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Note on the Prime Number Theorem

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Abstract

In this paper we prove the prime number theorem using the properties of the zeta function. The purpose of the present paper is to complete the proof given by Greene and Krantz [GRK] in which they omitted the proofs of some lemmas.

1 Introduction

Let $\pi(n)$ denote the number of primes not exceeding n. Then the prime number theorem asserts that

$$\lim_{n\to\infty} \frac{\pi(n)}{\left(\frac{n}{\log n}\right)} = 1.$$

Gauss conjectured this formula when he was fourteen years old. It was J. Hadamard and C. de la Vallee Poussin who in 1896 independently proved the prime number theorem. They used complex analysis—in particular an analysis of the Riemann zeta function. The purpose of the present paper is to complete the proof due to Greene and Krantz [GRK].

2 Preliminaries

For $\operatorname{Re} z > 1$, define

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

$\zeta(z)$ is called Riemann's zeta function. $\zeta(z)$ is holomorphic in $\{z \mid \operatorname{Re} z > 1\}$. It is known that $\zeta(z)$ has the following properties. We omit the proof.

(R.1) $\zeta(z)$ continues holomorphically to $\mathbb{C}\backslash\{1\}$.

(R.2) $\zeta(z)$ has a simple pole at $z = 1$ with residue 1.

(R.3) The only zeros of $\zeta(z)$ not in the set $\{z \mid 0 \leq \operatorname{Re} z \leq 1\}$ are at $-2n$ $(n \in \mathbb{N})$.

Lemma 1 *$\zeta(z)$ has no zero on $\{z \mid \operatorname{Re} z = 1\}$.*

Proof Suppose $\zeta(1+it_0)=0$ for some $t_0 \in \mathbb{R}$, $t_0 \neq 0$. Define

$$\Phi(z)=\zeta^3(z)\cdot\zeta^4(z+it_0)\cdot\zeta(z+2it_0).$$

Then there exist holomorphic functions h_1 and h_2 in a neighborhood of 1 such that

$$\Phi(z)=\left(\frac{1}{z-1}+h_1(z)\right)^3((z-1)h_2(z))^4\zeta(z+2it_0)$$

in a neighborhood of $z=1$. Hence Φ is expressed by

$$\Phi(z)=\alpha_1(z-1)^k+\alpha_2(z-2)^{k+1}+\cdots \quad (\alpha_1\neq 0, k\geq 1)$$

in a neighborhood of $z=1$. Then

$$\frac{\Phi'(z)}{\Phi(z)}=\frac{\alpha_1 k+\cdots}{(z-1)\{\alpha_1+\alpha_2(z-1)+\cdots\}}=\frac{k}{z-1}+h_3(z)$$

where h_3 is holomorphic in a neighborhood of $z=1$. Then there exists $\varepsilon_0>0$ such that

$$\operatorname{Re}\frac{\Phi'(x)}{\Phi(x)}>0 \tag{1}$$

for $1<x<1+\varepsilon_0$.

On the other hand, we obtain

$$\begin{aligned}\frac{\Phi'(x)}{\Phi(x)} &= \frac{3\zeta'(x)}{\zeta(x)}+\frac{4\zeta'(x+it_0)}{\zeta(x+it_0)}+\frac{\zeta'(x+2it_0)}{\zeta(x+2it_0)}\\ &= \sum_{n=2}^{\infty}\Lambda(n)\{-3e^{-x\log n}-4e^{-(x+it_0)\log n}-e^{-(x+2it_0)\log n}\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname{Re}\frac{\Phi'(x)}{\Phi(x)} &= \sum_{n=2}^{\infty}\Lambda(n)e^{-x\log n}\{-3-4\cos(t_0\log n)-\cos(2t_0\log n)\}\\ &= -2\sum_{n=2}^{\infty}\Lambda(n)e^{-x\log n}(\cos(t_0\log n)+1)^2\leq 0.\end{aligned}$$

This contradicts (1).

3 Proof of the prime number theorem

Definition Define

$$G(z)=-\left(\frac{\zeta'(z)}{\zeta(z)}+\frac{z}{z-1}\right)\frac{1}{z}.$$

Theorem 1 *$G(z)$ is holomorphic on $\{z \mid \operatorname{Re} z\geq 1\}$.*

Proof From the properties of the zeta function (R.1), (R.2) and Lemma 1, it is sufficient to show that $G(z)$ is holomorphic at $z = 1$. It follows from the property (R.2) and the Laurent expansion that

$$\zeta(z) = \frac{1}{z-1} + h(z),$$

where h is an entire function. For z near 1,

$$\begin{aligned}\frac{\zeta'(z)}{\zeta(z)} &= \frac{-\frac{1}{z-1} + (z-1)h'(z)}{1 + (z-1)h(z)} \\ &= \left\{-\frac{1}{z-1} + (z-1)h'(z)\right\} \sum_{n=0}^{\infty}(-(z-1)h(z))^n \\ &= -\frac{1}{z-1} + g(z),\end{aligned}$$

where g is holomorphic in a neighborhood of 1. Then

$$-\left(\frac{\zeta'(z)}{\zeta(z)} + \frac{z}{z-1}\right)\frac{1}{z} = -(1+g(z))\frac{1}{z}$$

is holomorphic at $z = 1$. □

Theorem 2 *For* $\operatorname{Re} z > 1$,

$$\frac{1}{\zeta(z)} = \prod_{p \in P}\left(1 - \frac{1}{p^z}\right),$$

where $P = \{2, 3, 5, \cdots\} = \{p_1, p_2, p_3, \cdots\}$ *is the set of positive primes.*

Proof Since $\sum_{n=1}^{\infty} n^{-z}$ converges for Re z > 1, $\sum_{p \in P} p^{-z}$ converges, and hence

$$\prod_{p \in P}\left(1 - \frac{1}{p^z}\right)$$

converges. For $\varepsilon > 0$, there exists a natural number N such that

$$\sum_{n=N+1}^{\infty}\left|\frac{1}{n^z}\right| < \varepsilon.$$

Since $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, we have

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = \sum_{n=1}^{\infty}\frac{1}{n^z} - \sum_{n=1}^{\infty}\frac{1}{(2n)^z} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^z}.$$

Let A_n be the set of all positive integers which are divisible by at least one of $p_1, \cdots, p_n$. Then we obtain

$$\begin{aligned}\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{2^z}\right)\zeta(z) &= \left(1-\frac{1}{3^z}\right)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^z} \\ &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^z}-\sum_{n=1}^{\infty}\frac{1}{\{3(2n-1)\}^z} \\ &= 1+\frac{1}{5^z}+\cdots=\sum_{q}\frac{1}{q^z},\end{aligned}$$

where the summation $\sum_q \frac{1}{q^z}$ is taken over all elements q of $\mathbb{N}-A_3$. Continuing in this manner, we obtain

$$\begin{aligned}&\left(1-\frac{1}{(p_N)^z}\right)\left(1-\frac{1}{(p_{N-1})^z}\right)\cdots\left(1-\frac{1}{2^z}\right)\zeta(z) \\ &=1+\frac{1}{(p_{N+1})^z}+\cdots \\ &=\sum_{r}\frac{1}{r^z},\end{aligned}$$

where the summation $\sum_r \frac{1}{r^z}$ is taken over all elements r of $\mathbb{N}-A_N$. Thus we have

$$\left|\left(\prod_{j=1}^{N}\left(1-\frac{1}{(p_j)^z}\right)\right)\zeta(z)-1\right|<\sum_{n=N+1}^{\infty}\frac{1}{|n^z|}<\varepsilon.$$

Therefore we have proved that

$$\frac{1}{\zeta(z)}=\lim_{N\to\infty}\prod_{j=1}^{N}\left(1-\frac{1}{(p_j)^z}\right)=\prod_{p\in P}\left(1-\frac{1}{p^z}\right).$$

□

Definition Define $\Lambda:\{n\in\mathbb{Z}\mid n>0\}\to\mathbb{R}$ by

$$\Lambda(m)=\begin{cases}\log p & (\text{if } m=p^k, p\in P, k\in\mathbb{N}) \\ 0 & (\text{otherwise})\end{cases}.$$

Then we have the following:

Theorem 3 *For* $\operatorname{Re} z>1$ *we have*

$$-\frac{\zeta'(z)}{\zeta(z)}=\sum_{n=2}^{\infty}\Lambda(n)e^{-z\log n}.$$

Proof By Theorem 1, we have

$$-\log\zeta(z)=\sum_{p\in P}\log(1-p^{-z})=\sum_{p\in P}\log(1-e^{-z\log p}).$$

Consequently,

$$\begin{aligned}-\frac{\zeta'(z)}{\zeta(z)} &= \sum_{p\in P}\frac{(\log p)e^{-z\log p}}{1-e^{-z\log p}}=\sum_{p\in P}(\log p)\sum_{k=1}^{\infty}(e^{-z\log p})^k\\ &= \sum_{k=1}^{\infty}\sum_{p\in P}(\log p)e^{-z\log p^k}=\sum_{n=2}^{\infty}\Lambda(n)e^{-z\log n}\end{aligned}$$

□

Definition For $x>0$, $x\in\mathbb{R}$, define

(1) $\psi(x)=\sum_{n\le x}\Lambda(n)$,

(2) for $p\in P$, $m_x(p)$ denotes the greatest integer k such that $p^k\le x$.

Lemma 2 *For* $x\ge 3$,

$$\frac{\psi(x)}{x}\le\frac{\pi(x)}{\frac{x}{\log x}}\le\frac{1}{\log x}+\frac{\psi(x)}{x}\left(\frac{\log x}{\log x-2\log\log x}\right).$$

Proof Since $p^{m_x(p)}\le x$, we obtain $m_x(p)\le\log x/\log p$. Then

$$\begin{aligned}\psi(x) &= \sum_{n\le x}\Lambda(n)=\sum_{p^k\le x}\log p=\sum_{p\le x}m_x(p)\log p\\ &\le \sum_{p\le x}\log x=\pi(x)\log x.\end{aligned}$$

This proves the left side inequality. Let $1<y<x$. Then

$$\begin{aligned}\pi(x) &\le \pi(y)+\sum_{y<p\le x}1\le\pi(y)+\sum_{y<p\le x}\frac{\log p}{\log y}\\ &\le \pi(y)+\frac{1}{\log y}\sum_{p\le x}\log p\le\pi(y)+\frac{\psi(x)}{\log y}.\end{aligned}$$

Put $y=\frac{x}{\log^2 x}<x$. Then

$$\begin{aligned}\pi(x) &\le \pi\left(\frac{x}{\log^2 x}\right)+\frac{\psi(x)}{\log x-2\log\log x}\\ &< \frac{x}{\log^2 x}+\frac{\psi(x)}{\log x}\left(\frac{\log x}{\log x-3\log\log x}\right).\end{aligned}$$

This proves the right side inequality. □

Definition For $u>0$, $u\in\mathbb{R}$, define $K(u)=\psi(e^u)e^{-u}$.

The following lemma follows easily from Lemma 2.

Lemma 3 *The prime number theorem holds if and only if* $\lim_{u\to\infty} K(u) = 1$.

Lemma 4 *For* $\operatorname{Re} z > 1$,

$$-\frac{\zeta'(z)}{\zeta(z)} = z\int_0^\infty \psi(e^u)e^{-zu}du.$$

Proof Since $\psi(n) = \psi(n-1) + \Lambda(n)$, we have by Theorem 2

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=2}^\infty \Lambda(n)e^{-z\log n} = \sum_{n=2}^\infty \psi(n)e^{-z\log n} - \sum_{n=3}^\infty \psi(n-1)e^{-z\log n}$$

$$= \sum_{n=2}^\infty \psi(n)(e^{-z\log n} - e^{-z\log(n+1)} = \sum_{n=2}^\infty \psi(n)\int_{\log n}^{\log(n+1)} ze^{-zu}du.$$

Since $\psi(e^u) = \psi(n)$ for $n < e^u < n+1$, we have

$$\begin{aligned} -\frac{\zeta'(z)}{\zeta(z)} &= z\sum_{n=2}^\infty \int_{\log n}^{\log(n+1)} \psi(e^u)e^{-zu}du \\ &= z\int_{\log 2}^\infty \psi(e^u)e^{-zu}du = z\int_0^\infty \psi(e^u)e^{-zu}du \end{aligned}$$

□

Theorem 4 *For* $\operatorname{Re} z > 1$,

$$G(z) = \int_0^\infty (K(u)-1)e^{-(z-1)u}du.$$

Proof By Lemma 4 we have

$$G(z) = \int_0^\infty \psi(e^u)e^{-zu}dz - \frac{1}{z-1} = \int_0^\infty \psi(e^u)e^{-zu}dz - \int_0^\infty e^{-(z-1)u}du$$

$$= \int_0^\infty (\psi(e^u)e^{-u} - 1)e^{-(z-1)u}du = \int_0^\infty (K(u)-1)e^{-(z-1)u}du$$

□

Lemma 5

$$\int_{-\infty}^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi.$$

Proof

$$\int_{-\infty}^\infty \left(\frac{\sin t}{t}\right)^2 dt = \int_{-\infty}^\infty \left(-\frac{1}{t}\right)' \sin^2 t dt = \int_{-\infty}^\infty \frac{\sin 2t}{t}dx = \pi.$$

□

Lemma 6 *For $\lambda > 0$, $s \in \mathbb{R}\backslash\{0\}$,*

$$\int_{-2}^{2} \frac{\lambda}{2}\left(1-\frac{|t|}{2}\right) e^{i\lambda st} dt = \lambda \frac{\sin^2 \lambda s}{(\lambda s)^2}.$$

Proof

$$\begin{aligned}
\int_{-2}^{2} \frac{\lambda}{2}\left(1-\frac{|t|}{2}\right) e^{i\lambda st} dt &= \int_{0}^{2} \lambda\left(1-\frac{t}{2}\right) \cos \lambda st dt \\
&= \lambda \int_{0}^{2} \left(1-\frac{t}{2}\right) \left(\frac{\sin \lambda st}{\lambda s}\right)' dt \\
&= \frac{1}{2s} \int_{0}^{2} \sin(\lambda st) dt \\
&= \lambda \frac{\sin^2 \lambda s}{(\lambda s)^2}.
\end{aligned}$$

□

Theorem 5 *For $\lambda > 1$, $y > 0$, $0 < \varepsilon < 1$, we have*

$$\left| \int_{-y\lambda}^{\infty} \left(K\left(y+\frac{v}{\lambda}\right)-1\right) e^{-\varepsilon(y+v\lambda^{-1})} \left(\frac{\sin v}{v}\right)^2 dv \right| \leq \frac{C(\lambda)}{y}, \tag{2}$$

where $C(\lambda)$ is a constant which depends only on λ.

Proof With the change of variable $u = y + \frac{v}{\lambda}$,

$$\begin{aligned}
I &= \int_{-y\lambda}^{\infty} \left(K\left(y+\frac{v}{\lambda}\right)-1\right) e^{-\varepsilon(y+v\lambda^{-1})} \left(\frac{\sin v}{v}\right)^2 dv \\
&= \int_{0}^{\infty} (K(u)-1)e^{-\varepsilon u} \left(\frac{\sin(\lambda(u-y))}{\lambda(u-y)}\right)^2 \lambda du \\
&= \int_{0}^{\infty} (K(u)-1)e^{-\varepsilon u} \left(\int_{-2}^{2} \frac{\lambda}{2}\left(1-\frac{|t|}{2}\right) e^{i\lambda(y-u)t} dt\right) du.
\end{aligned}$$

Since $K(u) = \psi(e^u)e^{-u} \leq u$ and

$$\int_{0}^{\infty} |K(u)-1| e^{-\varepsilon u} du \leq \int_{0}^{\infty} (1+u)e^{-\varepsilon u} du < \infty,$$

by Fubuni's theorem we obtain

$$\begin{aligned}
I &= \int_{-2}^{2} \left(\int_{0}^{\infty} (K(u)-1)e^{-((1+\varepsilon+i\lambda t)-1)u} du\right) \frac{\lambda}{2}\left(1-\frac{|t|}{2}\right) e^{i\lambda yt} dt \\
&= \int_{-2}^{2} \left\{G(1+\varepsilon+i\lambda t)\frac{\lambda}{2}\left(1-\frac{|t|}{2}\right)\right\} e^{i\lambda yt} dt
\end{aligned}$$

Let

$$E_\lambda = \{x + iy \mid 1 \le x \le 2,\ -2\lambda \le y \le 2\lambda\}.$$

Define

$$M_1(\lambda) = \sup_{z\in E_\lambda} |G(z)| + \sup_{z\in E_\lambda} |G'(z)|$$

and

$$M_2(\lambda) = M_1(\lambda)\lambda^2.$$

For $0 \le t \le 2$, define

$$f(t) = G(1+\varepsilon+i\lambda t)\frac{\lambda}{2}\left(1-\frac{t}{2}\right).$$

Then

$$f'(t) = G'(1+\varepsilon+i\lambda t)i\frac{\lambda^2}{2}\left(1-\frac{t}{2}\right) - G(1+\varepsilon+i\lambda t)\frac{\lambda}{4}.$$

Hence

$$|f(t)| \le M_2(\lambda), \quad |f'(t)| \le M_2(\lambda).$$

Then

$$\begin{aligned}\left|\int_0^2 f(t)e^{i\lambda yt}dt\right| &= \left|\int_0^2 f(t)\left(\frac{1}{i\lambda y}e^{i\lambda yt}\right)' dt\right| \\ &= \left|\left[\frac{1}{i\lambda y}e^{i\lambda yt}f(t)\right]_0^2 - \int_0^2 f'(t)\frac{e^{i\lambda yt}}{i\lambda y}dt\right| \\ &\le \frac{4M_2(\lambda)}{\lambda y}.\end{aligned}$$

Similarly, we obtain

$$\left|\int_{-2}^0 f(t)e^{i\lambda yt}dt\right| \le \frac{4M_2(\lambda)}{\lambda y}.$$

Define $C(\lambda) = 8M_2(\lambda)/\lambda$. Then $I \le C(\lambda)/y$. This completes the proof of Theorem 5. □

Corollary 1 *For all* $\lambda > 1$ *and* $y > 0$,

$$\left|\int_{-y\lambda}^\infty \left(K\left(y+\frac{v}{\lambda}\right)-1\right)\left(\frac{\sin v}{v}\right)^2 dv\right| \le \frac{C(\lambda)}{y}. \tag{3}$$

Proof It follows from Theorem 5 that

$$\begin{aligned}&\int_{-y\lambda}^\infty K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 e^{-\varepsilon(y+\lambda^{-1}v)}dv \\ &\le \int_{-y\lambda}^\infty \left(\frac{\sin v}{v}\right)^2 e^{-\varepsilon(y+\lambda^{-1}v)}dv + \frac{C(\lambda)}{y} \le \pi + \frac{C(\lambda)}{y}\end{aligned}$$

By the monotone convergence theorem, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0+}\int_{-y\lambda}^{\infty} K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 e^{-\varepsilon(y+\lambda^{-1}v)}dv\\&=\int_{-y\lambda}^{\infty}\lim_{\varepsilon\to 0+} K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 e^{-\varepsilon(y+\lambda^{-1}v)}dv\\&=\int_{-y\lambda}^{\infty} K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 dv.\end{aligned}$$

Hence

$$\int_{-y\lambda}^{\infty} K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 dv \le \pi + \frac{C(\lambda)}{y}.$$

Therefore,

$$K\left(y+\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2$$

is integrable on $[-y\lambda,\infty)$. Define

$$f_\varepsilon(v) = \left(K\left(y+\frac{v}{\lambda}\right)-1\right)\left(\frac{\sin v}{v}\right)^2 e^{-\varepsilon(y+\lambda^{-1}v)}.$$

Then

$$|f_\varepsilon(v)| \le \left(K\left(y+\frac{v}{\lambda}\right)+1\right)\left(\frac{\sin v}{v}\right)^2$$

for $v\in[-\lambda y,\infty)$. Lebesgue's dominated convergence theorem tells us that letting $\varepsilon\to 0$ in (2) gives

$$\left|\int_{-y\lambda}^{\infty}\left(K\left(y+\frac{v}{\lambda}\right)-1\right)\left(\frac{\sin v}{v}\right)^2 dv\right| \le \frac{C(\lambda)}{y}.$$

□

Lemma 7 *For $y>0$, $\lambda>1$, $-\sqrt{\lambda}\le v\le\sqrt{\lambda}$, we have*

(1) $K\left(y-\frac{1}{\sqrt{\lambda}}\right) \le K\left(y+\frac{v}{\lambda}\right)e^{\frac{2}{\sqrt{\lambda}}}$

(2) $K\left(y+\frac{1}{\sqrt{\lambda}}\right) \ge K\left(y+\frac{v}{\lambda}\right)e^{-\frac{2}{\sqrt{\lambda}}}$.

Proof Since $\psi(u)$ is increasing, we have

$$K\left(y-\frac{1}{\sqrt{\lambda}}\right)e^{y-\frac{1}{\sqrt{\lambda}}} = \psi(e^{y-\frac{1}{\sqrt{\lambda}}}) \le \psi(e^{y+\frac{v}{\lambda}}) = K\left(y+\frac{v}{\lambda}\right)e^{y+\frac{v}{\lambda}}.$$

Therefore we have

$$K\left(y-\frac{1}{\sqrt{\lambda}}\right) \le K\left(y+\frac{v}{\lambda}\right)e^{\frac{2}{\sqrt{\lambda}}}.$$

This proves (1). (2) is proved in the same way. □

Lemma 8 *For* $y > 1$, $\lambda > 1$,

$$\left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv\right) K\left(y - \frac{1}{\sqrt{\lambda}}\right) \leq e^{\frac{2}{\sqrt{\lambda}}} \left(\frac{C(\lambda)}{y} + \pi\right).$$

Proof We denote the left side of the above inequality by I_1. Then by Lemma 7(1) and Corollary 1 we obtain

$$\begin{aligned} I_1 &\leq e^{\frac{2}{\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 K\left(y + \frac{v}{\lambda}\right) dv \\ &\leq e^{\frac{2}{\sqrt{\lambda}}} \int_{-\infty}^{\infty} \left(\frac{\sin v}{v}\right)^2 K\left(y + \frac{v}{\lambda}\right) dv \\ &= e^{\frac{2}{\sqrt{\lambda}}} \int_{-\lambda y}^{\infty} \left(\frac{\sin v}{v}\right)^2 K\left(y + \frac{v}{\lambda}\right) dv \\ &\leq e^{\frac{2}{\sqrt{\lambda}}} \left\{\int_{-\lambda y}^{\infty} \left(K\left(y + \frac{v}{\lambda}\right) - 1\right) \left(\frac{\sin v}{v}\right)^2 dv + \pi\right\} \\ &\leq e^{\frac{2}{\sqrt{\lambda}}} \left(\frac{C(\lambda)}{y} + \pi\right). \end{aligned}$$

□

Lemma 9 *$K(x)$ is a bounded function.*

Proof Suppose K is unbounded. Then there exists a sequence $\{x_j\}$ such that $x_j \to \infty$ and $K(x_j) \to \infty$. Put $x_j + \frac{1}{\sqrt{\lambda}} = y_j$. Then by Lemma 8 we obtain

$$K(x_j) = K\left(y_j - \frac{1}{\sqrt{\lambda}}\right) \leq \left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv\right)^{-1} e^{\frac{2}{\sqrt{\lambda}}} \left(\frac{C(\lambda)}{y_j} + \pi\right).$$

Letting $j \to \infty$ gives

$$\infty \leq \left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv\right)^{-1} e^{\frac{2}{\sqrt{\lambda}}} \pi.$$

This is a contradiction. □

Lemma 10 *For any sequence $x_j \to \infty$ such that $\{K(x_j)\}$ has a limit,*

$$\lim_{j \to \infty} K(x_j) \leq 1.$$

Proof Put $x_j + \frac{1}{\sqrt{\lambda}} = y_j$. By Lemma 8, we have

$$\begin{aligned} K(x_j) &= K\left(y_j - \frac{1}{\sqrt{\lambda}}\right) \\ &\leq e^{\frac{2}{\sqrt{\lambda}}} \left\{\frac{C(\lambda)}{y_j} + \pi\right\} \left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv\right)^{-1}. \end{aligned}$$

Then

$$\lim_{j\to\infty} K(x_j) \leq e^{\frac{2}{\sqrt{\lambda}}}\pi \left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv\right)^{-1}.$$

Letting $\lambda \to \infty$ yields $\lim_{j\to\infty} K(x_j) \leq 1$.

□

Lemma 11 *For any sequence $x_j \to \infty$ such that $\{K(x_j)\}$ has a limit,*

$$\lim_{j\to\infty} K(x_j) \geq 1.$$

Proof Put $x_j - \frac{1}{\sqrt{\lambda}} = y_j$. We may assume that $y_j > 1$, $\lambda > 1$. Then it follows from Lemma 7(2) that

$$\begin{aligned} K(x_j)e^{\frac{2}{\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv &= K\left(y_j + \frac{1}{\sqrt{\lambda}}\right) e^{\frac{2}{\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv \\ &\geq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 K\left(y_j + \frac{v}{\lambda}\right) dv \\ &= \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 \left(K\left(y_j + \frac{v}{\lambda}\right) - 1\right) dv \\ &\quad + \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv. \end{aligned}$$

By Lemma 9, there exists $M > 0$ such that $K(x) < M$. Put

$$A = [\sqrt{\lambda}, \infty) \cup [-\lambda y_j, -\sqrt{\lambda}].$$

Then

$$\begin{aligned} K(x_j)e^{\frac{2}{\sqrt{\lambda}}} \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv &\geq \int_{-\lambda y_j}^{\infty} \left(\frac{\sin v}{v}\right)^2 \left(K\left(y_j + \frac{v}{\lambda}\right) - 1\right) dv \\ &\quad - \int_A \left(\frac{\sin v}{v}\right)^2 \left(K\left(y_j + \frac{v}{\lambda}\right) - 1\right) dv \\ &\quad + \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv \\ &\geq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv - (M+1) \int_{|v|\geq\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv \\ &\quad - \left| \int_{\lambda y_j}^{\infty} \left(\frac{\sin v}{v}\right)^2 \left(K\left(y_j + \frac{v}{\lambda}\right) - 1\right) dv \right| \end{aligned}$$

$$\geq \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv - (M+1)\int_{|v|\geq\sqrt{\lambda}} \left(\frac{\sin v}{v}\right)^2 dv$$
$$-\frac{C(\lambda)}{y_j}.$$

Letting $j \to \infty$ yields

$$\begin{aligned}\varliminf_{j\to\infty} K(x_j) \;&\geq\; e^{-\frac{2}{\sqrt{\lambda}}}\left(\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}}\left(\frac{\sin v}{v}\right)^2 dv\right)^{-1} \\ &\quad\times\left\{\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}}\left(\frac{\sin v}{v}\right)^2 dv - (M+1)\int_{|v|\geq\sqrt{\lambda}}\left(\frac{\sin v}{v}\right)^2 dv\right\}.\end{aligned}$$

Letting $\lambda \to \infty$ gives

$$\varliminf_{j\to\infty} K(x_j) \geq 1.$$

□

Theorem 6 (Prime Number Theorem) *Let $\pi(n)$ denote the number of primes not exceeding n. Then*

$$\lim_{n\to\infty} \frac{\pi(n)}{\left(\frac{n}{\log n}\right)} = 1.$$

Proof By Lemma 3, it is sufficient to show that

$$\lim_{x\to\infty} K(x) = 1.$$

Suppose that $\lim_{x\to\infty} K(x)$ either does not exist or does not equal 1. Then there exists a sequence $\{x_j\}$ such that $\{K(x_j)\}$ does not converge to 1 and $x_j \to \infty$. Then there exists $\varepsilon > 0$ such that

$$|K(x_j) - 1| \geq \varepsilon \tag{4}$$

for infinitely many j. We may assume that $\{x_j\}$ satisfies (4). Since $\{K(x_j)\}$ is bounded by Lemma 9, there exists a convergent subsequence $\{K(x_{j_n})\}$. Let $\lim_{n\to\infty} K(x_{j_n}) = \alpha$. By Lemma 10 and Lemma 11, $\alpha = 1$. But it follows from (4) that $|\alpha - 1| \geq \varepsilon$, which is a contradiction. □

References

[GRK] R. E. Greene and S. G. Krantz, *Function theory of one complex variable*, John Wiley & Sons, Inc., 1997.