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NAOSITE: Nagasaki University’s Academic Output SITE
An Inequality on the Exponential Function

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Abstract
In this paper we prove an inequality concerning the exponential function.

1 Introduction and Proof

The purpose of the present paper is to prove the following theorem:

**Theorem 1** For \(0 \leq \varepsilon \leq \pi\) and \(\theta \in \mathbb{R}\),

\[
|1 - e^{\varepsilon e^{i \theta}}| \geq 1 - e^{-\varepsilon}.
\]

**Proof** Since

\[
|1 - e^{\varepsilon e^{i \theta}}|^2 = 1 - 2e^{\varepsilon \cos \theta} \cos(\varepsilon \sin \theta) + e^{2\varepsilon \cos \theta},
\]

it is sufficient to show that

\[
f(\theta) = e^{2\varepsilon \cos \theta} - 2e^{\varepsilon \cos \theta} \cos(\varepsilon \sin \theta) - e^{-2\varepsilon} + 2e^{-\varepsilon} \geq 0
\]

for \(0 < \theta < 2\pi\). A simple calculation shows that

\[
f'(\theta) = 2\varepsilon \sin \theta e^{2\varepsilon \cos \theta} \left( \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon \cos \theta}} - 1 \right).
\]

We put

\[
g(\theta) = \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon \cos \theta}}.
\]

Then

\[
g'(\theta) = -\frac{\varepsilon \sin(\varepsilon \sin \theta) + \varepsilon \sin \theta \cos(\varepsilon \sin \theta)}{\sin^2 \theta e^{\varepsilon \cos \theta}}.
\]

First we assume that \(0 < \theta < \pi\). Put \(t = \varepsilon \sin \theta\) and

\[
h(t) = t \cos t - \sin t.
\]

Then \(0 \leq t \leq \pi\). Since \(h'(t) = -t \sin t \leq 0\) and \(h(0) = 0\), we have \(h(t) \leq 0\) \((0 \leq t \leq \pi)\). Therefore \(g'(\theta) \leq 0\) \((0 < \theta < \pi)\) and

\[
g(0) = \lim_{\theta \to 0} \frac{\sin(\theta + \varepsilon \sin \theta)}{\sin \theta e^{\varepsilon \cos \theta}} = \frac{1 + \varepsilon}{e^\varepsilon} \leq 1.
\]
Hence $g(\theta) \leq 1$ for $0 < \theta < \pi$. Then

$$f'(\theta) = 2\pi e^{2\cos \theta} \sin \theta(g(\theta) - 1) \leq 0.$$  

Therefore, $f(\theta)$ is monotonically decreasing for $\theta \in [0, \pi]$. Since $f(\pi) = 0$, it follows that $f(\theta) \geq 0$ for $\theta \in [0, \pi]$. A similar argument tells us that $g(\theta)$ is monotonically increasing on $[\pi, 2\pi]$, and $g(2\pi) = g(0) \leq 1$, and hence $g(\theta) \leq 1$ for $\theta \in [\pi, 2\pi]$. Therefore $f(\theta)$ is monotonically increasing for $\theta \in [\pi, 2\pi]$ and $f(\pi) = 0$. Thus we obtain $f(\theta) \geq 0$ for $\theta \in [\pi, 2\pi]$. This completes the proof of Theorem 1.