{align*}
\sigma_n &= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2\tau_{xy} \sin \alpha \cos \alpha \tag{2.27} \\
\tau_{nt} &= (\sigma_x - \sigma_y) \sin \alpha \cos \alpha + \tau_{xy} (\sin^2 \alpha - \cos^2 \alpha) \tag{2.28}
\end{align*}
\]
The bending and twisting moments at point $A$ are as follows:

$$M_n = \int_{-h/2}^{h/2} \sigma_n zdz, \quad M_{nt} = \int_{-h/2}^{h/2} \tau_{nt} zdz$$  \hspace{1cm} (2.29)

Using the expressions of $\sigma_n$ and $\tau_{nt}$, the above equations are written as:

$$M_n = M_x \cos^2 \alpha + M_y \sin^2 \alpha + 2M_{xy} \sin \alpha \cos \alpha$$  \hspace{1cm} (2.30)

$$M_{nt} = M_{xy} (\sin^2 \alpha - \cos^2 \alpha) + (M_y - M_x) \sin \alpha \cos \alpha$$  \hspace{1cm} (2.31)

Therefore, the restrained conditions $M_n=0, \ \theta_t=0, \ w=0$ can be presented as follows:

$$M_n = M_x \cos^2 \alpha + M_y \sin^2 \alpha + 2M_{xy} \sin \alpha \cos \alpha = 0$$

$$\theta_t = \theta_x \sin \alpha - \theta_y \cos \alpha = 0$$

$$w = 0$$  \hspace{1cm} (2.32)

With no-dimensional expressions, the following can be obtained:
Here, 

\[ X_M = \frac{a}{D_0 (1 - v^2)} M_\alpha, \quad X_\alpha = \theta_\alpha \]  

By using the above three restrained conditions and the boundary conditions of the equivalent rectangular plate, the integral constants and the reactions can be determined.

### 2.5 Comparison of FEM and the Proposed Method

In order to compare the proposed method with FEM (Finite Element Method), the isotropic square plate is chosen. Its properties are given as \( E_1/E_2 = 1, \ G_{12}/E_2 = 0.385, \ G_{13}/E_2 = 0.385, \ G_{23}/E_2 = 0.385, \ \nu_{12} = 0.3. \) Due to the symmetry, only the 1/4 plate is used to obtain the deflections at the central point of the whole plate for SSSS and CCCC plates and bending moments for CCCC plate as shown in Figure 2.14 (a) and (b).

\[ X_{\alpha} = X_4 \cos^2 \alpha + X_3 \sin^2 \alpha + 2X_3 \sin \alpha \cos \alpha = 0 \]

\[ X_\alpha = X_7 \sin \alpha - X_6 \cos \alpha = 0 \]

\[ X_8 = 0 \]  

Figure 2.14 Deflection and bending moments
Figures 2.15 and 2.16 show the unknown quantities at every discrete point for the FEM and the proposed method, respectively. It can be concluded if the divisional number is $m$ and the FEM is used, the numbers of the unknown quantities are $3m^2$ for SSSS plate and $3m^2 - 2m$ for CCCC plate. If the proposed method is used, the number of the unknown quantities is $6m-1$ for both the SSSS and CCCC plates. So for the same divisional number $m$, the numbers of the unknown quantities of the FEM and the proposed method are different. The unknown quantities of the proposed method are fewer than those of the FEM when $m \geq 3$.

Table 2.1 and 2.2 show the number of elements, the number of the unknown quantities, and the numerical results of deflection and bending moments for SSSS plates and CCCC plates.
Table 2.1 The central deflection $w_c$ of SSSS and CCCC isotropic plates under uniform and concentrated loads

<table>
<thead>
<tr>
<th>Element</th>
<th>Unknowns</th>
<th>Uniform L. $(w_c/qa^2/D) \times 1000$</th>
<th>Concent. L. $(w_c/Pa^2/D) \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS plate</td>
<td>CCCC plate</td>
</tr>
<tr>
<td>1/4 part</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SSSS</td>
<td>FEM</td>
<td>3.939</td>
<td>1.433</td>
</tr>
<tr>
<td>CCCC</td>
<td>P. M.</td>
<td>4.033</td>
<td>1.077</td>
</tr>
<tr>
<td>2×2</td>
<td>12</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>4×4</td>
<td>48</td>
<td>40</td>
<td>23</td>
</tr>
<tr>
<td>6×6</td>
<td>108</td>
<td>96</td>
<td>35</td>
</tr>
<tr>
<td>8×8</td>
<td>192</td>
<td>176</td>
<td>47</td>
</tr>
<tr>
<td>10×10</td>
<td>59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12×12</td>
<td>71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14×14</td>
<td>83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16×16</td>
<td>95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>[46]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

P.M.: Proposed Method

Table 2.2 Bending moments of CCCC isotropic plate under uniform load

<table>
<thead>
<tr>
<th>Element</th>
<th>Unknowns</th>
<th>$M_x/qa^2 \times 10$</th>
<th>$-M_y/qa^2 \times 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>FEM [47]</td>
<td>P. M. [47]</td>
</tr>
<tr>
<td>1/4 Part</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2×2</td>
<td>8</td>
<td>0.278</td>
<td>0.272</td>
</tr>
<tr>
<td>3×3</td>
<td>21</td>
<td>0.249</td>
<td>0.256</td>
</tr>
<tr>
<td>4×4</td>
<td>40</td>
<td>0.240</td>
<td>0.248</td>
</tr>
<tr>
<td>5×5</td>
<td>65</td>
<td>0.236</td>
<td>0.242</td>
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<tr>
<td>6×6</td>
<td>35</td>
<td>0.238</td>
<td>0.238</td>
</tr>
<tr>
<td>8×8</td>
<td>47</td>
<td>0.234</td>
<td>0.234</td>
</tr>
<tr>
<td>10×10</td>
<td>59</td>
<td>0.232</td>
<td>0.232</td>
</tr>
<tr>
<td>Exact</td>
<td>[46]</td>
<td>0.231</td>
<td>0.513</td>
</tr>
</tbody>
</table>
From Table 2.1, it can be seen that the tendency of the change of the deflection with the divisional number $m$ is different for the uniform load and the concentrated load in the proposed method. The uniform load can be estimated accurately even for the small divisional number $m$, but the concentrated load can’t. For the small number $m$, the concentrated load is underestimated as a uniform load in a definite area. Therefore, the deflection obtained by the proposed method is smaller than the exact solution. With the increase of the number $m$, the concentrated load can be satisfied more and more accurately and the present result is more and more close to the exact solution.

From Table 2.2, it can be seen that with the increase of the number $m$, the proposed method has more accuracy than that of FEM. Compared with the FEM, the proposed method has the advantage of using less unknown quantities to obtain satisfactory accuracy and the proposed method is closer to the exact solution than FEM. Both the proposed method and the FEM agree well with the exact solution for the bending isotropic plate with a concentrated load and a uniform load.

### 2.6 Conclusions

The main work in this chapter is described as follows:

1. The fundamental differential equations are established for the orthotropic plates with variable thickness and a concentrated load combined with a uniform load. By converting the differential equations into integral equations and applying numerical integration, the discrete solutions for elastic bending problems based on the Mindlin plate theory are obtained.

2. The above discrete method is extended to analyze the free vibration problems for the orthotropic plates by using Green function. The characteristic equation of the free vibration of orthotropic plates with variable thickness is obtained.

3. The proposed method does not require prior assumption of the shape of the deflection mode of the orthotropic plate and it can be considered as a semi-analytical method. With the increase of the numbers of dividing lines, the discrete solutions tend to analytical solutions theoretically. The discrete solutions are only related to the quantities of the discrete points on
the boundary. So it can make the calculation more easily, quickly and accurately.

(4) By translating the irregular-shaped and variable thickness plate into the equivalent rectangular plate, the bending and vibration problems of the irregular-shaped plate can be solved using the general discrete method developed by this chapter.

(5) Compared with FEM, the proposed method does not require prior assumption of the shape of the deflection mode of the plate. On the other hand, the proposed method has the advantage of using less unknown quantities to obtain satisfactory accuracy.

2.7 Appendix A

\[ A_{p1} = \gamma_{p1}, \quad A_{p2} = 0, \quad A_{p3} = \gamma_{p2}, \quad A_{p4} = \gamma_{p3}, \quad A_{p5} = 0, \quad A_{p6} = \overline{D}_{12}\gamma_{p4} + \overline{D}_{22}\gamma_{p5} + \overline{D}_{26}\gamma_{p6} \]

\[ A_{p7} = \overline{D}_{16}\gamma_{p4} + \overline{D}_{26}\gamma_{p5} + \overline{D}_{66}\gamma_{p6}, \quad A_{p8} = k(\overline{A}_{44}\gamma_{p7} + \overline{A}_{45}\gamma_{p8}) \]

\[ B_{p1} = 0, \quad B_{p2} = \mu\gamma_{p1}, \quad B_{p3} = \mu\gamma_{p3}, \quad B_{p4} = 0, \quad B_{p5} = \mu\gamma_{p2} \]

\[ B_{p6} = \mu(\overline{D}_{16}\gamma_{p4} + \overline{D}_{26}\gamma_{p5} + \overline{D}_{66}\gamma_{p6}), \quad B_{p7} = \mu(\overline{D}_{11}\gamma_{p4} + \overline{D}_{12}\gamma_{p5} + \overline{D}_{16}\gamma_{p6}) \]

\[ B_{p8} = \mu k(\overline{A}_{45}\gamma_{p7} + \overline{A}_{55}\gamma_{p8}) \]

\[ C_{p_{kl}} = \mu\gamma_{p3} + \mu \overline{D} \delta_{{kl}} \gamma_{p7}, \quad C_{p_{kl}} = \mu\gamma_{p2} + \mu \overline{D} \delta_{{kl}} \gamma_{p8}, \quad C_{p_{kl}} = \mu\overline{D} \delta_{{kl}} \gamma_{p6}, \quad C_{p_{kl}} = \mu\overline{D} \delta_{{kl}} \gamma_{p5} \]

\[ C_{p_{kl}} = \mu \overline{D} \delta_{{kl}} \gamma_{p4}, \quad C_{p_{kl}} = -\mu k(\overline{A}_{44}\gamma_{p7} + \overline{A}_{45}\gamma_{p8}), \quad C_{p_{kl}} = -\mu k(\overline{A}_{45}\gamma_{p7} + \overline{A}_{55}\gamma_{p8}) \]

\[ C_{p_{kl}} = 0, \quad [\gamma_{pt}] = [\rho_{pt}]^{-1} \]
Appendix B

\[ a_{i0i1} = a_{3i0i2} = a_{4i0i3} = 1, a_{6i0i4} = a_{7i0i5} = a_{8i0i6} = 1, \quad b_{20j1} = b_{30j2} = b_{50j3} = 1, \]

\[ b_{60j4} = b_{70j5} = b_{80j6} = 1, \quad b_{30002} = 0, \]

\[ a_{pijkl} = \sum_{i=1}^{8} \left\{ \sum_{k=0}^{i} \beta_{ik} A_{pl} \left[ a_{ik0jd} - a_{skld} (1 - \delta_{ki}) \right] + \sum_{l=0}^{j} \beta_{jl} B_{pl} \left[ a_{i0ljd} - a_{ijjd} (1 - \delta_{lj}) \right] \right. \]

\[ + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{plkl} a_{skld} (1 - \delta_{ki} \delta_{lj}) \right\}, \]

\[ b_{pijkl} = \sum_{i=1}^{8} \left[ \sum_{k=0}^{i} \beta_{ik} A_{pl} \left[ b_{ik0gd} - b_{skgd} (1 - \delta_{ki}) \right] + \sum_{l=0}^{j} \beta_{jl} B_{pl} \left[ b_{i0lgd} - b_{ilgd} (1 - \delta_{lj}) \right] \right. \]

\[ + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{plkl} b_{skgd} (1 - \delta_{ki} \delta_{lj}) \right\} \]

\[ \tilde{q}_{pijkl} = \sum_{i=1}^{8} \left[ \sum_{k=0}^{i} \beta_{ik} A_{pl} \left[ \tilde{q}_{ik0} - \tilde{q}_{sk} (1 - \delta_{ki}) \right] + \sum_{l=0}^{j} \beta_{jl} B_{pl} \left[ \tilde{q}_{i0l} - \tilde{q}_{il} (1 - \delta_{lj}) \right] \right. \]

\[ + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{plkl} \tilde{q}_{sk} (1 - \delta_{ki} \delta_{lj}) \right\} - \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} A_{pl} \tilde{q}_{sk} \]

\[ \tilde{q}_{pijkl} = \sum_{i=1}^{8} \left[ \sum_{k=0}^{i} \beta_{ik} A_{pl} \left[ \tilde{q}_{ik0} - \tilde{q}_{sk} (1 - \delta_{ki}) \right] + \sum_{l=0}^{j} \beta_{jl} B_{pl} \left[ \tilde{q}_{i0l} - \tilde{q}_{il} (1 - \delta_{lj}) \right] \right. \]

\[ + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{plkl} \tilde{q}_{sk} (1 - \delta_{ki} \delta_{lj}) \right\} - A_{p1} t_{q} u_{j1}. \]
Appendix C

The following is the equation (2.7)

\[ \frac{\partial X_2}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} + q + P \delta(\eta - \eta_0) \delta(\zeta - \zeta_0) = 0 \]  \hspace{1cm} (2.7-a)

\[ \frac{\partial X_3}{\partial \eta} + \frac{\partial X_4}{\partial \zeta} - \mu X_1 = 0 \]  \hspace{1cm} (2.7-b)

\[ \frac{\partial X_5}{\partial \eta} + \frac{\partial X_6}{\partial \zeta} - \mu X_2 = 0 \]  \hspace{1cm} (2.7-c)

\[ \overline{D}_{11} \frac{\partial X_5}{\partial \eta} + \overline{D}_{12} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{16} (\frac{\partial X_2}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta}) - \mu \overline{DX}_5 = 0 \]  \hspace{1cm} (2.7-d)

\[ \overline{D}_{12} \frac{\partial X_5}{\partial \eta} + \overline{D}_{22} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{26} (\frac{\partial X_2}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta}) - \mu \overline{DX}_4 = 0 \]  \hspace{1cm} (2.7-e)

\[ \overline{D}_{16} \frac{\partial X_5}{\partial \eta} + \overline{D}_{26} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{66} (\frac{\partial X_2}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta}) - \mu \overline{DX}_3 = 0 \]  \hspace{1cm} (2.7-f)

\[ k \overline{A}_{44} (\frac{\partial X_8}{\partial \zeta} + \mu X_8) + k \mu \overline{A}_{45} (\frac{\partial X_8}{\partial \eta} + X_7) - \mu \overline{DT}X_1 = 0 \]  \hspace{1cm} (2.7-g)

\[ k \overline{A}_{45} (\frac{\partial X_8}{\partial \zeta} + \mu X_8) + k \mu \overline{A}_{55} (\frac{\partial X_8}{\partial \eta} + X_7) - \mu \overline{DT}X_2 = 0 \]  \hspace{1cm} (2.7-h)

Equations (2.7-d)-(2.7-f) can be expressed as a matrix form:

\[ \begin{bmatrix} \mu \overline{DX}_5 \\ \mu \overline{DX}_4 \\ \mu \overline{DX}_3 \end{bmatrix} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{16} \\ \overline{D}_{12} & \overline{D}_{22} & \overline{D}_{26} \\ \overline{D}_{16} & \overline{D}_{26} & \overline{D}_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial X_7}{\partial \eta} \\ \frac{\partial X_6}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \end{bmatrix} \]  \hspace{1cm} (C-1)

Using the inverse matrix \( [S] = [\overline{D}]^{-1} \), the equation (C-1) can be expressed as:

\[ \begin{bmatrix} \frac{\partial X_7}{\partial \eta} \\ \frac{\partial X_6}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \mu \overline{DX}_5 \\ \mu \overline{DX}_4 \\ \mu \overline{DX}_3 \end{bmatrix} \]  \hspace{1cm} (C-2)

Following two equations can be obtained:

\[ \mu \frac{\partial X_7}{\partial \eta} = \mu \overline{D} \{ S_{11} X_5 + S_{12} X_4 + S_{16} X_3 \} \]
\[
\frac{\partial X_6}{\partial \zeta} = \mu \overline{D}\{S_{12}X_5 + S_{22}X_4 + S_{26}X_3\} \quad (C-3)
\]

Equation (C-3) can be rewritten as:

\[
\frac{\partial X_7}{\partial \eta} = \overline{D}\{S_{11}X_5 + S_{12}X_4 + S_{16}X_3\} \quad (C-4-1)
\]

\[
\frac{\partial X_6}{\partial \zeta} = \mu \overline{D}\{S_{12}X_5 + S_{22}X_4 + S_{26}X_3\} \quad (C-4-2)
\]

Integrating the equation (C-4-1) on the line \([\eta,0]\), the numerical integration can be derived:

\[
\int_0^\eta \frac{\partial X_7}{\partial \eta} d\eta = \int_0^\eta \overline{D}\{S_{11}X_5 + S_{12}X_4 + S_{16}X_3\} d\eta
\]

\[
X_{7/0} - X_{700} = \sum_{k=0}^{i} \beta_{ik} \overline{D}_{k0} S_{11}X_{5k0} + S_{12}X_{4k0} + S_{16}X_{3k0} \quad (C-5)
\]

\[
\sum_{k=1}^{i} \beta_{ik} \overline{D}_{k0} S_{11}X_{5k0} = -\beta_{i0} \overline{D}_{00} S_{11}X_{500} - \sum_{k=0}^{i} \beta_{ik} \overline{D}_{k0} \{S_{12}X_{4k0} + S_{16}X_{3k0}\} + (X_{7/0} - X_{700}) \quad (C-6)
\]

Using the matrix to express the right-hand side of equation (C-6):

\[
\begin{bmatrix}
\beta_{i1} & 0 & 0 & \cdots & 0 \\
\beta_{i2} & \beta_{i2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{i1} & \beta_{i2} & \beta_{i3} & \cdots & \beta_{iu}
\end{bmatrix}
\begin{bmatrix}
\overline{D}_{10} S_{11} X_{510} \\
\overline{D}_{20} S_{11} X_{520} \\
\overline{D}_{30} S_{11} X_{530} \\
\vdots \\
\overline{D}_{i0} S_{11} X_{5i0}
\end{bmatrix}
= A_1
\]

\[
\begin{bmatrix}
\overline{D}_{10} S_{11} X_{510} \\
\overline{D}_{20} S_{11} X_{520} \\
\overline{D}_{30} S_{11} X_{530} \\
\vdots \\
\overline{D}_{i0} S_{11} X_{5i0}
\end{bmatrix}
= \begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_u
\end{bmatrix}
\]

Unknown quantities of \(\overline{D}_{i0} S_{11} X_{5i0}\) can be derived as:

\[
\begin{bmatrix}
\overline{D}_{10} S_{11} X_{510} \\
\overline{D}_{20} S_{11} X_{520} \\
\overline{D}_{30} S_{11} X_{530} \\
\vdots \\
\overline{D}_{i0} S_{11} X_{5i0}
\end{bmatrix}
= \begin{bmatrix}
\beta_{i1} & 0 & 0 & \cdots & 0 \\
\beta_{i1} & \beta_{i2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{i1} & \beta_{i2} & \beta_{i3} & \cdots & \beta_{iu}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
\vdots \\
A_u
\end{bmatrix}
\]

where \([b_y] = [\beta_y]^{-1}\)  

\[
\overline{D}_{i0} S_{11} X_{5i0} = \sum_{k=1}^{i} \beta_{ik} A_k
\]

\[
= b_{i1} - \beta_{i0} \overline{D}_{00} S_{11} X_{500} - \sum_{k=0}^{i} \beta_{ik} \overline{D}_{k0} \{S_{12}X_{4k0} + S_{16}X_{3k0}\} + (X_{7/0} - X_{700})
\]

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\[\begin{align*}
&+ b_{12} \left[- \beta_{20} \overline{D}_{00} S_{11} X_{500} - \sum_{k=0}^{2} \beta_{2k} \overline{D}_{k40} \{ S_{12} X_{440} + S_{16} X_{340} \} + (X_{720} - X_{700}) \right] \\
&+ b_{13} \left[- \beta_{30} \overline{D}_{00} S_{11} X_{500} - \sum_{k=0}^{3} \beta_{3k} \overline{D}_{k40} \{ S_{12} X_{440} + S_{16} X_{340} \} + (X_{730} - X_{700}) \right] \\
&\vdots \\
&+ b_{i} \left[- \beta_{i0} \overline{D}_{00} S_{11} X_{500} - \sum_{k=0}^{i} \beta_{ik} \overline{D}_{k40} \{ S_{12} X_{440} + S_{16} X_{340} \} + (X_{7i0} - X_{700}) \right]
\end{align*}\]

\[\overline{D}_{i0} S_{11} X_{5i0} = X_{500} (1 - \overline{D}_{i0} S_{11}) \sum_{k=1}^{i} b_{ik} \beta_{k0}\]

Therefore,

\[\begin{align*}
X_{5i0} &= X_{500} \left( \frac{\overline{D}_{00} S_{11}}{\overline{D}_{i0} S_{11}} \sum_{k=1}^{i} b_{ik} \beta_{k0} \right) \\
&+ \left\{ X_{400} \left( \frac{\overline{D}_{00} S_{12}}{\overline{D}_{i0} S_{11}} \right) + X_{300} \left( \frac{\overline{D}_{00} S_{16}}{\overline{D}_{i0} S_{11}} \right) \right\} \sum_{k=1}^{i} b_{ik} \beta_{k0} + X_{700} \left( \frac{1}{\overline{D}_{i0} S_{11}} \right) \sum_{k=1}^{i} b_{ik} \\
&+ \left\{ X_{410} \left( \frac{\overline{D}_{00} S_{12}}{\overline{D}_{i0} S_{11}} \right) + X_{310} \left( \frac{\overline{D}_{00} S_{16}}{\overline{D}_{i0} S_{11}} \right) \right\} \sum_{k=1}^{i} b_{ik} \beta_{k1} + X_{710} \left( \frac{b_{i1}}{\overline{D}_{i0} S_{11}} \right) \\
&\vdots \\
&+ \left\{ X_{4i0} \left( \frac{\overline{D}_{00} S_{12}}{\overline{D}_{i0} S_{11}} \right) + X_{3i0} \left( \frac{\overline{D}_{00} S_{16}}{\overline{D}_{i0} S_{11}} \right) \right\} \sum_{k=1}^{i} b_{ik} \beta_{ki} + X_{7i0} \left( \frac{b_{i1}}{\overline{D}_{i0} S_{11}} \right) 
\end{align*}\] (C-9)

Rewriting the above equation in the simple form:

\[X_{5i0} = b_{5i003} X_{500} + \sum_{k=0}^{i} (a_{5_{04}k2} X_{34k0} + a_{5_{04}k3} X_{44k0} + a_{5_{04}k4} X_{74k0})\] (C-10)

Here \( b_{5i003} = \frac{\overline{D}_{00} S_{11}}{\overline{D}_{i0} S_{11}} \sum_{k=1}^{i} b_{ik} \beta_{k0} \)
The discrete solutions expressed in equation (2.16) are

\[
X_{pji} = \sum_{d=1}^{6} \left( \sum_{f=0}^{5} a_{pji\eta} X_{rf0} + \sum_{g=0}^{5} b_{pji\eta} X_{sg0} \right) + q_{pji} + q_{pji} P
\]

Where \((Q_\eta) = X_1\), \((Q_\zeta) = X_2\), \((M_{xy}) = X_3\), \((M_x) = X_4\), \((M_y) = X_5\), \((\theta_x) = X_6\), \((\theta_y) = X_7\) and \((w) = X_8\) are integral constants.

These constants along the edges \(\eta = 0\) and \(\zeta = 0\) are illustrated in Figure (C-1). There are only six integral constants at each discrete point. \(X_1\) and \(X_4\) do not exist along the edge \(\eta = 0\). \(X_2\) and \(X_5\) do not exist along the edge \(\zeta = 0\). So it is necessary to find the relations between the existent and un-existent constants. The coefficient \(a_{5i03}\) is used as an example to explain the meaning of foots of the coefficient.

- \(a_{x00\zeta}\): the coefficient of the integral constants along the edge \(\zeta = 0\);
- \(b_{x00\eta}\): the coefficient of the integral constants along the edge \(\eta = 0\);
- \(a_{5i03}\): the coefficient of the constants \(X_5\);
- \(a_{5i0\eta}\): the coefficient at point \((\eta,0)\);
- \(a_{5i0\zeta}\): the coefficient of the integral at point \((\eta,0)\);
- \(a_{5i02}\): the coefficient of the constant \(X_4\) at point \((\eta,0)\);
From the equation (C-4-2), using the same manner, the following equation and coefficients can be obtained:

\[ X_{40} = a_{40,03} X_{400} + \sum_{i=0}^{i} (b_{40,i2} X_{30i} + b_{40,i3} X_{50i} + b_{40,i4} X_{60i}) \]  \hspace{1cm} (C-11)

Here  \( a_{40,03} = -\frac{D_{00} S_{22}}{D_{0j} S_{22}} \sum_{i=1}^{i} b_{ij} \beta_{10} \)

\[ b_{40,j2} = -\frac{D_{02} S_{26}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j3} = -\frac{D_{02} S_{12}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j4} = -\frac{D_{02} S_{02}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],

\[ b_{40,j2} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j3} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j4} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],

\[ b_{40,j2} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j3} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],  \hspace{1cm} b_{40,j4} = -\frac{D_{0j} S_{22}}{D_{0j} S_{22}} \sum_{i=2}^{i} b_{ij} \beta_{12} \],
The equation (2.7-g)-(2.7-h) can be expressed by a matrix as follow:

\[
\begin{bmatrix}
\mu DT X_1 \\
\mu DT X_2
\end{bmatrix} = k \begin{bmatrix}
\frac{\partial X_{8}}{\partial \zeta} + \frac{\partial X_{8}}{\partial \eta}
\end{bmatrix}
\]

(C-12)

Using the inverse matrix \[A\]^{-1}, the equation (C-12) can be expressed as:

\[
k \begin{bmatrix}
\mu X_6 + \frac{\partial X_8}{\partial \zeta} \\
\mu X_7 + \frac{\partial X_8}{\partial \eta}
\end{bmatrix} = \begin{bmatrix}
T_{44} & T_{45} \\
T_{45} & T_{55}
\end{bmatrix} \begin{bmatrix}
\mu DT X_1 \\
\mu DT X_2
\end{bmatrix}
\]

(C-13)

The above equation rewritten as:

\[
k \frac{\partial X_8}{\partial \zeta} = \mu DT (T_{44} X_1 + T_{45} X_2) - k \mu X_6
\]

(C-14-1)

\[
k \frac{\partial X_8}{\partial \eta} = DT (T_{45} X_1 + T_{55} X_2) - k X_7
\]

(C-14-2)

Integrating the equation (C-14-2) on the line \[\eta,0\], with the same method described above, the following equation can be derived:

\[
X_{20} = b_{2000} X_{200} + \sum_{k=0}^{i} (a_{20k1} X_{1k0} + a_{20k5} X_{7k0} + a_{20k6} X_{8k0})
\]

(C-15)
With the same method, from equation (C-14-1), the following can be obtained:

\[ X_{10j} = a_{10j01} X_{100} + \sum_{l=0}^{j} (b_{10j1l} X_{20l} + b_{10j4l} X_{60l} + b_{10j6l} X_{80l}) \]  

(C-16)

\[ a_{10j01} = \frac{\bar{D} T_{00} T_{44}}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{10} \]

\[ b_{10j01} = \frac{-\bar{D} T_{00} T_{45}}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{10} , \quad b_{10j04} = \frac{k}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{12} , \quad b_{10j06} = -\frac{k}{\mu \bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \]

\[ b_{10j11} = -\frac{\bar{D} T_{01} T_{45}}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{11} , \quad b_{10j14} = \frac{k}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{12} , \quad b_{10j16} = -\frac{k}{\mu \bar{D} T_{0j} T_{44}} b_{j1} \]

\[ b_{10j21} = -\frac{\bar{D} T_{02} T_{45}}{\bar{D} T_{0j} T_{44}} \sum_{l=2}^{j} b_{jl} \beta_{12} , \quad b_{10j24} = \frac{k}{\bar{D} T_{0j} T_{44}} \sum_{l=2}^{j} b_{jl} \beta_{12} , \quad b_{10j26} = -\frac{k}{\mu \bar{D} T_{0j} T_{44}} b_{j2} \]

\[ b_{10j1} = -\frac{\bar{D} T_{00} T_{45}}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{11} , \quad b_{10j4} = \frac{k}{\bar{D} T_{0j} T_{44}} \sum_{l=1}^{j} b_{jl} \beta_{12} , \quad b_{10j6} = -\frac{k}{\mu \bar{D} T_{0j} T_{44}} b_{j1} \]

From equations (C-10), (C-11), (C-15) and (C-16), the following coefficients can be simply expresses as:

\[ a_{r0ad} = 1, \quad b_{s0j0d} = 1, \]

\[ a_{10j01} = \frac{\bar{D} T_{00} T_{44}}{\bar{D} T_{0j} T_{44}} \alpha_{j}, b_{10j1} = -\frac{\bar{D} T_{00} T_{45}}{\bar{D} T_{0j} T_{44}} \beta_{1j} , b_{10j4} = \frac{k}{\bar{D} T_{0j} T_{44}} \beta_{1j} , b_{10j6} = -\frac{k}{\mu \bar{D} T_{0j} T_{44}} \gamma_{j} \]

\[ b_{2001} = -\frac{\bar{D} T_{00} T_{55}}{\bar{D} T_{0j} T_{55}} \alpha_{i}, a_{20i1} = -\frac{\bar{D} T_{00} T_{45}}{\bar{D} T_{0j} T_{55}} \beta_{ik} , a_{20i4} = \frac{k}{\bar{D} T_{0j} T_{55}} \beta_{ik} , a_{20i6} = -\frac{k}{\mu \bar{D} T_{0j} T_{55}} \gamma_{ik} \]

\[ a_{40j03} = -\frac{D_{00} S_{22}}{D_{0j} S_{22}} \alpha_{j} , b_{40j2} = \frac{-D_{00} S_{26}}{D_{0j} S_{22}} \beta_{j2} , b_{40j3} = \frac{-D_{00} S_{26}}{D_{0j} S_{22}} \beta_{j2} , b_{40j4} = \frac{1}{\mu D_{0j} S_{22}} \gamma_{j} \]

\[ b_{5003} = \frac{-D_{00} S_{11}}{D_{0j} S_{11}} \alpha_{i} , a_{50i2} = \frac{-D_{00} S_{16}}{D_{0j} S_{11}} \beta_{ik} , a_{50i3} = \frac{-D_{00} S_{16}}{D_{0j} S_{11}} \beta_{ik} , a_{50i5} = \frac{1}{D_{0j} S_{11}} \gamma_{ik} \]

\[ \alpha_{j} = (-1)^{j}, \quad \bar{\alpha}_{j} = (-1)^{j}, \quad \bar{\beta}_{ik} = \delta_{ik} + (-1)^{(i+1)} \delta_{0k}, \quad \bar{\beta}_{jkl} = \delta_{jkl} + (-1)^{(j+1)} \delta_{0l}, \]
\[
\bar{y}_{ik} = \frac{4m(-1)^{(i+k)}}{1 + \delta_{ik} + \delta_{0k}}, \quad \bar{y}_{jl} = \frac{4n(-1)^{(j+l)}}{1 + \delta_{jl} + \delta_{0l}}, \quad \bar{y}_{ik} = \begin{cases} 0 & (i < k) \\ \bar{y}_{ik} & (i \geq k) \end{cases}, \quad \bar{y}_{jl} = \begin{cases} 0 & (j < l) \\ \bar{y}_{jl} & (j \geq l) \end{cases}
\]

\[
\bar{D}T_{i0} = \left( \frac{h_0}{h_{i0}} \right), \quad \bar{D}T_{0j} = \left( \frac{h_0}{h_{0j}} \right), \quad \bar{D}_{i0} = \left( \frac{h_0}{h_{i0}} \right)^3, \quad \bar{D}_{0j} = \left( \frac{h_0}{h_{0j}} \right)^3
\]

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{16} \\
S_{12} & S_{22} & S_{26} \\
S_{16} & S_{26} & S_{66}
\end{bmatrix} = \begin{bmatrix}
\bar{D}_{11} & \bar{D}_{12} & \bar{D}_{16} \\
\bar{D}_{12} & \bar{D}_{22} & \bar{D}_{26} \\
\bar{D}_{16} & \bar{D}_{26} & \bar{D}_{66}
\end{bmatrix}^{-1}, \quad \begin{bmatrix}
T_{44} & T_{45} \\
T_{45} & T_{55}
\end{bmatrix} = \begin{bmatrix}
\bar{A}_{44} & \bar{A}_{45} \\
\bar{A}_{45} & \bar{A}_{55}
\end{bmatrix}^{-1}
\]
Chapter 3

Experimental Results
3.1 Laser Holography Interferometry

The unique characteristic of holography is that both the phase and amplitude of the light waves from an object are recorded. The laser beam is split as a reference beam and an object beam by the beam splitter and they produce an interference pattern, and the pattern is recorded by the photographic film. If the hologram is illuminated once again with a same wave-length reference wave, the original object wave will be reconstructed.

In the experiment, the specimen on which the strain gauges to be affixed, are fixed in the fixture and acoustically excited by a siren. Oscillation frequencies at the maximum output of strain gauges are considered as resonance frequencies. These resonance vibration modes are obtained by forming a hologram on photographic plate, using the time-average method of holographic interferometry. The experimental device and principle are shown in Figure 3.1 and 3.2, respectively. Figure 3.3 shows the reconstruction of hologram [48].

Figure 3.1 Laser holography instrument
Figure 3.2 Principle of hologram

Figure 3.3 Reconstruction of hologram
3.2 Experimental Results

In this experiment, aluminum alloy plates are used as specimen and the dimensions and material properties of the plates are given in Table 3.1.

<table>
<thead>
<tr>
<th>Material</th>
<th>$a \times b \times h$ (mm)</th>
<th>$E$ (GPa)</th>
<th>$G$ (GPa)</th>
<th>$\nu$</th>
<th>$\rho$ (kN/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum Alloy</td>
<td>90×90×1</td>
<td>69.6</td>
<td>26.1</td>
<td>0.33</td>
<td>26.5</td>
</tr>
</tbody>
</table>

3.2.1 Plates with uniform thickness

Firstly, in order to investigate the convergence and accuracy of the analytical results, the isotropic uniform cantilever plates are used to obtain frequencies and vibration modes by holographic interferometry. For the aluminum alloy cantilever plates, the convergence and accuracy of the lowest six modes frequencies with number of divisions and the extrapolated values are shown in Table 3.2, together with the analytical results obtained by Ref.[49] and experimental results. Figure 3.4 shows the nodal patterns.

<table>
<thead>
<tr>
<th>$m=n$</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>105.8</td>
<td>257.0</td>
<td>679.4</td>
<td>868.2</td>
<td>979.3</td>
<td>1738</td>
</tr>
<tr>
<td>12</td>
<td>105.5</td>
<td>255.2</td>
<td>658.0</td>
<td>842.6</td>
<td>949.7</td>
<td>1674</td>
</tr>
<tr>
<td>Ex.</td>
<td>105.3</td>
<td>253.7</td>
<td>640.9</td>
<td>822.1</td>
<td>926.1</td>
<td>1622</td>
</tr>
<tr>
<td>Ref.[49]</td>
<td>105.7</td>
<td>260.3</td>
<td>648.3</td>
<td>828.2</td>
<td>947.0</td>
<td>1653</td>
</tr>
<tr>
<td>Experiment</td>
<td>100.0</td>
<td>245.0</td>
<td>580.0</td>
<td>815.0</td>
<td>890.0</td>
<td>1520</td>
</tr>
</tbody>
</table>

Ex.: The values obtained by using Richardson’s extrapolation formula.

![Figure 3.4 Nodal patterns for plates with uniform thickness (upper: experimental; lower: analytical)](image-url)
3.2.2 Plates with non-uniform thickness

Four types of plates with non-uniform thickness are used in the experiment, plate shapes refer to Figure 3.5. Table 3.3 shows vibration frequencies [51] and nodal patterns are shown in Figure 3.6-3.9 (the left is for analytical results and the right is for experimental results).

Figure 3.5 Specimen with non-uniform thickness (A, B, C, D) (holes used for fixing)
Table 3.3 Vibration frequencies of lowest five modes of plates A, B, C and D (Hz)

<table>
<thead>
<tr>
<th>m=12</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical -A</td>
<td>126.8</td>
<td>232.3</td>
<td>460.3</td>
<td>547.5</td>
<td>-</td>
</tr>
<tr>
<td>Experimental-A</td>
<td>115.0</td>
<td>208.0</td>
<td>387.0</td>
<td>524.0</td>
<td>-</td>
</tr>
<tr>
<td>Analytical -B</td>
<td>124.3</td>
<td>221.4</td>
<td>445.3</td>
<td>503.2</td>
<td>577.4</td>
</tr>
<tr>
<td>Experimental-B</td>
<td>127.0</td>
<td>268.0</td>
<td>424.0</td>
<td>542.0</td>
<td>575.0</td>
</tr>
<tr>
<td>Analytical-C</td>
<td>91.7</td>
<td>203.2</td>
<td>487.4</td>
<td>555.5</td>
<td>725.1</td>
</tr>
<tr>
<td>Experimental-C</td>
<td>95.0</td>
<td>206.0</td>
<td>497.0</td>
<td>580.0</td>
<td>761.0</td>
</tr>
<tr>
<td>Analytical-D</td>
<td>87.5</td>
<td>196.5</td>
<td>467.6</td>
<td>539.3</td>
<td>703.0</td>
</tr>
<tr>
<td>Experimental-D</td>
<td>89.0</td>
<td>183.0</td>
<td>477.0</td>
<td>580.0</td>
<td>670.0</td>
</tr>
</tbody>
</table>

Figure 3.6 Comparison of nodal patterns of analytical and experimental results of plate A
Figure 3.7 Comparison of nodal patterns of analytical and experimental results of plate B

Figure 3.8 Comparison of nodal patterns of analytical and experimental results of plate C
From above Figures, it is shown that for plates A and B, the nodal patterns of analytical results are almost the same as experimental results. All of nodal lines are symmetrical to the vertical axis. For plates C and D, because of the plates with unsymmetrical thickness, nodal lines are not symmetrical to the vertical axis. Inclined nodal lines are obtained both in analytical results and in experimental results. The analytical results are in agreement with experimental results.

3.2.3 Plates with a hole defect

Table 3.4 shows the characteristics of the plate with a hole defect A and defect B. Figure 3.10 shows a plate with a defect and Figure 3.11 shows the size of the defect, the defect is in the center of the plate.
Table 3.4 Characteristics of the specimen of plates with a hole defect

<table>
<thead>
<tr>
<th>Model</th>
<th>Dimension (mm)</th>
<th>Defect size (the ratio of length)*1</th>
<th>Defect thickness (mm) (the ratio of thickness)*2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Length × width × height</td>
<td>Length × width (mm)</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>90×90×1</td>
<td>15×15 (0.17)</td>
<td>0.1 (0.1)</td>
</tr>
<tr>
<td>B</td>
<td>90×90×1</td>
<td>15×15 (0.17)</td>
<td>0.5 (0.5)</td>
</tr>
</tbody>
</table>

*1 The ration of length = c/a (%)  
*2 The ration of thickness = (h₀-h)/h₀ (%)  

Nodal patterns are shown in Figure 3.12 and Figure 3.13 for analytical [50] and experimental results of defect A and B. From 1st to 5th modes, the nodal patterns are the same as the uniform plate. So the nodal patterns from 1st to 5th modes are not shown in Figure 3.12 and Figure 3.13. Vibration frequencies are shown in Figure 3.14 for defect A.
Figure 3.12 Comparison of nodal patterns of analytical and experimental results (defect A)

Figure 3.13 Comparison of nodal patterns of analytical and experimental results (defect-B)
(The uppers are analytical results and the lowers are experimental results)
From above analysis, it is shown that the vibration frequencies of the analytical results are in agreement with the experimental results. From the experimental result analysis, it is shown that the proposed discrete method has enough accuracy for plates with a defect.

3.3 Conclusions

In this chapter, in order to verify the proposed discrete method described in Chapter 2, the experiments have been done for isotropic plates with uniform thickness, with non-uniform thickness and with a hole defect. The analytical results have been compared with experimental results and reference results. From above experimental results analysis, it is shown that proposed discrete method has enough convergence and accuracy.
Chapter 4

Free Vibration Analysis of Isotropic Plates with Non-homogeneity
4.1 Free Vibration Analysis of Isotropic Plates with Multiple Point Supports

4.1.1 Introduction

In this section, the proposed discrete method is developed for analyzing the free vibration problem of isotropic plates with point supports. The method is a general method. It can be used to analyze the free vibration of isotropic plates with arbitrarily located point supports, various aspect ratio, variable thickness and general boundary conditions.

4.1.2 Simulating Point Supports by Using Concentrated Loads with Dirac’s Delta Function

In this section, the concentrated loads with Dirac’s delta functions are used to simulate the point supports which limit the displacements of the plate but do not offer constraint on the slopes. Considering the equations of equilibrium, the strain – displacement relations, the stress – strain relations and the load – stress relations, the fundamental differential equations of the isotropic plate having a concentrated load \( P \) at a point \((x_q, y_c)\) and the point supports \( P_{cd} \) at each discrete point \((x_d, y_d)\) (as shown in Figure 4.1) are as follows:

\[
\frac{\partial^2 Q_x}{\partial x^2} + \frac{\partial^2 Q_y}{\partial y^2} + \frac{P}{\delta} [\delta(x-x_q)\delta(y-y_c)] + \sum_{c=0}^{m} \sum_{d=0}^{n} P_{cd} \delta(x-x_c)\delta(y-y_d) = 0
\]
By introducing the non-dimensional expressions,

\[
\begin{align*}
[X_1, X_2] &= \frac{a^2}{D_o (1 - \nu^2)} [Q_x, Q_y], \\
[X_3, X_4, X_5] &= \frac{a}{D_o (1 - \nu^2)} [M_{xy}, M_y, M_x], \\
[X_6, X_7, X_8] &= [\theta_y, \theta_x, w/a].
\end{align*}
\]

Eq. (4.1) is rewritten as the following non-dimensional forms:

\[
\begin{align*}
\mu \frac{\partial X_2}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} + P \delta(\eta - \eta_d) \delta(\zeta - \zeta_d) + \sum_{c=0}^{m} \sum_{d=0}^{n} P_{cd} \delta(\eta - \eta_c) \delta(\zeta - \zeta_d) &= 0, \\
\mu \frac{\partial X_3}{\partial \eta} + \frac{\partial X_4}{\partial \zeta} &= 0, \\
\mu \frac{\partial X_5}{\partial \eta} + \frac{\partial X_3}{\partial \zeta} &= 0, \\
\mu \frac{\partial X_7}{\partial \eta} + \nu \frac{\partial X_6}{\partial \zeta} - IX_5 &= 0, \\
\nu \frac{\partial X_7}{\partial \eta} + \frac{\partial X_6}{\partial \zeta} - IX_4 &= 0, \\
\mu \frac{\partial X_8}{\partial \eta} + \frac{\partial X_2}{\partial \zeta} - JX_3 &= 0, \\
\frac{\partial X_6}{\partial \eta} + X_7 - HX_2 &= 0.
\end{align*}
\]
\[
\frac{\partial X_s}{\partial \zeta} + \mu X_s - \mu H X_1 = 0,
\] (4.2)

here \( \mu = b/a, I = \mu(1 - \nu^2)(h_0/h)^3, J = 2\mu(1 + \nu)(h_0/h)^3, H = ((1 + \nu)/5)(h_0/a)^2(h_0/h), \)
\[ P = \overline{Pa} \left(D_0(1 - \nu^2)\right), \quad P_{cd} = \overline{P_{cd}a} \left(D_0(1 - \nu^2)\right), \quad D_0 = E h_0^3/(12(1 - \nu^2)) \]
\( \) is the standard bending, \( h_0 \) is the standard thickness of the isotropic plate, \( h \) is the thickness of the isotropic plate, \( t = h/1.2 \).

In the above equations, the variable quantity \( \frac{h_0}{h} \) is separated and expressed only in the quantities \( I, J \) and \( H \), so that the equations can be used for the isotropic plates with continuously variable thickness or stepped thickness.

Eq. (4.2) can also be expressed as the following simple form:
\[
\sum_{s=1}^{8} \left\{ F_{1s} \frac{\partial X_s}{\partial \zeta} + F_{2s} \frac{\partial X_s}{\partial \eta} + F_{3s} X_s\right\} + P \delta(\eta - \eta_d)\delta(\zeta - \zeta_d)\delta_{1s} + \\
+ \sum_{c=0}^{n} \sum_{d=0}^{n} P_{cd} \delta(\eta - \eta_c)\delta(\zeta - \zeta_d)\delta_{1s} = 0
\] (4.3)

where \( t=1-8, \delta_{1s} \) is Kronecker’s delta, \( F_{111} = F_{124} = F_{133} = F_{156} = F_{167} = F_{188} = 1, \)
\( F_{146} = \nu, \quad F_{212} = F_{223} = F_{235} = F_{247} = F_{256} = \mu, \quad F_{257} = \mu \nu, \quad F_{278} = 1, \quad F_{321} = F_{332} = -\mu, \)
\( F_{345} = F_{354} = -I, \quad F_{363} = -J, \quad F_{372} = -H, \quad F_{377} = 1, \quad F_{381} = -\mu H, \quad F_{386} = \mu, \quad \) other \( F_{kn} = 0. \)

As chapter 2 has described, by integrating the Eq. (4.3) over the area \([i, j]\), the following integral equation is obtained:
\[
\sum_{s=1}^{8} \left\{ F_{1s} \int_{0}^{\zeta} \left[ X_s(\eta, \zeta) - X_s(\eta, 0)\right] d\eta + F_{2s} \int_{0}^{\zeta} \left[ X_s(\eta, \zeta) - X_s(0, \zeta)\right] d\eta\right\} + \\
\sum_{c=0}^{n} \sum_{d=0}^{n} P_{cd} \int_{0}^{\zeta} \left[ X_s(\eta, \zeta) - X_s(\eta, 0)\right] d\eta \delta_{1s} \]
\[
+ \sum_{c=0}^{n} \sum_{d=0}^{n} P_{cd} \int_{0}^{\zeta} \left[ X_s(\eta, \zeta) - X_s(0, \zeta)\right] d\eta \delta_{1s} = 0
\] (4.4)

Next, by applying the trapezoidal rule of the approximate numerical integration over the area \([i, j]\), the simultaneous equation for the unknown quantities \( X_{sij} = X_s(\eta, \zeta) \) at the main point \((i, j)\) of the area \([i, j]\) is obtained directly from Equation (4.4) as follows:
\[
\sum_{s=1}^{8} \left\{ F_{1s} \sum_{k=0}^{j} \beta_{ik} X_{skj} - X_{sk0} \right\} + F_{2s} \sum_{l=0}^{j} \beta_{lj} (X_{slj} - X_{s0l}) + F_{3s} \sum_{k=0}^{j} \sum_{l=0}^{j} \beta_{ik} \beta_{lj} X_{skl} \}
\]
\[ + P u_{ij} u_{ji} \delta_{tt} + \sum_{c=0}^{m} \sum_{d=0}^{n} P_{cd} u_{ic} u_{jd} \delta_{tt} = 0 \]  

(4.5)

where

\[ \beta_{ik} = \alpha_{ik} / m, \beta_{ji} = \alpha_{ji} / n, \alpha_{ik} = 1 - (\delta_{ik} + \delta_{ji}) / 2, \alpha_{ji} = 1 - (\delta_{ij} + \delta_{ji}) / 2, t = 1 - 8, i = 1 - m, j = 1 - n, u_{iq} = u(\eta_{i} - \eta_{q}), u_{jr} = u(\zeta_{j} - \zeta_{r}), u_{ic} = u(\eta_{i} - \eta_{c}), u_{jd} = u(\zeta_{j} - \zeta_{d}). \]

By retaining the quantities at main point \((i, j)\) on the left hand side of the equation, putting other quantities on the right hand side, the following equation can be obtained:

\[
\sum_{s=1}^{9} \left\{ F_{1s} \beta_{s} + F_{2s} \beta_{s} + F_{3s} \beta_{s} \beta_{s} \right\} X_{sij} \]

\[ = \sum_{s=1}^{9} \left\{ F_{1s} \sum_{k=0}^{i} \beta_{ik} [X_{s,ik} - X_{s,ik}(1 - \delta_{ik})] + F_{2s} \sum_{l=0}^{j} \beta_{jl} [X_{s,jl} - X_{s,jl}(1 - \delta_{jl})] \right. \]

\[ - F_{3s} \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} X_{s,kl} (1 - \delta_{ik}\delta_{jl}) \left. \right\} - P u_{ij} u_{ji} \delta_{tt} - \sum_{c=0}^{m} \sum_{d=0}^{n} P_{cd} u_{ic} u_{jd} \delta_{tt} \]  

(4.6)

By using the matrix transition, the solution \( X_{pij} \) of the above Eq. (4.6) is obtained:

\[ X_{pij} = \sum_{i=1}^{9} \left\{ \sum_{k=0}^{i} \beta_{ik} A_{pt} [X_{p,tk} - X_{p,tk}(1 - \delta_{ik})] + \sum_{l=0}^{j} \beta_{jl} B_{pt} [X_{p,tl} - X_{p,tl}(1 - \delta_{jl})] \right. \]

\[ + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{ptkl} X_{p,kl} (1 - \delta_{ik}\delta_{jl}) \left. \right\} - A_{t1} P u_{ij} u_{jr} - \sum_{c=0}^{m} \sum_{d=0}^{n} A_{t1} P_{cd} u_{ic} u_{jd} \]  

(4.7)

where \( p = 1 - 8, A_{pt}, B_{pt}, C_{ptkl} \) are given in Appendix A.

With the same consideration described in chapter 2, Eq. (4.7) is rewritten as follows:

\[ X_{pij} = \sum_{d=1}^{6} \left\{ \sum_{f=0}^{i} a_{pjfd} X_{rfc} + \sum_{g=0}^{j} b_{pjfg} X_{sog} \right\} + \bar{q}_{pij} P + \sum_{c=0}^{m} \sum_{d=0}^{n} \bar{a}_{pjcd} P_{cd} \]  

(4.8)

where \( a_{pjfd}, b_{pjfg}, \bar{q}_{pij} \) and \( \bar{a}_{pjcd} \) are given in Appendix B.

Eq. (4.8) gives the discrete solution of the fundamental differential Eq. (4.3) of the bending problem of an isotropic plate with a concentrated load and point supports, and the discrete Green function is chosen as \( X_{sij} a^{2} / [PD_{0}(1 - \nu^{2})] \), that is \( w(x_{0}, y_{0}, x, y) / P \).

### 4.1.3 Characteristic Equation for Vibration Analysis
By using the method of Chapter 2 described, the displacement amplitude \( w(x_0, y_0) \) of point \((x_0, y_0)\) of the isotropic plate during the free vibration is given as follows:

\[
\hat{w}(x_0, y_0) = \int_0^b \int_0^a \rho h \omega^2 \hat{w}(x, y) [w(x_0, y_0, x, y) / P] \, dx \, dy,
\]  

(4.9)

By using the numerical integration method and the following non-dimensional expressions:

\[
\lambda^4 = \frac{\rho_0 h_0 \omega^2 a^4}{D_0 (1 - \nu^2)}, \quad \Lambda = 1/(\mu \lambda^4),
\]

\[
H(\eta, \zeta) = \frac{\rho h(x, y)}{\rho_0}, \quad W(\eta, \zeta) = \frac{\hat{w}(x, y)}{a},
\]

\[
G(\eta_0, \zeta_0, \eta, \zeta) = \frac{w(x_0, y_0, x, y)}{a} \frac{D_0 (1 - \nu^2)}{P a},
\]

where \( \rho_0 \) is the standard mass density, the characteristic equation is obtained from the Eq. (4.9) as

\[
\begin{bmatrix}
K_{00} & K_{01} & K_{02} & \cdots & K_{0m} \\
K_{10} & K_{11} & K_{12} & \cdots & K_{1m} \\
K_{20} & K_{21} & K_{22} & \cdots & K_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_{m0} & K_{m1} & K_{m2} & \cdots & K_{mm}
\end{bmatrix}
= 0
\]

(4.10)

where:

\[
K_{ij} = \beta_{mj} \begin{bmatrix}
\beta_{n0} H_{j0} G_{i0} - \Lambda \delta_{ij} & \beta_{n1} H_{j1} G_{i0} & \beta_{n2} H_{j2} G_{i0} & \cdots & \beta_{nn} H_{jn} G_{i0} \\
\beta_{n0} H_{j0} G_{i1} & \beta_{n1} H_{j1} G_{i1} - \Lambda \delta_{ij} & \beta_{n2} H_{j2} G_{i1} & \cdots & \beta_{nn} H_{jn} G_{i1} \\
\beta_{n0} H_{j0} G_{i2} & \beta_{n1} H_{j1} G_{i2} & \beta_{n2} H_{j2} G_{i2} - \Lambda \delta_{ij} & \cdots & \beta_{nn} H_{jn} G_{i2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n0} H_{j0} G_{in} & \beta_{n1} H_{j1} G_{in1} & \beta_{n2} H_{j2} G_{in2} & \cdots & \beta_{nn} H_{jn} G_{in} - \Lambda \delta_{ij}
\end{bmatrix}
\]

4.1.4 Numerical Results

To investigate the validity of the proposed method, the frequency parameters are given
for isotropic plates with arbitrarily located point supports, various aspect ratios, general boundary conditions and variable thickness. In this section, three kinds of isotropic plates with variable thickness are studied and they are shown in Figure 4.2. The ratio of the thickness and length \( h_0 / a = 0.001 \) is adopted. All the convergent values of the frequency parameters are obtained for the isotropic plates by using Richardson’s extrapolation formula for two cases of divisional numbers \( m(=n) \). Some of the results are compared with those reported before.

Figure 4.2 Isotropic plates with variable thickness (a) variable thickness in one direction \( h=h_0(1+\alpha x/a) \); (b) variable thickness in two directions \( h=h_0(1+\alpha x/a)(1+\beta y/b) \) and (c) stepped thickness in one direction.

4.1.4.1 Isotropic plates with a central point support

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions \( m \) and \( n \) for a SSSS square isotropic plate with a central point support. The lowest 7 natural frequency parameters of the isotropic plate are shown in Figure 4.3. It shows a good convergency of the numerical results by the proposed method. After studying the figure, it is decided to obtain the convergent results of frequency parameter by using Richardson’s extrapolation formula for two cases of divisional numbers \( m(=n) \) of 14 and 16. By the same method, the suitable number of divisions \( m(=n) \) can be determined for the other isotropic plates.
Figure 4.3 The natural frequency parameter $\lambda$ versus the divisional number $m (= n)$ for the SSSS square isotropic plate with a central point support.

Table 4.1 shows the numerical values for the lowest 6 natural frequency parameter $\lambda$ of SSSS isotropic plates with a central point support. The aspect ratios $b/a = 0.5, 1.0, 2.0$ are considered. The results obtained by Huang and Thambiratnam [6] and Kim and Dickinson [7] are also shown in the table. The nodal lines of 6 modes of free vibration of the isotropic plates with $b/a = 0.5, 1.0$ are shown in Figure 4.4, which are identical to those obtained in Ref. [6] for the square isotropic plate. The mode with no nodal line is the third one for the square isotropic plate, but the second mode for plates with $b/a = 0.5$. 

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Table 4.1 Natural frequency parameter \( \lambda \) for SSSS isotropic plates with a central point support

<table>
<thead>
<tr>
<th>( b/a )</th>
<th>Reference</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ex.</td>
<td>7.191</td>
<td>7.191</td>
<td>7.466</td>
<td>9.095</td>
<td>10.158</td>
<td>11.585</td>
</tr>
<tr>
<td></td>
<td>Ref.[6]</td>
<td>7.19</td>
<td>7.19</td>
<td>7.44</td>
<td>9.10</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>14×14</td>
<td>4.598</td>
<td>5.066</td>
<td>6.740</td>
<td>7.621</td>
<td>7.316</td>
<td>8.043</td>
</tr>
<tr>
<td></td>
<td>16×16</td>
<td>4.586</td>
<td>5.027</td>
<td>6.714</td>
<td>7.514</td>
<td>7.286</td>
<td>7.884</td>
</tr>
<tr>
<td></td>
<td>Ex.</td>
<td>4.548</td>
<td>4.901</td>
<td>6.629</td>
<td>7.162</td>
<td>7.195</td>
<td>7.364</td>
</tr>
</tbody>
</table>

Ex.*: The convergent values of the frequency parameters obtained by using Richardson’s extrapolation formula.
Table 4.2 shows the numerical values for the lowest 6 natural frequency parameter $\lambda$ of CCCC isotropic plates with a central point support and aspect ratios $b/a = 0.5, 1.0, 2.0$. The present results of CCCC square isotropic plate are in good agreement with those obtained by Kim and Dickinson [7]. The nodal lines of 6 modes of the isotropic plates with $b/a = 0.5, 1.0$ are shown in Figure 4.5. It can be observed for the modes with nodal lines passing through the center of the isotropic plate, the frequency parameters and mode shapes are the same for the isotropic plates with or without a central point support. The phenomenon occurred in CCCC isotropic plate is as the same as that in SSSS isotropic plate. For the ratio $b/a = 1.0$, the nodal lines of modes of CCCC isotropic plates are as the same as those of SSSS isotropic plates. For the ratio $b/a = 0.5$, there are some changes of mode order in the third, fourth, fifth and sixth modes. From Tables 4.1 and 4.2, it can be noted the boundary conditions affect the frequency parameters considerably.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>References</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>0.5</td>
<td>$14 \times 14$</td>
<td>11.712</td>
</tr>
<tr>
<td></td>
<td>$16 \times 16$</td>
<td>11.674</td>
</tr>
<tr>
<td></td>
<td>Ex.</td>
<td>11.549</td>
</tr>
<tr>
<td>1</td>
<td>$14 \times 14$</td>
<td>8.966</td>
</tr>
<tr>
<td></td>
<td>$16 \times 16$</td>
<td>8.919</td>
</tr>
<tr>
<td>2</td>
<td>$14 \times 14$</td>
<td>5.856</td>
</tr>
<tr>
<td></td>
<td>$16 \times 16$</td>
<td>5.837</td>
</tr>
</tbody>
</table>

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4.1.4.2 Square isotropic plates with a point support on a corner

Table 4.3 shows the numerical values for the lowest 6 natural frequency parameter $\lambda$ of square isotropic plates with a point support on the corner $(a,b)$. Three kinds of boundary conditions are considered. Table 4.3 involves the other values obtained by Kim and Dickinson [7] and it shows satisfactory accuracy of the numerical results by the proposed method.

Table 4.3 Natural frequency parameter $\lambda$ for square isotropic plates with a point support on the corner $(a, b)$

<table>
<thead>
<tr>
<th>BC</th>
<th>References</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSFF</td>
<td>14×14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16×16</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ref.[7]</td>
<td></td>
</tr>
<tr>
<td>SCFF</td>
<td>14×14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16×16</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ex.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ref.[7]</td>
<td></td>
</tr>
<tr>
<td>CCFF</td>
<td>14×14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16×16</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.5 Nodal patterns for CCCC isotropic plates with a central point support.
4.1.4.3 CFFF square isotropic plates with two arbitrarily located point supports

Table 4.4 shows the numerical values for the lowest 6 natural frequency parameter \( \lambda \) of the CFFF square isotropic plate with two point supports. The points \((a/2, 0), (a/2, b)\), the points \((a/2, b/4), (a/2, 3b/4)\) and points \((a/4, 3b/4), (3a/4, b/4)\) are chosen as the positions of the two supports, respectively. In order to compare with the results obtained by Saliba [1] [2] and Kim and Dickinson [7], two kinds of Poisson’s ratio \( \nu = 0.333 \) and \( \nu = 0.3 \) are used. Table 4.4 shows satisfactory accuracy of the numerical results by the proposed method and the fundamental frequency parameter of isotropic plate with two point supports along the edge are lower than those of corresponding isotropic plates with interior two point supports. The optimal location of the point support in increasing the fundamental frequency parameter discussed here is at point \((a/2, b/4), (a/2, 3b/4)\).

<table>
<thead>
<tr>
<th>Mode sequence number</th>
<th>Position of point supports</th>
<th>( \nu )</th>
<th>References</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((a/2, 0), (a/2, b))</td>
<td>0.333</td>
<td>Ex.</td>
<td>2.570</td>
<td>4.154</td>
<td>5.205</td>
<td>6.451</td>
<td>7.440</td>
<td>8.037</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ref.[1]</td>
<td>2.525</td>
<td>4.099</td>
<td>5.162</td>
<td>6.364</td>
<td>7.295</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ref.[7]</td>
<td>2.538</td>
<td>4.127</td>
<td>5.166</td>
<td>6.420</td>
<td>7.394</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>((a/2, b/4), (a/2, 3b/4))</td>
<td>0.333</td>
<td>Ex.</td>
<td>3.225</td>
<td>4.118</td>
<td>5.184</td>
<td>7.054</td>
<td>7.561</td>
<td>8.011</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ref.[2]</td>
<td>3.173</td>
<td>4.086</td>
<td>5.228</td>
<td>6.999</td>
<td>7.424</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>((a/2, 0), (a/2, b))</td>
<td>0.3</td>
<td>Ex.</td>
<td>2.586</td>
<td>4.173</td>
<td>5.166</td>
<td>6.467</td>
<td>7.397</td>
<td>7.998</td>
</tr>
<tr>
<td></td>
<td>((a/2, b/4), (a/2, 3b/4))</td>
<td>0.3</td>
<td>Ex.</td>
<td>3.133</td>
<td>4.221</td>
<td>5.280</td>
<td>7.474</td>
<td>6.831</td>
<td>8.016</td>
</tr>
</tbody>
</table>
4.1.4.4 SSSS square isotropic plates with variable thickness in one direction

Table 4.5 presents the numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates with variable thickness in one direction shown in Figure 4.2(a). For the isotropic plate with a central point support, three cases of $\alpha = 0.1, 0.8, 1.2$ are considered. The numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates without point support are also shown and compared with the results of Appl and Byers [52]. The nodal lines of 6 modes of free vibration of these isotropic plates are shown in Figure 4.6. For the isotropic plates without point supports, the vertical nodal lines move to the thinner part with increase of the value of $\alpha$. The changes can be found in the third, the fourth and the sixth mode shapes. For the isotropic plates with a central point support, the straight lines change to curve lines with increase of the value of $\alpha$. The changes can be found in the first and fifth mode shapes. The obvious change can be also found in the third mode.

<table>
<thead>
<tr>
<th>cases of point support</th>
<th>$\alpha$</th>
<th>References</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td>with no point support</td>
<td>0.1</td>
<td>Ex.</td>
<td>4.660 7.367 7.367 9.318 10.402 10.406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ref.[52]</td>
<td>4.661 — — — — — —</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>Ex.</td>
<td>5.356 8.412 8.466 10.698 11.767 11.914</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ref.[52]</td>
<td>5.355 — — — — — —</td>
</tr>
<tr>
<td>with a central point</td>
<td>0.1</td>
<td>Ex.</td>
<td>7.360 7.367 7.657 9.318 10.405 11.858</td>
</tr>
<tr>
<td>support</td>
<td>0.8</td>
<td>Ex.</td>
<td>8.241 8.412 9.022 10.698 11.835 13.269</td>
</tr>
</tbody>
</table>
4.1.4.5 SSSS square isotropic plates with variable thickness in two directions

Table 4.6 presents the numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates with variable thickness in two directions shown in Figure 4.2(b). For the isotropic plate with a central point support, three kinds of combination of $\alpha$ and $\beta$ are considered. The numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates without point support are also shown and compared with the results of Singh and Saxena [18]. Table 4.6 shows the frequency parameter $\lambda$ increases with the increase of $\alpha$ or $\beta$ for both the isotropic plates with and without point support.
Table 4.6 Natural frequency parameter $\lambda$ for SSSS square isotropic plates with variable thickness in two directions

<table>
<thead>
<tr>
<th>Cases of point support</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>References</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
<th>6th</th>
</tr>
</thead>
<tbody>
<tr>
<td>With no point support</td>
<td>0.5</td>
<td>-0.5</td>
<td>Ex.</td>
<td>4.365</td>
<td>6.806</td>
<td>6.880</td>
<td>8.705</td>
<td>9.520</td>
<td>9.620</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
<td>Ex.</td>
<td>5.659</td>
<td>8.880</td>
<td>8.935</td>
<td>11.30</td>
<td>12.51</td>
<td>12.51</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ref.[18]</td>
<td>5.659</td>
<td>8.885</td>
<td>8.938</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With a central point</td>
<td>-0.5</td>
<td>-0.5</td>
<td>Ex.</td>
<td>4.995</td>
<td>5.302</td>
<td>5.776</td>
<td>6.702</td>
<td>7.319</td>
<td>8.005</td>
</tr>
<tr>
<td>support</td>
<td>0.5</td>
<td>-0.5</td>
<td>Ex.</td>
<td>6.583</td>
<td>6.869</td>
<td>7.431</td>
<td>8.706</td>
<td>9.566</td>
<td>10.57</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5</td>
<td>Ex.</td>
<td>8.684</td>
<td>8.945</td>
<td>9.510</td>
<td>11.30</td>
<td>12.51</td>
<td>13.97</td>
</tr>
</tbody>
</table>

4.1.4.6 SSSS square isotropic plates with stepped thickness in one direction

Table 4.7 presents the numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates with stepped thickness in one direction shown in Figure 4.2(c). For the isotropic plate with a central point support, two kinds of the ratios $c/a$ and $h_1/h_0$ are considered. The numerical values for the lowest 6 natural frequency parameter $\lambda$ of the SSSS square isotropic plates without point support are also shown and compared with the exact solutions obtained by Xiang and Wang [53]. It is noted the present results agree well with these exact solutions of isotropic plates without point support even for the higher frequency parameters. The frequency parameters increase with the increase of the ratio of $c/a$ or $h_1/h_0$. The effects of the ratios $c/a$ and $h_1/h_0$ on the frequency parameters are significant.
Table 4.7 Natural frequency parameter $\lambda$ for SSSS square isotropic plates with stepped thickness in one direction

<table>
<thead>
<tr>
<th>Point support</th>
<th>$c / a$</th>
<th>$h_i / h_0$</th>
<th>References</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>With no point support</td>
<td>0.25</td>
<td>0.5</td>
<td>Ex.</td>
<td>3.647</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ref.[53]</td>
<td>3.658</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.8</td>
<td>Ex.</td>
<td>4.198</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.8</td>
<td>Ex.</td>
<td>4.411</td>
</tr>
<tr>
<td>With a central point support</td>
<td>0.25</td>
<td>0.5</td>
<td>Ex.</td>
<td>5.227</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.8</td>
<td>Ex.</td>
<td>6.547</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.5</td>
<td>Ex.</td>
<td>6.223</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.8</td>
<td>Ex.</td>
<td>6.923</td>
</tr>
</tbody>
</table>

4.1.5 Conclusions

The proposed discrete method is extended for analyzing the free vibration problem of isotropic plate with point supports. A concentrated load with Dirac’s delta function is used to simulate the point support. The characteristic equation of the free vibration is gotten by using the Green function. The effects of positions of point supports, the variable thickness, the aspect ratio and the boundary conditions on the frequencies are considered. The following conclusions have been obtained: The fundamental frequency parameter of isotropic plate with two point supports along the edge is lower than those of corresponding isotropic plates with interior two point supports. For the isotropic plates without point supports, the vertical nodal lines move to the thinner part with the increase of the value of $\alpha$. For the isotropic plates with a central point support, the straight lines change to curve line with increase of the value of aspect ratio $\alpha$. The frequency parameters increase with the increase of the ratio of $c / a$ or $h_i / h_0$. The effects of the ratios $c / a$ and $h_i / h_0$ on the frequency parameters are significant. It also shows that the present results have a good convergence and satisfactory accuracy.
4.2 Free Vibration Analysis of Isotropic Plates Resting on Non-homogeneous Elastic Foundations

4.2.1 Introduction

In this section, the proposed discrete method is extended for analyzing the free vibration of isotropic plates resting on non-homogeneous elastic foundations. The fundamental differential equations of an isotropic plate on non-homogeneous foundations are established and satisfied exactly throughout the whole isotropic plate. By applying the characteristic equation, the behavior of the free vibration of the isotropic plates on foundations can be analyzed efficiently without a calculation by a trial and error method. The efficiency and accuracy of the proposed method for the free vibration of square isotropic plates on Winkler foundation are investigated.

4.2.2 Discrete Green Function of an Isotropic Plate Resting on Non-homogeneous Elastic Foundation

Figure 4.7 shows a square isotropic plate of length $a$, density $\rho$ and stepped thickness $h$ resting on non-homogeneous foundations of foundation modulus $k$. The thickness and the foundation modulus in the central square part are $h_2$ and $k_2$, and those for the other part are $h_1$ and $k_1$, respectively. In this section, the elastic foundation is modeled as a spring system and the intensity of the reaction of the foundation is assumed to be proportional to the deflection $w$ of the isotropic plate. By considering the reaction of the foundation as a kind of lateral load, the fundamental differential equations of the isotropic plate having a concentrated load $\bar{P}$ at a point $(x_q, y_q)$ and resting on a Winkler foundation of the foundation modulus $k$ are as follows:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{P}\delta(x - x_q)\delta(y - y_q) - kw = 0,$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0,$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0,$$

$$\frac{\partial \theta_y}{\partial x} + \nu\frac{\partial \theta_x}{\partial y} = \frac{M_x}{D},$$
\[
\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} = \frac{M_y}{D}, \qquad \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} = \frac{2 M_{xy}}{(1 - \nu) D},
\]

\[
\frac{\partial w}{\partial x} + \frac{\partial}{\partial y} (Q_x) = \frac{Q_y}{Gt}, \qquad \frac{\partial w}{\partial y} + \frac{\partial}{\partial x} (Q_y) = \frac{Q_y}{Gt}.
\]

By introducing the non-dimensional expressions,
\[
[x_1, x_2] = \frac{a^2}{D_0(1 - \nu^2)} [Q_y, Q_x],
\]
\[
[x_3, x_4, x_5] = \frac{a}{D_0(1 - \nu^2)} [M_{xy}, M_y, M_x],
\]
\[
[x_6, x_7, x_8] = [\theta_y, \theta_x, w/a],
\]

Figure 4.7 A square isotropic plate with stepped thickness resting on elastic foundations

By introducing the non-dimensional expressions,
the equation (4.11) is rewritten as the following non-dimensional forms:

\[
\begin{align*}
\mu \frac{\partial^2 X_2}{\partial \eta^2} + \frac{\partial X_1}{\partial \zeta} &+ P \delta(\eta - \eta_0) \delta(\zeta - \zeta_0) - \kappa X_8 = 0, \\
\mu \frac{\partial X_3}{\partial \eta} + \frac{\partial X_4}{\partial \zeta} &- \mu X_1 = 0, \\
\mu \frac{\partial X_5}{\partial \eta} + \frac{\partial X_3}{\partial \zeta} &- \mu X_2 = 0, \\
\mu \frac{\partial X_7}{\partial \eta} + \nu \frac{\partial X_6}{\partial \zeta} &- \overline{D} X_5 = 0, \\
\nu \mu \frac{\partial X_7}{\partial \eta} + \frac{\partial X_6}{\partial \zeta} &- \overline{D} X_4 = 0, \\
\mu \frac{\partial X_6}{\partial \eta} + \frac{\partial X_7}{\partial \zeta} &- \frac{2}{1-\nu} \overline{D} X_3 = 0, \\
\frac{\partial X_8}{\partial \eta} &+ \overline{X}_7 - \overline{H} X_2 = 0, \\
\frac{\partial X_8}{\partial \zeta} + \mu X_6 - \mu \overline{H} X_1 = 0,
\end{align*}
\]

(4.12)

where \(\mu = b/a, \overline{D} = \mu(1-\nu^2)(h_0/h)^3, \overline{H} = ((1+\nu)/5)(h_0/a)^2 h_0/h, P = \overline{P} a/(D_0(1-\nu^2)),\) \(D_0 = Eh_0^3/(12(1-\nu^2))\) is the standard bending rigidity, \(\kappa = \mu K/(1-\nu^2),\) \(K\) is the dimensionless modulus of the foundation, it is defined as follows:

\[K = ka^4/D_0,\]

In the above equation, the variable quantity \(h_0/h\) has been separated and expressed only in the quantities \(\overline{D}\) and \(\overline{H}\), so that the equation can be used for the isotropic plate with stepped thickness.

The equation (4.12) can also be expressed as the following simple form.

\[
\sum_{s=1}^{8} \left\{ F_{1s} \frac{\partial X_s}{\partial \zeta} + F_{2s} \frac{\partial X_s}{\partial \eta} + F_{3s} \overline{X}_s \right\} + P \delta(\eta - \eta_0) \delta(\zeta - \zeta_0) \delta_{1t} = 0 \\
(t = 1 \sim 8)
\]

(4.13)

where \(\delta_{1t}\) is Kronecker’s delta, \(F_{111} = F_{124} = F_{133} = F_{156} = F_{167} = F_{188} = F_{146} = \nu,\)

\[F_{212} = F_{223} = F_{235} = F_{247} = F_{266} = \mu, F_{257} = \mu \nu, F_{278} = 1, F_{318} = -\overline{k}, F_{321} = F_{332} = -\mu,\]

\(F_{345} = F_{354} = -\overline{D}, F_{363} = -2\overline{D}/(1-\nu), F_{372} = -\overline{H}, F_{377} = 1, F_{381} = -\mu \overline{H}, F_{386} = \mu,\)

other \(F_{1s} = 0.\)
Based on the same consideration with section 4.1, the discrete solution is written as follows.

\[
X_{\text{pij}} = \sum_{d=1}^{6} \left\{ \sum_{f=0}^{i} a_{\text{pijfd}} X_{\text{ef}} + \sum_{g=0}^{j} b_{\text{pijgd}} X_{\text{sg}} \right\} + \bar{q}_{\text{pij}} P, \tag{4.14}
\]

where \( a_{\text{pijfd}}, b_{\text{pijgd}} \) and \( \bar{q}_{\text{pij}} \) are given in Appendix B.

The equation (4.14) gives the discrete solution of the fundamental differential equation (4.13) of the bending problem of an isotropic plate resting on an elastic foundation and having a concentrated load, and the discrete Green function is chosen as \( X_{\text{sj}}a^2 /[PD_0(1 - \nu^2)] \), that is \( w(x_0,y_0,x,y)/\bar{P} \).

4.2.3 Numerical Results

To investigate the validity of the proposed method, the frequency parameters are given for the isotropic plate shown in Figure 4.7. The standard thickness \( h_0 \) is chosen as \( h_i \) and \( h_0/a = 1/1000 \) is used. Simply supported boundary conditions are considered. All the convergent values of the frequency parameters are obtained for simply supported square isotropic plates \( (\mu = b/a = 1) \) by using Richardson’s extrapolation formula for two cases of divisional numbers \( m (= n) \). Some of the results are compared with those reported previously.

4.2.3.1 A square isotropic plate on homogeneous foundations

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions \( m \) and \( n \) for a square isotropic plate with uniform thickness on homogeneous foundations. The isotropic plate is special case of the plate shown in Figure 4.7 with \( c/a = 0.0 \). Table 4.8 shows the numerical values for the lowest 4 natural frequency parameter \( \lambda \) of square isotropic plates on homogeneous foundation with \( K = 0, 10, 100, 1000, 10000 \). The results obtained by Matsunaga [13] and the exact values of the isotropic plate with \( K = 0 \) [54] are also shown in the Table 4.8. It can be seen that the numerical results of the proposed method have satisfactory accuracy. From this table, it can be also seen that the effect of the constant \( K \) on the fundamental frequency parameter is much more significant than that on higher frequency parameters, the
frequency parameters increase with increase of the constant $K$, and they increase quickly when $K$ is larger than 100.

Table 4.8 Natural frequency parameter $\lambda$ for a SSSS isotropic square plate on homogeneous foundations

<table>
<thead>
<tr>
<th>$K$</th>
<th>References</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12×12</td>
<td>4.575</td>
<td>7.336</td>
<td>7.336</td>
<td>9.311</td>
</tr>
<tr>
<td></td>
<td>16×16</td>
<td>4.564</td>
<td>7.272</td>
<td>7.272</td>
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4.2.3.2 A square isotropic plate with uniform thickness on non-homogeneous foundations

Table 4.9 shows the numerical values for the lowest 4 natural frequency parameter $\lambda$ of the isotropic plates shown in Figure 4.7 with $K_1 = 0$ or $K_2 = 0$, which is the case of the local uniformly distributed support. The side ratio of the local square part and the isotropic plate $c/a = 0.6$ and the thickness ratio $h_1/h_2 = 1.0$ are adopted. The convergent results of frequency parameter are obtained by using Richardson’s extrapolation formula for two cases of divisional numbers $m(= n)$ pointed in Table 4.9. The present results are compared with those obtained by Laura and Gutie’rrez [10] and Ju, Lee and Lee [15]. They are in good agreement.

Table 4.10 shows the numerical values for the lowest 4 natural frequency parameter $\lambda$ of the isotropic plate on non-homogeneous foundations with $h_1/h_2 = 1.0$, $c/a = 0.6$ and four kinds of combination of $K_1$ and $K_2$. The convergent results of frequency parameter are obtained by using Richardson’s extrapolation formula for two cases of divisional numbers $m(= n)$ pointed in Table 4.10. The present results are also in good agreement with those obtained by Laura and Gutie’rrez [10] and Ju, Lee and Lee [15]. From Tables 4.8~4.10, it can be seen the proposed method can be used to solve the problem of isotropic plates on homogeneous foundations, local uniformly distributed supports and non-homogeneous foundations.
Table 4.9 Natural frequency parameter \( \lambda \) for a SSSS square isotropic plate with the central part on local uniform supports (\( c/a = 0.6 \))

<table>
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Table 4.10 Natural frequency parameter $\lambda$ for a SSSS square isotropic plate with the central part on non-homogeneous foundations ($c / a = 0.6$)

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4.2.3.3 Square isotropic plates with stepped thickness in central square part

The numerical calculation is carried out for the isotropic plate shown in Figure 4.7 with \( K_1 = K_2 = 0 \), which is the case without foundations. The numerical values for the lowest 3 natural frequency parameter \( \lambda \) of the square isotropic plates with \( c/a = 0.5 \) and \( h_1/h_2 = 0.7, 0.8, 1.5 \) are presented in Table 4.11. The convergent results of frequency parameter are obtained by using Richardson’s extrapolation formula for two cases of divisional numbers \( m(= n) \) of 12 and 16. The present results are compared with those obtained by Ju, Lee and Lee [15]. It shows the present results have satisfactory accuracy. From Table 4.11, it can be noted that the frequency parameters decrease with the increase of the ratio \( h_1/h_2 \).

<table>
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4.2.3.4 Square isotropic plates with stepped thickness on non-homogeneous elastic foundations

As an application of the proposed method, some numerical results are presented for the isotropic plate with stepped thickness in the central square part resting on local uniformly distributed supports or non-homogeneous foundations. The ratios of \( h_1/h_2 = 0.8 \) and \( c/a = 0.5 \) are considered. The convergent results of frequency parameter of these isotropic plates are obtained by using Richardson’s extrapolation formula for two cases of divisional numbers 12 and 16 in Table 4.12 and 4.13. From these two tables, it can be
noted for the specific modulus of the foundation $K$ and the ratios of $c/a$ and $h_1/h_2$, the fundamental frequency parameters of the isotropic plate with the central part having higher foundation modulus are higher than those of the isotropic plate with the central part having lower foundation modulus and it can also be seen that the modulus of the foundation affects the frequency parameters greatly.

Table 4.12 Natural frequency parameter $\lambda$ for a SSSS square isotropic plate with stepped thickness in the central square part on local uniformly distributed supports

\( \left( h_1/h_2 = 0.8, c/a = 0.5 \right) \)

<table>
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<th>$K_2$</th>
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<th>2nd(3rd)</th>
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<td>7.613</td>
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Table 4.13 Natural frequency parameter $\lambda$ for a SSSS square isotropic plate with stepped thickness in the central square part on non-homogeneous foundations

\( \left( h_1/h_2 = 0.8, c/a = 0.5 \right) \)

<table>
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<tr>
<th>$K_1$</th>
<th>$K_2$</th>
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4.2.4 Conclusions

The proposed discrete method is extended for analyzing the free vibration problem of square isotropic plates with stepped thickness on the elastic foundations. The spring system is used to simulate the foundations. The characteristic equation of the free vibration is gotten by using the Green function. The results shown that the effect of the dimensionless modulus of the foundation $K$ on the fundamental frequency parameter is much more significant than that on higher frequency parameters, the frequency parameters increase with increase of the constant $K$, and they increase quickly when $K$ is larger than 100. The fundamental frequency parameters of the isotropic plate with the central part having higher foundation modulus are higher than those of the isotropic plate with the central part having lower foundation modulus and the modulus of the foundation affects the frequency parameter greatly. The results also shown that with the increase of the thickness ratio, the first and the second frequency parameters decrease for the isotropic plates with the specific modulus of the foundation. With increase of the value of the modulus of the foundation, the frequency parameters increase.
4.3 Appendix A

\[ A_{p1} = \gamma_{p1}, \quad B_{p1} = 0, \quad C_{p3kl} = \mu(\gamma_{p3} + k_{kl}\gamma_{p7}), \]

\[ A_{p2} = 0, \quad B_{p2} = \mu\gamma_{p1}, \quad C_{p2kl} = \mu\gamma_{p2} + k_{kl}\gamma_{p8}, \]

\[ A_{p3} = \gamma_{p2}, \quad B_{p3} = \mu\gamma_{p3}, \quad C_{p3kl} = J_{kl}\gamma_{p6}, \]

\[ A_{p4} = \gamma_{p3}, \quad B_{p4} = 0, \quad C_{p4kl} = I_{kl}\gamma_{p4}, \]

\[ A_{p5} = 0, \quad B_{p5} = \mu\gamma_{p2}, \quad C_{p5kl} = I_{kl}\gamma_{p5}, \]

\[ A_{p6} = \gamma_{p4} + \nu\gamma_{p5}, \quad B_{p6} = \mu\gamma_{p6}, \quad C_{p6kl} = -\mu\gamma_{p7}, \]

\[ A_{p7} = \gamma_{p6}, \quad B_{p7} = \mu(\nu\gamma_{p1} + \gamma_{p5}) \]

\[ A_{p8} = \gamma_{p7}, \quad B_{p8} = \gamma_{p8}, \quad C_{p8kl} = 0, \]

\[ \bar{\gamma}_{11} = \beta_{ii}, \quad \bar{\gamma}_{12} = \mu\beta_{ij}, \quad \bar{\gamma}_{22} = -\mu\beta_{jj}, \quad \bar{\gamma}_{23} = \beta_{ii}, \quad \bar{\gamma}_{25} = \mu\beta_{ji}, \quad \bar{\gamma}_{31} = -\mu\beta_{ij}, \]

\[ \bar{\gamma}_{33} = \mu\beta_{jj}, \quad \bar{\gamma}_{34} = \beta_{ii}, \quad \bar{\gamma}_{44} = -I_{ij}\beta_{jj}, \quad \bar{\gamma}_{46} = \beta_{ii}, \quad \bar{\gamma}_{47} = \mu\nu\beta_{jj}, \quad \bar{\gamma}_{55} = -I_{ij}\beta_{ji}, \quad \bar{\gamma}_{56} = \nu\beta_{ii}, \]

\[ \bar{\gamma}_{57} = \mu\beta_{jj}, \quad \bar{\gamma}_{59} = -J_{ij}\beta_{ii}, \quad \bar{\gamma}_{66} = \mu\beta_{jj}, \quad \bar{\gamma}_{67} = \beta_{ii}, \quad \bar{\gamma}_{71} = -\mu\beta_{ij}, \quad \bar{\gamma}_{76} = \mu\beta_{jj}, \quad \bar{\gamma}_{78} = \beta_{ii}, \]

\[ \bar{\gamma}_{82} = -\mu\beta_{ij}, \quad \bar{\gamma}_{87} = \beta_{ij}, \quad \bar{\gamma}_{88} = \beta_{jj}, \quad \text{other } \bar{\gamma}_{pk} = 0, \quad [\gamma_{pk}] = [\bar{\gamma}_{pk}]^{-1}, \quad \beta_{ij} = \beta_{ii}\beta_{jj}. \]
Appendix B

\[ a_{10/1} = a_{30/2} = a_{40/3} = 1, \quad a_{60/4} = a_{210/5} = a_{60/6} = 1 \]

\[ b_{20/1} = b_{30/2} = b_{50/3} = 1, \quad b_{60/4} = b_{20/5} = b_{60/6} = 1, \]

\[ b_{30002} = 0 \]

\[ a_{pijd} = \sum_{t=1}^{\infty} \left\{ \sum_{k=0}^{t} \beta_{ik} A_{pt}[a_{tk0jd} - a_{tkjd}(1 - \delta_{kl})] + \sum_{l=0}^{t} \beta_{jl} B_{pt}[a_{10ljd} - a_{1ljd}(1 - \delta_{lj})] \right\} + \sum_{k=0}^{t} \sum_{l=0}^{t} \beta_{ik} \beta_{jl} C_{pkl} a_{skjd}(1 - \delta_{kl}) \delta_{lj} \}
\]

\[ b_{pijd} = \sum_{t=1}^{\infty} \left\{ \sum_{k=0}^{t} \beta_{ik} A_{pt}[b_{tk0jd} - b_{tkjd}(1 - \delta_{kl})] + \sum_{l=0}^{t} \beta_{jl} B_{pt}[b_{10ljd} - b_{1ljd}(1 - \delta_{lj})] \right\} + \sum_{k=0}^{t} \sum_{l=0}^{t} \beta_{ik} \beta_{jl} C_{pkl} b_{skjd}(1 - \delta_{kl}) \delta_{lj} \}
\]

\[ \bar{q}_{pji} = \sum_{t=1}^{\infty} \left\{ \sum_{k=0}^{t} \beta_{ik} A_{pt}[\bar{q}_{tk0} - \bar{q}_{tk}(1 - \delta_{kl})] + \sum_{l=0}^{t} \beta_{jl} B_{pt}[\bar{q}_{1lj} - \bar{q}_{1lj}(1 - \delta_{lj})] \right\} + \sum_{k=0}^{t} \sum_{l=0}^{t} \beta_{ik} \beta_{jl} C_{pkl} \bar{q}_{skl}(1 - \delta_{kl}) \delta_{lj} - A_{pt} u_{ikl} u_{jlr} \]

\[ \bar{q}_{pji} = \sum_{t=1}^{\infty} \left\{ \sum_{k=0}^{t} \beta_{ik} A_{pt}[-\bar{q}_{tko} - \bar{q}_{tk}(1 - \delta_{kl})] + \sum_{l=0}^{t} \beta_{jl} B_{pt}[-\bar{q}_{1ljc} - \bar{q}_{1lj}(1 - \delta_{lj})] \right\} + \sum_{k=0}^{t} \sum_{l=0}^{t} \beta_{ik} \beta_{jl} C_{pkl} \bar{q}_{sklc}(1 - \delta_{kl}) \delta_{lj} - \gamma_{pt} u_{ikl} u_{jlr} \]

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Chapter 5

Free Vibration Analysis of Orthotropic Plates with Non-homogeneity
5.1 Free Vibration Analysis of Orthotropic Plates

5.1.1 Introduction

In chapter 2, the discrete solution for bending and free vibration problems of orthotropic plates with variable thickness and a concentrated load combined with uniform load was obtained. In this section, the convergence, efficiency and accuracy of proposed method described in Chapter 2 will be investigated.

5.1.2 Numerical Results

An $xyz$ coordinate system (shown in Figure 5.1) is used in this section.

![Figure 5.1 An orthotropic rectangular plate and ordinate](image)

The proposed method described in Chapter 2 is used to analyze the bending and free vibration problems of orthotropic plate with uniform or non-uniform thickness. In order to compare the present results with those previously reported, the orthotropic material is chosen as E-glass/epoxy material. Its properties are given as $E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = G_{13} = G_{23} = 12.0$ GPa, $\nu_{12} = 0.23$. The ratios of $E_1 / E_2 = 2.448$, $G_{12} / E_2 = 0.484$, $G_{13} / E_2 = 0.484$, and $G_{23} / E_2 = 0.484$. The thickness functions are chosen as $h = h_0 (1 + \alpha x / a)$ for variable thickness in one direction.
5.1.2.1 The Efficiency of the Proposed Discrete Method

Due to the symmetry, only the 1/4 plate is used to obtain the deflections at the central point \( C(a/2, b/2) \) of the whole plate for SSSS and CCCC orthotropic plates (\( \mu = 1 \)).

![Figure 5.2](image1.png)
(a) Orthotropic SSSS \((3m^2)\)  
(b) Orthotropic CCCC \((3m^2-2m)\)

Figure 5.2 Unknown quantities at discrete points for the finite element method.

![Figure 5.3](image2.png)
(a) Orthotropic SSSS \((6m-1)\)  
(b) Orthotropic CCCC \((6m-1)\)

Figure 5.3 Unknown quantities at discrete points for the proposed method.

Figures 5.2 and Figure 5.3 show the unknown quantities at every discrete point for the FEM and the proposed method, respectively. It can be concluded if the divisional number is \( m \) and the FEM is used, the numbers of the unknown quantities are \( 3m^2 \) for SSSS orthotropic plate and \( 3m^2 - 2m \) for CCCC orthotropic plate. If the proposed method is used, the number of the unknown quantities is \( 6m - 1 \) for both the SSSS and CCCC orthotropic plates. So for the same divisional number \( m \), the numbers of the unknown quantities of the FEM and the proposed method are different. The unknown quantities of the proposed method are fewer than those of the FEM when \( m \geq 3 \).
5.1.2.2 The Convergence of the Numerical Solution

In order to examine the convergence, numerical calculation is carried out by varying the number of divisions \( m \) and \( n \). The lowest six natural frequency parameters of a CSCS orthotropic square plate with variable thickness in one direction \( (\alpha = 0.4) \) are shown in Figure 5.4. It shows a good convergence of the numerical results by the proposed method. It can be also noticed that convergent results of frequency parameter can be obtained by using Richardson’s extrapolation formula for two cases of divisional numbers \( m(= n) \) of 12 and 16. By the same method, the suitable number of divisions \( m(= n) \) can be determined for the other plates. In this section, all the convergent values of frequency parameter are obtained by using Richardson’s extrapolation formula for two cases of divisional numbers 12 and 16.

Figure 5.4 The natural frequency parameter \( \lambda \) versus the divisional number \( m(=n) \) for CSCS orthotropic square plate with variable thickness in one direction \( (\alpha=0.4) \)
5.1.2.3 The Accuracy of the Numerical Solutions

Table 5.1 presents numerical values for the lowest 3 natural frequency parameter $\hat{\nu}$ of CSSS, SSSS, SSFS orthotropic plates (shown in Figure 5.5). Taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 1.0, 2.0$ are considered. The results obtained by Bert and Malik [25] are also shown in above table. It can be seen that the numerical results of the proposed method have satisfactory accuracy. The results presented in Table 5.1 are limited to orthotropic plates with two opposite edges having the same thickness and simply supported boundary conditions.

![Figure 5.5 An orthotropic rectangular plate with variable thickness in one direction.](image)

5.1.3 Conclusions

In order to verify the proposed discrete method described in Chapter 2, the free vibration analysis of orthotropic rectangular plates has been made. A comparison of the proposed method and the FEM is made for orthotropic plates. The efficiency of the proposed method has been investigated. The numerical results show that the proposed method has a good convergence and satisfactory accuracy.
Table 5.1 Natural frequency parameter $\lambda$ for CSSS, SSSS and SSFS orthotropic plates with variable thickness in one direction

<table>
<thead>
<tr>
<th>B.C.</th>
<th>$b/a$</th>
<th>$\alpha$</th>
<th>Source</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
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<td>Ex.</td>
<td>5.572</td>
<td>7.548</td>
<td>9.249</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>Ref.[25]</td>
<td>5.574</td>
<td>7.553</td>
<td>9.263</td>
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<td></td>
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<td>10.050</td>
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</tr>
<tr>
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<td></td>
<td>Ref.[25]</td>
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<td>8.281</td>
<td>10.068</td>
<td></td>
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<tr>
<td></td>
<td>0.8</td>
<td>Ex.</td>
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<td>8.905</td>
<td>10.745</td>
<td></td>
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<tr>
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<td>2.0</td>
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<td>Ex.</td>
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<td>5.572</td>
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<tr>
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<td>Ex.</td>
<td>4.902</td>
<td>7.253</td>
<td>8.374</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ref.[25]</td>
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<td>7.256</td>
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<td>5.945</td>
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</tr>
<tr>
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<tr>
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<td>Ref.[25]</td>
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<tr>
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<td>Ex.</td>
<td>2.519</td>
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5.2 Free Vibration Analysis of Orthotropic Rectangular Plates with General Boundary Conditions

5.2.1 Introduction

In this section, using the discrete method developed in Chapter 2 to analyze the free vibration problem of orthotropic rectangular plates with general boundary conditions. By comparing the numerical results obtained by the proposed method with those previously published, the efficiency and accuracy of the proposed method are investigated. The frequency parameters are obtained for the orthotropic plates with general boundary conditions and variable thickness in one or two directions. The model shapes are given for some of the square orthotropic plates with three kinds of thickness variations.

5.2.2 Differential Equation with a Concentrated Load \( \overline{P} \) and Discrete Green Function

Using the same manner as described in Chapter 2, the differential equations of the plate with a concentrated load \( \overline{P} \) at point \((x_i, y_i)\) are established as follows:

\[
\begin{align*}
\mu \frac{\partial X_2}{\partial \eta} + \frac{\partial X_1}{\partial \zeta} + P \delta(\eta - \eta_i) \delta(\zeta - \zeta_i) &= 0, \\
\mu \frac{\partial X_3}{\partial \eta} + \frac{\partial X_4}{\partial \zeta} - \mu X_1 &= 0, \\
\mu \frac{\partial X_5}{\partial \eta} + \frac{\partial X_3}{\partial \zeta} - \mu X_2 &= 0, \\
\overline{D}_{11} \mu \frac{\partial X_7}{\partial \eta} + \overline{D}_{12} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{16} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \overline{D} X_5 &= 0, \\
\overline{D}_{12} \mu \frac{\partial X_7}{\partial \eta} + \overline{D}_{22} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{26} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \overline{D} X_4 &= 0, \\
\overline{D}_{16} \mu \frac{\partial X_7}{\partial \eta} + \overline{D}_{26} \frac{\partial X_6}{\partial \zeta} + \overline{D}_{66} \left( \frac{\partial X_7}{\partial \zeta} + \mu \frac{\partial X_6}{\partial \eta} \right) - \mu \overline{D} X_3 &= 0, \\
k \overline{A}_{44} \delta(\xi - \xi_i) + k \mu \overline{A}_{45} \left( \frac{\partial X_8}{\partial \eta} + X_7 \right) - \mu \overline{D} X_1 &= 0, \\
k \overline{A}_{45} \delta(\xi - \xi_i) + k \mu \overline{A}_{55} \left( \frac{\partial X_8}{\partial \eta} + X_7 \right) - \mu \overline{D} X_2 &= 0.
\end{align*}
\]
The discrete solution of the fundamental differential equation (5.1) is as follow:

\[ X_{pq} = \sum_{d=1}^{5} \left( \sum_{f=0}^{4} a_{pqfd} X_{rf0} + \sum_{g=0}^{5} b_{pqgpd} X_{s0g} \right) + \bar{q}_{p} P, \]  

(5.2)

where \( a_{pqfd}, b_{pqgpd} \) and \( \bar{q}_{p} \) are given in the Appendix B of Chapter 2.

The above Eq. (5.2) gives the discrete solution of the fundamental differential equation (5.1) of the bending problem of an orthotropic plate under a concentrated load, and the discrete Green function is chosen as \( X_{8q}/[P(a/ Dq(1 - \nu_{12} \nu_{21})]. \)

5.2.3 Boundary Conditions of a Rectangular Orthotropic Plate

The boundary conditions along the edges \( \zeta = 0 \) and \( 1 \) are as follows:

\[ \theta_y = \theta_x = w = 0 \quad \text{for a clamped edge}; \]
\[ M_y = \theta_x = w = 0 \quad \text{for a simply supported edge}; \]
\[ Q_y = M_{xy} = M_y = 0 \quad \text{for a free edge}; \]

The boundary conditions along the edges \( \eta = 0 \) and \( 1 \) are as follows:

\[ \theta_y = \theta_x = w = 0 \quad \text{for a clamped edge}, \]
\[ M_x = \theta_y = w = 0 \quad \text{for a simply supported edge}, \]
\[ Q_x = M_{xy} = M_x = 0 \quad \text{for a free edge}. \]

5.2.4 Numerical Results

The proposed method is used to obtain the frequency parameters and mode shapes for orthotropic plate with variable thickness and various boundary conditions. E-glass/epoxy material \( (E_1 = 60.7 GP_a, E_2 = 24.8 GP_a, G_{12} = 12.0 GP_a, \nu_{12} = 0.23) \) is used when no properties of material are appointed specially. The thickness functions are chosen as \( h = \bar{h}_o(1 + \alpha x / a) \) and \( h = \bar{h}_o(1 + \alpha x / a)(1 + \beta y / b) \) for variable thickness in one and two directions, respectively. The ratio of the length and thickness \( a/ \bar{h}_o = 100 \) is adopted.

5.2.4.1 Variable Thickness in One Direction

To show the accuracy of the proposed method and to investigate the effects of the boundary conditions, aspect ratios and variable thickness on the frequency parameters, the
lowest six frequency parameters are calculated for 14 kinds of boundary conditions with
taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 0.5, 1.0, 2.0$.

Numerical values for the lowest six natural frequency parameters $\lambda$ of CSCS, CSSS,
SSSS, SSFS, SCSC and SSCS of orthotropic plates are given in Tables 5.2–5.4. It can be
seen that the frequency parameters increase with increase of the taper ratio $\alpha$ for the
plate with specific boundary condition and aspect ratio $b/a$, and decrease with the
increase of aspect ratio $b/a$ for the plate with the same boundary condition and taper
ratio $\alpha$. The effect of boundary condition on the frequency parameters can be observed by
comparing the corresponding results presented in Tables 5.2 and 5.3. In these two tables,
the highest frequency parameters can be obtained for CSCS plates, then successively for
CSSS, SSSS and SSFS. It shows that with decrease of boundary constraints, which results
in decrease of stiffness, frequency parameters decrease significantly. From Tables 5.2 and
5.4, it can be found that even for the plates with the same aspect ratio, taper ratio and the
same boundary condition, which are two opposite edges clamped and two opposite edges
simply supported, the results of CSCS and SCSC orthotropic plates are quite different. For
CSCS plate, the longitudinal direction of the orthotropic material is coincident with the
simply supported edges, but for SCSC plate, it is coincident with the clamped edges. It
shows that the direction of principle material axes also influences the frequency parameters
greatly. For the square plates with specific taper ratio, the fundamental frequency of CSCS
plate is higher than that of SCSC plate. By comparing the results shown in Tables 5.2 and
5.3, it can be also noticed that the frequency parameters of CSSS and SSCS plate are the
same for the case of uniform thickness ($\alpha = 0.0$), but different for the cases of variable
thickness $\alpha = 0.4, 0.8$. The fundamental frequencies of SSCS plates with $\alpha = 0.4, 0.8$
are higher than those of CSSS plates for $b/a = 1, 2$, but lower for $b/a = 0.5$. The results
obtained by Bert and Malik [25] are also shown in the above tables. It can be seen that the
numerical results of the proposed method have satisfactory accuracy. The results presented
in Tables 5.2–5.4 are limited to orthotropic plates with two opposite edges having the same
thickness and simply supported boundary conditions.
Table 5.2 Natural frequency parameter $\lambda$ for CSCS and CSSS orthotropic plates with variable thickness in one direction

<table>
<thead>
<tr>
<th>B.C.</th>
<th>$b/a$</th>
<th>$\alpha$</th>
<th>References</th>
<th>Mode sequence number</th>
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<td></td>
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Table 5.3 Natural frequency parameter $\lambda$ for SSSS and SSFS orthotropic plates with variable thickness in one direction

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Table 5.4 Natural frequency parameter $\lambda$ for SCSC and SSCS orthotropic plates with variable thickness in one direction

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105
The nodal patterns of the lowest six modes of the above plates with \( b/a = 1 \) are shown in Figures 5.6–5.11. With change of the boundary conditions, the order of some mode shapes changes. From Figures 5.7 and 5.11, it is noticed that the vertical nodal lines tend to be close to simply supported edges. In Figure 5.9, the vertical nodal lines are close to the free edges. These show the vertical nodal lines have the trend to be close to the edge with less boundary constraint. The trend can be found in the third, fifth and sixth modes in Figures 5.7 and 5.11, and in the second, fourth and fifth modes in Figure 5.9. From Figures 5.6–5.11, it can be noted that with increase of taper ratio, the vertical nodal lines move to the thinner part of the plates. Obvious change can be seen in the third, fifth and sixth modes in Figure 5.6, and corresponding modes in other figures.

![Nodal patterns](image)

Figure 5.6 Nodal patterns for CSCS orthotropic square plates with variable thickness in one direction.
Figure 5.7 Nodal patterns for CSSS orthotropic square plates with variable thickness in one direction

Figure 5.8 Nodal patterns for SSSS orthotropic square plates with variable thickness in one direction
Figure 5.9 Nodal patterns for SSFS orthotropic square plates with variable thickness in one direction

Figure 5.10 Nodal patterns for SCSC orthotropic square plates with variable thickness in one direction
Tables 5.5 and 5.6 present the numerical results for the lowest six natural frequency parameter $\lambda$ of the CCCC, CCSC, SCFC and FCCC orthotropic plates with taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 0.5, 1.0, 2.0$. The results obtained by the proposed method are compared with those of Ashour [26]. It can be noticed that the present results agree well with Ashour’s results for CCCC orthotropic plates. For CCSC orthotropic plates with $b/a = 1.0, 2.0$, they still agree each other, but for orthotropic plates with $b/a = 0.5$, difference can be found. The difference can also be found in the results for SCFC plates, especially in those for FCCC plates. Due to the lack of the published information on orthotropic plates with variable thickness, no other suitable references can be used for comparison. But comparing the results of CCCC and FCCC shown in Tables 5.5 and 5.6, respectively, it is inferred Ashour may have lost some of the lower frequency parameters for FCCC plates. According to the conclusions obtained earlier, the frequencies of FCCC plates should be lower than those corresponding results of CCCC plates. So the fundamental frequency of FCCC plate with $\alpha = 0.4$ and $b/a = 0.5$ is expected to be lower than 11.132 and his results is 16.429. In order to confirm the accuracy of the proposed method further, the lower three frequency parameters are calculated by using the solution obtained by Hearmon [55] for uniform E-glass/epoxy plates with $b/a = 0.5$. These results are also shown in Table 5.5. Although the results presented in Tables 5.5 and 5.6 are not limited to plates with two opposite edges simply supported, they are still limited to orthotropic plates with two opposite edges having the same boundary conditions and the same thickness.
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Table 5.6 Natural frequency parameter $\lambda$ for SCFC and FCCC orthotropic plates with variable thickness in one direction

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As an application of the proposed method, the numerical results are presented for the orthotropic plates with general boundary conditions. The number of the combination of the boundary conditions is too large to be considered completely, so the numerical results of the lowest six natural frequency parameter $\lambda$ are given only for CCCS, SSSC, SSCC and FFCF plates with taper ratios $\alpha = 0.0, 0.4, 0.8$ and aspect ratios $b/a = 0.5, 1.0, 2.0$. These results are shown in Table 5.7.

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5.2.4.2 Variable Thickness in Two Directions

As another application of the proposed method, the numerical results are given for the plates with linearly variable thickness in two directions. Table 5.8 presents the results for the plates with six kinds of boundary conditions and four kinds of thickness variation.

At last, the numerical results are given for orthotropic plate made of graphite-epoxy material \( (E_1/E_2 = 40.0, \ G_{12}/E_2 = 0.5, \ v_{12} = 0.25 ) \) and glass–epoxy material \( (E_1/E_2 = 4.67, \ G_{12}/E_2 = 0.5, \ v_{12} = 0.26 ) \). CFFF and SSSS orthotropic plates with variable thickness are considered. In Table 5.9, the results of plates with uniform thickness or variable thickness in one direction are also given and compared with those obtained by Liew et al. [56] and Lam et al. [57]. These results are in good agreement.
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Table 5.9 Natural frequency parameter $\lambda$ for isotropic or orthotropic plates with variable thickness

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5.2.5 Conclusions

The proposed discrete method is extended for analyzing the free vibration problem of orthotropic rectangular plates with general boundary conditions. The effects of the boundary conditions, aspect ratios and variable thickness in one and two directions on the
frequencies are considered. It is found that the frequency parameters increase with increase of the taper ratio $\alpha$ for the orthotropic plate with CSSS, SSSS, SSFS boundary condition and aspect ratio $b/a$, and decrease with the increase of aspect ratio $b/a$ for the plate with the same boundary condition and taper ratio $\alpha$. It shown that with increase of boundary constraints, which results in decrease of stiffness, frequency parameters decrease significantly. The results by the proposed method have been compared with those previously reported. It shows that the proposed results have a good convergence and satisfactory accuracy.

5.3 Free Vibration Analysis of Orthotropic Square Plates with a Hole

5.3.1 Introduction

This section extends to analyze the free vibration of orthotropic square plates with a hole (using the method described in section 5.2). By considering the hole as an extremely thin part of a plate, the free vibration problem of a plate with a hole can be transformed into the free vibration problem of its equivalent square plates with non-uniform thickness. The effects of side to thickness ratio, hole side to plate side ratio and the variation of the thickness in one direction or two directions on the frequency properties are studied. The convergence and accuracy of the proposed method are investigated.

5.3.2 Equivalent Square Plate of a Square Plate with a Square Hole

A square plate with a hole can be transformed into an equivalent square plate with non-uniform thickness (shown in Figure 5.12) by considering the hole as an extremely thin part of the plate theoretically. The thickness of the actual part of original square plate is expressed as $h$, and the thickness of the extremely thin part of the equivalent square plate is expressed as $h_i$. The thickness of the plate along the border line between the actual part and the extremely thin part is chosen as $(h + h_i)/2$. In this section, numerical results are obtained for a simply supported square plate with a central square hole.
5.3.3 Numerical Results

The convergence and accuracy of numerical solutions are investigated for simply supported isotropic and orthotropic plates with holes for the cases of uniform thickness and variable thickness in one or two directions. The material properties of isotropic and orthotropic plates are shown in Table 5.10.

In order to examine the convergence, numerical calculation is carried out by varying the number of divisions \( m \) and \( n \). The lowest 5 natural frequency parameters of an isotropic square plate with a square hole are shown in Figure 5.13. It can be noticed that the results of \( m=12 \) are close to the convergent results. In order to get more exact results, the Richardson’s extrapolation formula is used for two cases of divisional numbers \( m(=n) \) of 12 and 16. In this section, the convergent results are those results obtained by the Richardson’s extrapolation formula. Table 5.11 is used to determine the suitable thickness ratio \( h/h_i \) of the original and extremely thin parts. The ratio \( h/h_i \) should be chosen as very large value theoretically, but in fact, no good results can be gotten due to the precision problem of the computer. So in Table 5.11, the \( h/h_i \) is chosen as 2, 6, 12, 18. It is found the results of \( h/h_i = 12 \) are very close to the results of \( h/h_i = 18 \). So it is sufficient to set the thickness ratio \( h/h_i = 12 \).

By the same method, the number of divisions \( m(=n) \) and the thickness ratio \( h/h_i \)
can be determined for the other plates. The lowest 5 natural frequencies and mode shapes of the square plates with square holes are presented for the cases of uniform thickness and variable thickness.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\frac{E_1}{E_2}$</th>
<th>$\frac{G_{12}}{E_2}$</th>
<th>$\frac{G_{13}}{E_2}$</th>
<th>$\frac{G_{23}}{E_2}$</th>
<th>$\nu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>isotropic</td>
<td>1</td>
<td>0.385</td>
<td>0.385</td>
<td>0.385</td>
<td>0.3</td>
</tr>
<tr>
<td>orthotropic</td>
<td>40</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Figure 5.13 The fundamental frequency versus the divisional number $m (=n)$ for SSSS isotropic square plate with a square hole and uniform thickness ($c/a = 0.5$, $a/h = 100$, $h/h_i = 12$).
Table 5.11 The natural frequency parameter $\lambda$ of SSSS isotropic square plate with a square hole and uniform thickness for various thickness ratio $h/h_i$

$\left( c/a = 0.5, a/h = 100, m = n = 16 \right)$

<table>
<thead>
<tr>
<th>$h/h_i$</th>
<th>Source</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Present</td>
<td>1st</td>
</tr>
<tr>
<td>12</td>
<td>Present</td>
<td>4.768</td>
</tr>
<tr>
<td>18</td>
<td>Present</td>
<td>4.767</td>
</tr>
</tbody>
</table>

5.3.3.1 Plate with Uniform Thickness

Numerical values for the lowest 5 natural frequency parameters $\lambda$ of SSSS isotropic thin square plate with a square hole are given in Table 5.12 with the FEM values obtained by Ali and Atwal [30], Kaushal and Bhat [58] and the exact solutions [54]. For the plate with $c/a = 0$, by comparing the present results and FEM results [52, 47] with the exact solutions [54], it can be found the present results are better than FEM results [52, 47]. For the plate with small hole such as $c/a < 0.3$, the fundamental and fifth frequency parameters obtained by the proposed method are higher than those FEM results, and the other frequency parameters shows sometimes higher and sometimes lower. For the plate with $c/a = 0.4$, the fundamental and higher frequency parameters obtained by the proposed method are lower and higher than those results [52, 47], respectively. For the plate with $c/a = 0.5$, all the present results are lower than FEM results in references [52, 47]. From Table 5.12, it is noticed the results obtained by different methods are some different but the maximum error is smaller than 5 percent. The effects of the hole size on the first 5 frequencies for the SSSS isotropic thin square plate can be found from Table 5.12. It might be noted that the variations of the fundamental and higher frequencies with hole size are quite different. As the ratio $c/a$ increases, the fundamental frequency first decreases a little, then increases. For $c/a = 0.5$, the fundamental frequency of the plate is higher than the corresponding frequency for the plate without a hole. But as the ratio $c/a$ increases, the second, third and fourth frequencies firstly increase a little, and then decrease.
For $c/a = 0.5$, these frequencies are lower than the corresponding frequencies for the plate without a hole. The fifth frequency monotonously decreases with the increase of $c/a$.

Table 5.13 presents the numerical results for the lowest 5 natural frequency parameters $\lambda$ of the SSSS orthotropic thin and moderately thick square plates with a square hole of side ratio $c/a = 0.5$. By comparing with the results of Reddy [32], the accuracy of the present results is investigated. Table 5.13 shows the side-to–thickness ratio $a/h$ affects the frequency considerately. The nodal patterns of the 5 modes of the plates are shown in Figure 5.14. It can be noted when $a/h$ changes from 100 to 10, the 1st, 2nd, 4th and 5th mode shapes don’t change but the 3rd mode shape changes a lot.

**Table 5.12** The first five frequencies versus the ratio $c/a$ for SSSS isotropic square plate with a square hole and uniform thickness ($a/h = 100$, $h/h_i = 12$)

<table>
<thead>
<tr>
<th>$c/a$</th>
<th>Source</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>0</td>
<td>Present</td>
<td>4.548</td>
</tr>
<tr>
<td></td>
<td>Ref.[58]</td>
<td>4.529</td>
</tr>
<tr>
<td>0.1</td>
<td>Present</td>
<td>4.544</td>
</tr>
<tr>
<td>0.2</td>
<td>Present</td>
<td>4.485</td>
</tr>
<tr>
<td></td>
<td>Ref.[30]</td>
<td>4.397</td>
</tr>
<tr>
<td></td>
<td>Ref.[58]</td>
<td>4.482</td>
</tr>
<tr>
<td>0.3</td>
<td>Present</td>
<td>4.507</td>
</tr>
<tr>
<td>0.4</td>
<td>Present</td>
<td>4.588</td>
</tr>
<tr>
<td>0.5</td>
<td>Present</td>
<td>4.822</td>
</tr>
</tbody>
</table>
Table 5.13 Natural frequency parameter $\lambda$ for SSSS orthotropic square plate with a square hole and uniform thickness for various thickness ($c/a = 0.5$, $h/h_i = 12$)

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Source</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>100</td>
<td>Present</td>
<td>7.098</td>
</tr>
<tr>
<td></td>
<td>Ref.[32]</td>
<td>7.160</td>
</tr>
<tr>
<td>10</td>
<td>Present</td>
<td>6.473</td>
</tr>
<tr>
<td></td>
<td>Ref.[32]</td>
<td>6.537</td>
</tr>
</tbody>
</table>

Figure 5.14 Nodal patterns for SSSS orthotropic square plate with a square hole and uniform thickness ($c/a = 0.5$, $h/h_i = 12$).

5.3.3.2 Plate with Variable Thickness in One Direction

In order to investigate the accuracy of the proposed method for the plate with variable thickness, numerical values for the lowest 5 natural frequency parameters $\lambda$ of SSSS isotropic thin square plate with variable thickness in one direction are given in Table 5.14 with the results of Appl and Byers [52]. In this section, variable thickness in one direction varies linearly along the $y$–direction according to the equation $h(x, y) = h_0(1 + ay/a)$. From Table 5.14, it can be seen the method described can be also used to solve the vibration problem of the plate with variable thickness.

As application of the proposed method, the numerical results for the lowest 5 natural frequency parameters $\lambda$ of SSSS orthotropic thin and moderately thick square plates with a square hole of side ratio $c/a = 0.5$ and variable thickness in one direction are presented.
in Tables 5.15 and 5.16. From these tables, it can be noticed that the frequency parameters will increase with the increase of $\alpha$. The nodal patterns of the 5 modes of the plates are shown in Figures 5.15 and 5.16. In both figures, the horizontal nodal lines of the second modes move to the thinner parts of the plates with the increase of $\alpha$.

Table 5.14 Natural frequency parameter $\lambda$ for SSSS isotropic square plate with variable thickness in one direction ($a/h_0 = 100$, $h_0/h_t = 12$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Source</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>Present</td>
<td>4.660</td>
<td>7.363</td>
<td>7.363</td>
<td>9.312</td>
<td>10.390</td>
</tr>
<tr>
<td></td>
<td>Ref.[59]</td>
<td>4.661</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>0.8</td>
<td>Present</td>
<td>5.354</td>
<td>8.406</td>
<td>8.439</td>
<td>10.685</td>
<td>11.747</td>
</tr>
<tr>
<td></td>
<td>Ref.[59]</td>
<td>5.335</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 5.15 Natural frequency parameter $\lambda$ for SSSS orthotropic square plate with a square hole and variable thickness in one direction ($c/a = 0.5$, $a/h_0 = 100$, $h_0/h_t = 12$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Source</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>Present</td>
<td>7.098</td>
<td>8.539</td>
<td>9.070</td>
<td>10.528</td>
<td>13.979</td>
</tr>
<tr>
<td></td>
<td>Ref.[32]</td>
<td>7.160</td>
<td>—</td>
<td>—</td>
<td>10.598</td>
<td>—</td>
</tr>
<tr>
<td>0.1</td>
<td>Present</td>
<td>7.267</td>
<td>8.749</td>
<td>9.299</td>
<td>10.782</td>
<td>14.045</td>
</tr>
<tr>
<td>0.8</td>
<td>Present</td>
<td>8.217</td>
<td>10.110</td>
<td>10.571</td>
<td>12.388</td>
<td>15.710</td>
</tr>
</tbody>
</table>

Table 5.16 Natural frequency parameter $\lambda$ for SSSS orthotropic square plate with a square hole and variable thickness in one direction ($c/a = 0.5$, $a/h_0 = 10$, $h_0/h_t = 12$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Source</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>Present</td>
<td>6.473</td>
<td>7.634</td>
<td>7.841</td>
<td>9.240</td>
<td>10.373</td>
</tr>
<tr>
<td></td>
<td>Ref.[32]</td>
<td>6.537</td>
<td>—</td>
<td>—</td>
<td>9.139</td>
<td>—</td>
</tr>
<tr>
<td>0.1</td>
<td>Present</td>
<td>6.584</td>
<td>7.755</td>
<td>7.917</td>
<td>9.368</td>
<td>10.431</td>
</tr>
<tr>
<td>0.8</td>
<td>Present</td>
<td>7.206</td>
<td>8.456</td>
<td>8.565</td>
<td>10.151</td>
<td>10.784</td>
</tr>
</tbody>
</table>

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Figure 5.15 Nodal patterns for SSSS orthotropic square plate with a square hole and variable thickness in one direction ($c/a = 0.5, a/h_0 = 100, h_0/h = 12$)

Figure 5.16 Nodal patterns for SSSS orthotropic square plate with a square hole and variable thickness in one direction ($c/a = 0.5, a/h_0 = 10, h_0/h = 14$)
5.3.3.3 Plate with Variable Thickness in Two Directions

The numerical results for the lowest 5 natural frequency parameter $\lambda$ of the SSSS orthotropic thin and moderately thick square plate with a square hole of side ratio $c/a = 0.5$ and variable thickness in two directions are presented in Tables 5.17 and 5.18. The thickness of the plate varies in the $x,y$-directions according to the sinusoidal function given by $h(x,y) = h_0(1 - \alpha \sin \pi x / a)(1 - \alpha \sin \pi y / a)$. Three cases of $\alpha = 0.1$, $\alpha = 0.3$ and $\alpha = 0.5$ are considered. It shows that the frequency parameters will decrease with the increase of $\alpha$. The nodal patterns of the 5 modes of the plates are shown in Figures 5.17 and 5.18.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Source</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>0.1</td>
<td>Present</td>
<td>6.839</td>
</tr>
<tr>
<td>0.3</td>
<td>Present</td>
<td>6.289</td>
</tr>
<tr>
<td>0.5</td>
<td>Present</td>
<td>5.691</td>
</tr>
</tbody>
</table>

Table 5.17 Natural frequency parameter $\lambda$ for SSSS orthotropic square plate with a square hole and variable thickness in two directions ($c/a = 0.5$, $a/h_0 = 10$, $h_0 / h_i = 14$)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Source</th>
<th>Mode sequence number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>0.1</td>
<td>Present</td>
<td>6.331</td>
</tr>
<tr>
<td>0.3</td>
<td>Present</td>
<td>5.995</td>
</tr>
<tr>
<td>0.5</td>
<td>Present</td>
<td>5.543</td>
</tr>
</tbody>
</table>

Table 5.18 Natural frequency parameter $\lambda$ for SSSS orthotropic square plate with a square hole and variable thickness in two directions ($c/a = 0.5$, $a/h_0 = 10$, $h_0 / h_i = 16$)
Figure 5.17 Nodal patterns for SSSS orthotropic square plate with a square hole and variable thickness in two directions ($ c/a = 0.5, a/h_0 = 100, h_0/h_t = 14 $)

Figure 5.18 Nodal patterns for SSSS orthotropic square plate with a square hole and variable thickness in two directions ($ c/a = 0.5, a/h_0 = 10, h_0/h_t = 16 $)
5.3.4 Conclusions

The discrete method is extended for analyzing the free vibration problem of simply supported orthotropic square plate with a square hole. An equivalent square plate is used to obtain the dynamic characteristics of a plate with a hole. The frequency parameters and their mode shapes are shown for simply supported thin and moderately thick plates with a hole for isotropic and orthotropic cases. It can be found that the transverse shear deformation effect is much more pronounced in orthotropic plate than in isotropic plate. The effects of the variation of the thickness in one and two directions on the frequencies are considered. The results by the proposed method have been compared with those previously reported. It shows that the present results have a good convergence and satisfactory accuracy. Although numerical results are given for only simply supported plates, the proposed method is a general method and can be used to solve the vibration problem of plates with different boundary conditions.
Chapter 6

General Conclusions
In the thesis, a discrete method has been proposed for the free vibration problems of plates with non-homogeneity. In order to investigate the efficiency and accuracy of the method, the following analysis has been carried out. For the isotropic plates, 1) Free vibration analysis of rectangular plates with multiple point supports and 2) Free vibration problems of square plates with stepped thickness on non-homogeneous elastic foundation have been investigated. For the orthotropic plates, 1) Convergence, accuracy and efficiency for vibration analysis of orthotropic plates, 2) Free vibration analysis of orthotropic rectangular plates with general boundary conditions and 3) Free vibration analysis of orthotropic square plates with a hole have been investigated. The main conclusions are summarized as follows:

In chapter 1, a review of published literature, the purpose and the outline of this thesis have been presented. Because the fundamental differential equations of the problem of non-homogeneous plates are the simultaneous partial differential equations with variable coefficients, it is almost impossible to obtain the analytical solutions. In order to analyze the free vibration problems with non-homogeneity, many numerical methods were used. Almost all the methods need the assumption of deflection at first. For some cases, they may encounter some difficulties such as the phenomenon of shear locking, the corner problem, complicated boundary conditions and etc. On the other hand, the problems of plates with non-homogeneity are generally more difficult than those of ordinary plates because the boundary is more difficult to be described easily and accurately and there is different modulus for the different materials for the non-homogeneous plates. All these make the calculation more complex and need more computer memory space and time. So in order to understand the dynamic behavior of such structure components with non-homogeneity which is different from homogeneous isotropic structure component, much more precise and more efficient method is needed even for the analytical method of a thin plate. It means in order to design and manufacture these new types of structure components safely and economically, it is indispensable to develop a more precise and efficient method for analyzing those structural mechanics characteristics. It is the purpose of this thesis.

In chapter 2, the basic theory has been given. At first, in order to obtain discrete solution
equations of orthotropic plates with variable thickness and distributed load combined with a concentrated load, the fundamental differential equations of the orthotropic plates with variable thickness were converted into integral equations. By applying numerical integration, the discrete solutions based on the Mindlin plate theory have been obtained. Secondly, the discrete method is extended to analyze the free vibration of orthotropic plates with variable thickness by using the Green function. The discrete form solution for deflection of the orthotropic plate with a concentrated load give the discrete type Green function of the plate. The characteristic equation of free vibration of orthotropic rectangular plate with non-uniform thickness and a concentrated load combined with uniform load has been obtained. By adding some parts, an irregular-shaped and variable thickness plate can be finally considered as an equivalent rectangular plate with variable thickness and point supports. Under the concept of the equivalent rectangular plate, the problems of irregular shaped plates, such as plates with holes, can be translated into the problems of equivalent rectangular plates. And then, all the analysis can be carried out on the equivalent rectangular plate. At last, comparison of FEM and proposed method has been made. Some numerical results have been analyzed for SSSS isotropic plate and CCCC isotropic plate under uniform load and a concentrated load. It is concluded that the proposed method does not require prior assumption of the shape of the deflection mode of the plate. The proposed method has the advantage of using less unknown quantities to obtain satisfactory accuracy. So it can make the calculation more easily, quickly and accurately.

In chapter 3, in order to verify the proposed discrete method described in Chapter 2, the experimental results has been obtained with a laser holographic interferometry. Aluminum alloy plates are used as specimen. Isotropic plates with uniform thickness, with non-uniform thickness and with a hole defect have been investigated. The experimental results have been compared with analytical results and reference results. From the experimental results analysis, it is shown that proposed discrete method has enough convergence and accuracy.

In chapter 4, for an isotropic plate, which is considered as a special case of orthotropic plates with the axial modulus \( E_1 = E_2 = E \) and the shear modulus
The discrete method described in Chapter 2 has been used for analyzing the free vibration problems of isotropic plates. There are two sections involved.

In the first section, the discrete method is used for analyzing free vibration of isotropic rectangular plates with variable thickness and multiple point supports. Six numerical results have been carried out. There are (1) isotropic rectangular plates with a central point support; (2) isotropic square plates with a point support on a corner; (3) CFFF isotropic square plates with two arbitrarily located point supports; (4) SSSS isotropic square plates with variable thickness in one direction; (5) SSSS isotropic square plates with variable thickness in two directions and (6) SSSS isotropic square plates with stepped thickness in one direction.

The conclusions have been obtained as follows: The fundamental frequency parameter of isotropic plate with two point supports along the edge is lower than those of corresponding plates with interior two point supports. For the isotropic plates without point supports, the vertical nodal lines move to the thinner part with the increase of the value of \( \alpha \). For the isotropic plates with a central point support, the straight lines change to curve line with increase of the value of aspect ratio \( \alpha \). The frequency parameters increase with the increase of the ratio of \( c/a \) or \( h_1/h_0 \). The effects of the ratios \( c/a \) and \( h_1/h_0 \) on the frequency parameters are significant.

In the second section, the proposed discrete method is used for analyzing the free vibration problem of isotropic square plates with stepped thickness on non-homogeneous elastic foundations. The spring system is used to simulate the foundations. Four numerical results have been carried out. There are: (1) an isotropic square plate on homogeneous foundations; (2) an isotropic square plate with uniform thickness on non-homogeneous foundations; (3) isotropic square plates with stepped thickness in central square part and (4) isotropic square plates with stepped thickness on non-homogeneous elastic foundations.

The following conclusions have been obtained: The effect of the dimensionless modulus of the foundation \( K \) on the fundamental frequency parameter is much more significant than that on higher frequency parameters, the frequency parameters increase with increase of the constant \( K \), and they increase quickly when \( K \) is larger than 100. The fundamental
frequency parameters of the isotropic plate with the central part having higher foundation modulus are higher than those of the isotropic plate with the central part having lower foundation modulus and the modulus of the foundation affects the frequency parameter greatly. The results also shown that with the increase of the thickness ratio, the first and the second frequency parameters decrease for the isotropic plates with the specific modulus of the foundation. With increase of the value of the modulus of the foundation, the frequency parameters increase.

Above results also shown that proposed method has a good convergence and satisfactory accuracy.

In chapter 5, the discrete method described in Chapter 2 is used for analyzing the free vibration problems of orthotropic plates. There are three sections involved.

In the first section, in order to verify the proposed method described in Chapter 2, vibration analysis of orthotropic rectangular plates with non-uniform thickness has been investigated. This is the general method for analyzing the bending and free vibration of orthotropic plates. The efficiency, convergence and accuracy of the proposed method described in Chapter 2 have been investigated.

In the second section, the discrete method is developed for analyzing the free vibration problem of orthotropic rectangular plates with general boundary conditions. The efficiency and accuracy for the free vibration problem of tapered orthotropic rectangular plates with general boundary conditions have been investigated. The effects of the boundary conditions, the aspect ratio and variable thickness on the frequency parameter have been discussed. Some new data and mode shapes for the plates with general boundary condition and variable thickness in one or two directions have been given.

From the above numerical results, the following conclusions have been obtained: (1) For an orthotropic rectangular plate with variable thickness in one direction: The frequency parameters increase with increase of the taper ratio $\alpha$ for the plate with CSSS, SSSS, SSFS boundary condition and aspect ratio $b/a$, and decrease with the increase of aspect ratio $b/a$ for the plate with the same boundary condition and taper ratio $\alpha$. The highest frequency parameters were obtained for CSSS plates, then successively for SSSS and SSFS. It shown that with increase of boundary constraints, which results in decrease of
stiffness, frequency parameters decrease significantly. The vertical nodal lines have the
tendency to be close to the edge with less boundary constraint. With increase of taper ratio,
the vertical nodal lines move to the thinner part of the plates gradually. (2) For the
orthotropic rectangular plate with variable thickness in two directions: The numerical
results were shown that the boundary conditions and the taper ratio affect the frequency
parameter greatly. With the increase of the boundary constraint and the taper ratio, the
frequency parameter would increase.

In third section, the proposed method is used for analyzing the free vibration problem of
simply supported orthotropic square plate with a square hole. In this section, a square plate
with a square hole is transformed into an equivalent square plate with non-uniform
thickness by considering the hole as an extremely thin part of the equivalent plate.
Therefore, the dynamic characteristics of the plate with a hole have been obtained by
analyzing the equivalent plate. The effects of the side to thickness ratio, hole side to plate
side ratio and variation of the thickness on the frequency properties have been considered.
Some numerical analyses have been carried out for the simply supported orthotropic square
plate with a square hole. It has been fund that the transverse shear deformation effect is
much more pronounced in orthotropic plate than in isotropic plate. It shows that the present
results have a good convergence and satisfactory accuracy. Although numerical results
were given for only simply supported plates, the proposed method is a general method and
can be used to solve the vibration problem of plates with different boundary conditions.

In chapter 6, the main work and conclusions have been summarized.
Bibliography


Conference on Heterogeneous Material Mechanics, Study on characteristics of vibration of cantilever plates with variable thickness and a hole defect. (in press)


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