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An Inequality on the Exponential Function

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Abstract

In this paper we prove an inequality concerning the exponential function.

1 Introduction and Proof

The purpose of the present paper is to prove the following theorem:

**Theorem 1** For $0 < \varepsilon \leq \pi$ and $\theta \in \mathbb{R}$,

$$|1 - e^{\varepsilon e^{i\theta}}| \geq 1 - e^{-\varepsilon}.$$  

**Proof** Since

$$|1 - e^{\varepsilon e^{i\theta}}|^2 = 1 - 2e^{\varepsilon e^{i\theta}}\cos(\varepsilon \sin \theta) + e^{2\varepsilon e^{i\theta}},$$

it is sufficient to show that

$$f(\theta) = e^{2\varepsilon \cos \theta} - 2e^{\varepsilon \cos \theta} \cos(\varepsilon \sin \theta) - e^{-2\varepsilon} + 2e^{-\varepsilon} \geq 0$$

for $0 < \theta < 2\pi$. A simple calculation shows that

$$f'(\theta) = 2\varepsilon e^{2\varepsilon \cos \theta} \left( \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon e^{i\theta}}} - 1 \right).$$

We put

$$g(\theta) = \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon e^{i\theta}}}. $$

Then

$$g'(\theta) = \frac{-\varepsilon \sin(\varepsilon \sin \theta) + \varepsilon \sin \theta \cos(\varepsilon \sin \theta)}{\sin^2 \theta e^{\varepsilon e^{i\theta}}}.$$

First we assume that $0 < \theta < \pi$. Put $t = \varepsilon \sin \theta$ and

$$h(t) = t \cos t - \sin t.$$  

Then $0 \leq t \leq \pi$. Since $h'(t) = -t \sin t \leq 0$ and $h(0) = 0$, we have $h(t) \leq 0$ ($0 \leq t \leq \pi$). Therefore $g'(\theta) \leq 0$ ($0 < \theta < \pi$) and

$$g(0) = \lim_{\theta \to 0} \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon e^{i\theta}}} = \frac{1 + \varepsilon}{e^{\varepsilon}} \leq 1.$$
Hence $g(\theta) \leq 1$ for $0 < \theta < \pi$. Then

$$f'(\theta) = 2ze^{2e\cos \theta} \sin \theta(g(\theta) - 1) \leq 0.$$  

Therefore, $f(\theta)$ is monotonically decreasing for $\theta \in [0, \pi]$. Since $f(\pi) = 0$, it follows that $f(\theta) \geq 0$ for $\theta \in [0, \pi]$. A similar argument tells us that $g(\theta)$ is monotonically increasing on $[\pi, 2\pi]$, and $g(2\pi) = g(0) \leq 1$, and hence $g(\theta) \leq 1$ for $\theta \in [\pi, 2\pi]$. Therefore $f(\theta)$ is monotonically increasing for $\theta \in [\pi, 2\pi]$ and $f(\pi) = 0$. Thus we obtain $f(\theta) \geq 0$ for $\theta \in [\pi, 2\pi]$. This completes the proof of Theorem 1.