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On a Method of Solution for the Coupled Hill Type Equations and Its Application to the Study of the Stability of Nonlinear Vibrations

by

Kazuo TAKAHASHI* and Kiyokatsu KAWAHARA**

In this paper, a method of the stability analysis for large amplitude steady state response of a nonlinear beam and flat plate under periodic excitation. The nonlinearity is attributed to the membrane tension which is developed when the beam and plate deflections are not small in comparison to their thickness. This problem is analyzed by the application of a Galerkin method, in which the effect of multi-mode participation is considered, and an unspecified function of the time resulting in nonlinear coupled ordinary differential equations of motion is solved by the harmonic balance method and the Newton Raphson method.

The stability question is investigated by studying the behavior of a small perturbation of the steady state response. The perturbation equations of the present method of solution reduce to the coupled Hill type equation. Assuming the solution of the form as a product of characteristic component and a Fourier series which represents the periodicity of motion and application of the harmonic balance method can transform the stability problem into the eigenvalue problem of a nonsymmetric matrix. After a proper transformation, the eigenvalues can be calculated on a digital computer by the QR double step method.

The effectiveness and the accuracy of the proposed method are examined for a Mathieu equation whose stability has been worked out in detail and the application to stability analysis of the nonlinear vibrations of beams are presented.

1. Introduction

Nonlinear vibrations of beams and flat plates have been studied by many authors1-5). The common approach is to assume some form for the spatial solution and then solve the ordinary differential equation that results for time variable. Earlier authors have used a single spatial function in their approach, and there have been very little effort in investigation of this problem using multiple mode solutions. This fact is primarily in the difficulty in solving the coupled nonlinear differential equations. Most coupled problems considered require numerical techniques at some point in the solution. Recently, with the advent of a high speed digital computer, there have been several papers6-10) which discuss the multiple mode problem. The methods of analysis to study the coupled nonlinear ordinary differential equations for time variable are the harmonic balance method6-9) and the method of averaging10). However, the only and reliable

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method that does not lose its usefulness when
the amplitude of vibration becomes large is
the harmonic balance method. This method
involves assuming a solution of summed har-
monics, substituting this solution into the dif-
ferential equations, and equating the coef-
ficients of the harmonics to zero. The resulting
set of nonlinear algebraic equations may be
solved on a digital computer by the Newton
Raphson method. According to the multiple
mode participation and the harmonic balance
method, nonlinear harmonic, higher harmonic,
subharmonic and coupling responses of beams
and flat plates are predicted.

Although a solution has been found to the
steady state problem, there is neither as-
surance that these solutions are unique nor
that they are stable solutions because of the
nonlinearity of the equations. Thus, it will be
necessary to check the stability of these so-
lutions and to investigate the possibility of
further steady state solutions. The stability
question will be investigated by studying the
behavior of a small perturbation of the steady
state response. The perturbation equations re-
duce to the coupled Hill type equations in the
case of the stability problem of the present so-
lution. The stability has received much less
attention, although Benett and Eisley analyzed the
coupled Mathieu equation by a di-
rect numerical application of Floquet theory,
and Busby investigated by means of the
method of averaging. However, as for the
coupled Hill type equations, it seems that there
exists no convenient method.

The present work is concerned with the
stability analysis of steady state response of
multiple degree-of-freedom systems. Assuming
a solution of the form as a product of a char-
acteristic component and a vector which has
period components, expanding this vector into
a Fourier series and substituting the series into
the perturbation equations and using the har-
monic balance method, we can obtain a system
of homogeneous algebraic equations. Then,
an investigation of stability of the trivial
solution now reduces to study the eigenvalue
problem of a nonsymmetric matrix for de-
termining the characteristic exponents. Then,
problem leads to find the condition under which
any of the subtraction of the eigenvalue from
the damping constant have a positive real part
or not. This method directly determines the
stability of the point under consideration.

Numerical results are presented at first for
a undamped and damped Mathieu equation
whose stability has been worked out in detail
for checking the accuracy and compared with
various methods of solution. In addition, the
stability of nonlinear steady state response of
the hinged-hinged beam for single mode ap-
proach and two mode approach is performed
by the present method. The stabilities of the
amplitude of harmonic, higher harmonic and
subharmonic response are investigated.

2. Equations of Motion
(1) A Straight Beam

The equations of motion describing the
transverse vibration of a straight beam, which
axially restrained in moving and large de-
flection is permitted as follows

\[ p = -\frac{EA}{2l} \int_0^l \left( \frac{\partial^2 y}{\partial x^2} \right)^2 dx, \]  

(1)

\[ L(y,P) = E I \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + P \frac{\partial^2 y}{\partial x^2} \]  

(2)

where \( E \) denotes Young's modulus, \( I \) the
moment inertia of the cross section, \( \rho \) the mass
density, \( A \) the cross sectional area, \( \ell \) the beam
length, \( y \) the transverse deflection, \( t \) the time,
\( x \) the axial coordinate, \( c \) the linear viscous
damping coefficient, \( P \) the load intensity of
the external force, and \( \Omega \) the circular frequen-
cy of the external force, and \( P \) the only
nonlinear term due to the effect of the trans-
verse displacement on the axial force caused
by immovable supports. The beam vibrates
with moderately large amplitude but the
curvatures are assumed small and Hooke's
law applied at all times.
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(2) A Circular Plate (with Radial Symmetry)

The nonlinear equations governing large amplitude motions of a circular plate with radial symmetry are taken to be

$$\nabla^4 F = - \frac{E}{r} \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2}, \quad (3)$$

$$L(w, F) = \gamma \nabla^4 w + \rho \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} - \frac{h}{r} \frac{\partial}{\partial r} \left( \frac{\partial F \partial w}{\partial r} \right) - p \cos \Omega t = 0, \quad (4)$$

where $F$ denotes Airy's stress function defined by $\sigma_r = \frac{1}{2} \frac{\partial F}{\partial r}$, $\sigma_\theta = \frac{1}{2} \frac{\partial F}{\partial \theta}$, $\sigma_r$ the radial stress, $\sigma_\theta$ the circumferential stress, $r$ the radial coordinate, $w$ the transverse displacement, $t$ the time, $D = Eh^3 / 12(1 - \nu^2)$ the flexural rigidity, $h$ the plate thickness, $\nu$ Poisson's ratio.

3. Multiple Degree-of-Freedom Approach

A normal mode solution is assumed as follows

$$y = r \sum_{i=1}^{n} X_i(x) T_i(t) \quad \text{for the case of a beam}, \quad (5)$$

$$w = h \sum_{i=1}^{n} R_i(r) T_i(t) \quad \text{for the case of a circular plate},$$

where $r$ is the radius of gyration of the beam, $T_i(t)$ an unknown function of the time, and $X_i(x), R_i(r)$ spatial variables satisfying the geometric boundary conditions of the beam or the circular plate, which denote the associated linear problem. Substituting the second equation of Eq.(5) into Eq.(3), the desired representation of the stress function $F$ can be given by

$$F = - \frac{E}{4} \sum_{i=1}^{n} \sum_{i=1}^{n} N_{ij} \sum_{j=2}^{n} \sum_{j=2}^{n} a_i^j \bar{d}_j \left( \frac{\zeta}{2^i + j - 2} \right)^{N_{ij} \zeta^2}, \quad (6)$$

where $N_{ij} = (i + j - 2)(2i + j - 2 - \nu) / (1 - \nu)$ for the immovable edge, $N_{ij} = i + j - 2$ for the movable edge, $M_{ij} = (i - 1)(2j - 3)(j - 1) / [(i + j - 2)(i + j - 3)]^2$, $\zeta = r/a$.

Applying a Galerkin method to Eq.(2) or (4), the following set of nonlinear ordinary differential equations is obtained

$$\ddot{T}_n + 2\kappa_n a_n \dot{T}_n + \frac{\alpha_n}{T_n} + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \beta_{klm} T_k = 0, \quad (7)$$

where $\alpha_n = (\lambda_n / \lambda_1)^2$, $\beta_n = \int_0^1 X_n d\xi$, $\beta_{klm} = \int_0^1 dX_n dX_l d\xi \int_0^1 \frac{d^2X_m}{d\xi^2} X_n d\xi / (2 \lambda \delta_n)$.

Equation (7) will be necessary to get the solution approximately. The harmonic balance method is used here. For Duffing type of Eq.(7), a solution of the form is assumed as

$$\ddot{w} + \omega^2 w + \beta \dot{w} + \alpha \ddot{w} = \omega \cos \Omega t + \beta \dot{w} + \alpha \ddot{w}, \quad (8)$$

where $\alpha$ and $\beta$ are amplitude components and $\Omega$ the frequency ratio defined by $\Omega / \omega_1$.

Substituting Eq.(8) into Eq.(7) and applying the harmonic balance method, a set of nonlinear algebraic equations will be obtained and solved by the Newton Raphson method on a high speed digital computer for a proper initial guess.

4. Stability Analysis

The stability question will be investigated by studying the behavior for a small perturbation of the steady state response. Let $T_n = T_n + \delta_n$, where $T_n$ is the steady state solution for Eq.(7) and $\delta_n$ is a small perturbation of the n-th mode. Substituting this equation into Eq.(7), and retaining only first order terms, the
Following is obtained

\[ \delta_n + 2 \alpha_n \delta_n + \alpha_n^2 \delta_n + \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{m=1}^{m} \]

\[ \beta_{klm} (T_k T_l T_m \delta_k + T_k T_m \delta_l + T_l T_m \delta_m) = 0. \]

Substituting Eq. (8) into Eq. (9), the perturbation equations will take the form of the matrix notation

\[ E \delta + 2 \mathbf{H} \delta + \mathbf{A} \delta + \sum_{k=1}^{m} (\mathbf{B}_k \cos 2k \omega \tau + \mathbf{C}_k \sin 2k \omega \tau) \delta = 0, \]  

where \( E \) is the consistent mass matrix which reduces to unit matrix in this case. \( \mathbf{H} \) the damping matrix, \( \mathbf{A} \) the linear stiffness matrix, \( \mathbf{B}_k \) and \( \mathbf{C}_k \) the geometric stiffness matrix, \( \delta \) the vector of generalized coordinates, \( \dot{\delta} \) the generalized velocity vector and \( \ddot{\delta} \) the generalized acceleration vector. Eq. (10) is the coupled Hill type equation. Let introduce the matrix transformation \( \delta = e^{-\beta \omega \tau} \delta \) where \( \delta \) is a vector with component which must be determined. Taking into account that \( e^{-\beta \omega \tau} \) is a nonsingular matrix and assuming that the damping matrix is the scalar matrix \( H = h \mathbf{E} \) where is the damping constant which is the same for all forms of vibration, Eq. (10) can be written in the form

\[ \ddot{\delta} + \left( \mathbf{A} - \mathbf{H}^2 + \sum_{k=1}^{m} (\mathbf{B}_k \cos 2k \omega \tau + \mathbf{C}_k \sin 2k \omega \tau) \right) \delta = 0. \]  

We seek the solution of Eq. (11) in the form

\[ \delta = e^{i \omega \tau} \left\{ \frac{1}{2} \mathbf{b}_0 + \sum_{k=1}^{m} (a_k \sin 2k \omega \tau + b_k \cos 2k \omega \tau) \right\}, \]  

where \( a_k \) and \( b_k \) are some vectors not dependent on time.

Substituting the series Eq. (12) in the differential equation (11) and applying the harmonic balance method, we obtain a set of homogeneous algebraic equations

\[ G \mathbf{X} = 0, \]  

where \( G \) is the coefficient matrix, and \( \mathbf{X} \) vector of generalized coordinates.

In order for this system to have solutions other than the trivial ones, the determinant of the coefficient must be equal to zero. Thus, we obtain the equation for determining the characteristic components as

\[ \text{det} (G) = 0. \]  

From the property of the matrix \( G \), Eq. (14) may be decomposed into the following three matrices as

\[ \text{det} (G) = \text{det} (M_0 - \lambda M_1 - \lambda^2 M_2) = 0, \]  

where \( M_0 \), \( M_1 \) and \( M_2 \) are the coefficient matrices of the constant, first and second power of \( \lambda \), respectively.

To obtain the value of \( \lambda \) in Eq. (15), we employ the method which obtains the eigenvalue of the following double size matrix

\[ \begin{bmatrix} \mathbf{E} & \mathbf{X} \\ \mathbf{M}_2^{-1} \mathbf{M}_1 - \mathbf{M}_2^{-1} \mathbf{M}_0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \]  

where \( \mathbf{Y} = \lambda \mathbf{X} \).

As the matrix of Eq. (16) is a generally nonsymmetric matrix with real element, the eigenvalues consist of pairs of complex number. The nonsymmetric matrix is converted to the general Hessenberg form and the eigenvalue is evaluated using the QR double step method. Therefore, the stability is determined point by point for both increasing and decreasing frequency. If any of the subtraction of the eigenvalue from the damping constant have a positive real part and the corresponding basic solution is unbounded as \( \tau \to \infty \) and the solution is unstable. On the other hand, if all of the subtraction of the eigenvalue from the damping constant have negative real part, then \( e^{(\lambda - \mu)\tau} \to 0 \) as \( \tau \to \infty \) and the solution is stable.

5. Stability of the Solution of the Mathieu Equation

For the purpose of examining the accuracy of the present solution, in this section, we carry out such a discussion for the special case of the damped Mathieu equation

\[ \ddot{\delta} + 2 \beta \dot{\delta} + (\alpha + \epsilon \cos \tau) \delta = 0, \]  

where \( \beta = h \sqrt{\alpha} \).

Introducing transformation \( \delta = e^{\omega \tau} \bar{\delta} \) gives

\[ \ddot{\bar{\delta}} + (\alpha - \beta^2 + \epsilon \cos \tau) \bar{\delta} = 0. \]  

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Eq. (12) may be rewritten in this case as follows
\[ \ddot{\bar{y}} = e^{i\tau} \left\{ \frac{1}{2} b_0 + \sum_{k=1}^{\infty} (a_k \sin k \tau + b_k \cos k \tau) \right\}. \]  
(19)

Substituting Eq. (19) into Eq. (18) and equating coefficients of identical \( \sin k \tau, \cos k \tau \), we obtain the following set of recurrence relations for the \( a_k \) and \( b_k \)

\[ \frac{1}{2} (\lambda^2 + a - \beta^2) b_0 + \frac{1}{2} e b_1 = 0, \]

where

\[
G = \begin{bmatrix}
\lambda^2 + a - \beta^2 & \delta & 0 & 0 & 0 & 0 \\
\delta/2 & \lambda^2 + a - \beta^2 - 1 & \delta/2 & 0 & 2 \lambda & 0 \\
0 & \delta/2 & \lambda^2 + a - \beta^2 - 4 & \delta/2 & 4 \lambda & 0 \\
0 & 0 & \delta/2 & \lambda^2 + a - \beta^2 - 9 & 0 & 6 \lambda \\
0 & -2 \lambda & 0 & 0 & \lambda^2 + a - \beta^2 - 1 & \delta/2 \\
0 & 0 & -4 \lambda & 0 & \delta/2 & \lambda^2 + a - \beta^2 - 4 & \delta/2 \\
0 & 0 & 0 & -6 \lambda & 0 & \delta/2 & \lambda^2 + a - \beta^2 - 9 \\
\end{bmatrix}.
\]

Consequently, Eq. (16) has the form by using Eq. (21) as

\[ AZ = \lambda Z, \]
(22)

where

\[
A = \begin{bmatrix}
0 & E \\
M_2^{-1} M_1 & -M_1^{-1} M_0 \\
\end{bmatrix}, \quad Z = \begin{bmatrix} X \ Y \end{bmatrix}^T.
\]

\[
Y = \lambda X,
\]

\[
M_0 = \begin{bmatrix}
\delta/2 & \alpha - \beta^2 - 1 & \delta/2 & 0 & 0 & 0 \\
0 & \delta/2 & \alpha - \beta^2 - 4 & \delta/2 & 0 & 0 \\
0 & 0 & \delta/2 & \alpha - \beta^2 - 9 & 0 & 0 \\
0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 1 & \delta/2 \\
0 & 0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 4 \\
0 & 0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 9 \\
0 & 0 & 0 & 0 & 0 & \delta/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

\[
M_1 = \begin{bmatrix}
\delta/2 & \alpha - \beta^2 - 1 & \delta/2 & 0 & 0 & 0 \\
0 & \delta/2 & \alpha - \beta^2 - 4 & \delta/2 & 0 & 0 \\
0 & 0 & \delta/2 & \alpha - \beta^2 - 9 & 0 & 0 \\
0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 1 & \delta/2 \\
0 & 0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 4 \\
0 & 0 & 0 & 0 & \delta/2 & \alpha - \beta^2 - 9 \\
0 & 0 & 0 & 0 & 0 & \delta/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
M_2 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}.
\]
The stability regions of the Mathieu equation will be determined by using Eq.(22). We show the undamped and damped ($h=0.1$) stability region for a Mathieu equation in Figs. 1 and 2, respectively, in which the shaded region are the stable regions. In these figures, the thicker solid lines show the two term solution $(b_0, b_1, a_1)$ and thinner lines show the three term solution $(b_0, b_1, b_2, a_1, a_2)$. In Fig.1, dotted lines show the stability boundaries which are determined from the condition under which the differential equation has periodic systems with $2\pi$ or $4\pi$ according to the Floquet theory. Boundaries of the principal region of the stability marked by $S_{1/2}$ and $C_{1/2}$ have periodic solutions with period $4\pi$ and the boundaries of the second region of the stability marked by $S_1$ and $C_1$ have periodic solution with period $2\pi$. As these results obtained by the Floquet theory are converged values, we can estimate these results as the exact solution for comparison of the present solution with the established solution.

Approximate region of the stability of the Mathieu equation for small $\epsilon$ obtained by the use of the perturbation method are also shown in Fig.1.

As can be seen in Fig.1 in the case of the undamped case, the difference between the two term solution and the three term solution of the present solution does not noted for not too large values of $\epsilon$ and a certain difference is coming out with increase of $\epsilon$. However, more than four term solution almost coincides with the three term solution. Therefore, the three term solution will be assumed to be the converged value.

The two term solution of the stability boundary $S_1$ is equal to unity independently of the magnitude of $a$. But, as the three term solution agrees with the exact solution, the stability boundary $S_1$ of the present method is omitted from Fig.1. If we compare the present solution with the exact solution, it will be seen that the present solution agrees well with the exact solution as to the stability boundaries $S_{1/2}$ and $S_1$ and do not agree well as to the stability boundaries $C_{1/2}$ and $C_1$ in the range where $\epsilon$ is not small. The result of this method overestimates the area of the stability range. However, the present solution gives a satisfactory results compared with the first term solution of the perturbation method which is usually used in the stability analysis of a nonlinear system.

As to the damped Mathieu equation as shown in Fig.2, the present solution is assumed to give a similar result as well as undamped case. In this case, the stability boundary corresponding to $C_{1/2}$ is not affected by the
number of the term.

6. Stability of the Harmonic, Higher Harmonic and Subharmonic Response of the Nonlinear System

(1) Single Degree-of-Freedom System

For the case of the symmetric vibration about the center of a hinged-hinged beam, the equation for the first mode can be obtained from Eq.(7) in the following form

\[ T + 2h^2 T + T + 0.25 T^3 = \frac{4}{\pi} \frac{\bar{\omega}}{\omega} \cos \frac{\omega}{\pi} \tau. \]  (23)

Employing the first two harmonics as

\[ T = a_1 \cos \omega \tau + a_3 \cos 3 \omega \tau + b_1 \sin \omega \tau + b_3 \sin 3 \omega \tau. \]  (24)

Substituting Eq.(24) into Eq.(23) and using the harmonic balance method, the following set of four nonlinear algebraic equations is obtained as

\[ (1 - \bar{\omega}^2) a_1 + 2h \bar{\omega} b_1 + \frac{3}{16} (a_1^2 + a_3^2 + 2a_1 a_3 + 2a_1^2) \]
\[ + a_1 b_1^2 - a_3 b_3^2 + 2a_1 b_3^2 + 2a_3 b_1 b_3) = \frac{4}{\pi} \frac{\bar{\omega}}{\omega}, \]
\[ (1 - 9 \bar{\omega}^2) a_3 + 6h \bar{\omega} b_3 + \frac{1}{16} (a_3^2 + 3a_1^2 + 6a_3 a_1) \]
\[ - 3a_1 b_1^2 + 3a_3 b_3^2 + 6a_3 b_1 b_3) = 0, \]
\[ (1 - \bar{\omega}^2) b_1 - 2h \bar{\omega} b_1 + \frac{3}{16} (b_1^2 - b_3^2 b_1 + 2b_3 b_1^2 \]
\[ + 2a_1 b_3^2 + a_3 b_1^2 + a_3 b_3^2 - 2a_3 b_1 b_3) = 0, \]  (25)
\[ (1 - 9 \bar{\omega}^2) b_3 - 6h \bar{\omega} a_3 + \frac{1}{16} (-b_1^2 + 3b_3^2 + \]
\[ 6b_1^2 b_3 + 3a_1^2 b_3 + 6a_1^2 b_3 + 3a_3^2 b_3) = 0. \]

Fig. 3 shows the undamped overall solution \((h = 0)\) of Eq.(25) near the first natural frequency range in the rectangular coordinate graph, where the abscissa indicates the frequency ratio \(\bar{\omega}\) and the ordinate indicates the absolute value of the amplitude, i.e., \(A = |a_1 + a_3|\) which occurs at \(\bar{\omega} = n \pi (n = 0, 1, 2, \cdots)\). In Fig.3, the solid lines correspond to amplitude in phase with the external force and the broken lines correspond to amplitude out of phase with external force. The amplitudes marked by \(\bigcirc\) correspond to stable points and the amplitude marked by \(\times\) correspond to unstable points.

Resonance which occurs in the neighborhood of \(\bar{\omega} = 1.0\) presents a harmonic solution. As shown in Fig.3, the amplitude transmits from a stable point to an unstable point where points at which the resonance curves have a vertical tangent \((\bar{\omega} = 1.42\) when \(\beta = 70\) and \(\bar{\omega} = 1.28\) when \(\beta = 30\)) in the \(A, \bar{\omega}\) plane. The locus of the vertical tangents of the response curves
in the $\Omega, \omega$ plane maps on the boundary $C_{\gamma_2}$ between a stable and unstable region of the $\alpha, \epsilon$ plane as shown in Fig.1. These results coincide with the foregoing conclusion based on physical ground and the accuracy of the present method seems to be satisfactory.

The resonance which occurs in the neighborhood of $\omega=0.3$ shows the higher harmonic whose frequency is three times as many as the frequency of the external force. This response occurs continuously with the increase or decrease of the frequency. The amplitudes in phase with the external force transmit from the stable point to the unstable point where the response curves have vertical tangents. The study of the stability of this higher harmonic is obtained by using the three term solution of the harmonic balance method. According to the two term solution, all the amplitudes are stable. This means that the variational equation which is calculated by the amplitude components requires to present precisely the variational equation of the nonlinear differential equation.

The unstable amplitude of the harmonic response in the neighborhood of $\omega = 0.6$ corresponds to the second instability region of the Mathieu equation. As the unstable region is considerably narrow, this amplitude becomes stable when damping term is taken into account.

The resonance which results through bifurcation from the harmonic response in the neighborhood of $\omega = 3.0$ is the subharmonic response of order $1/3$. The amplitude is defined by $A = |a_{1/3} + a_1|$ where we assume the solution $T = a_1 \cos \omega T + a_{1/3} \cos \omega T / 3$. The amplitude out of phase with the external force is stable and the amplitude in phase with the external force is unstable.

Fig. 4 Amplitude frequency curves of a hinged-hinged beam (single degree-of-freedom, damped case)

Fig. 4 shows the frequency response curve when the viscous damping term is included. We use the instantaneous maximum value of the amplitude as the definition of the amplitude in this case. The damping constant $h$ is taken to 5% in the case of the harmonic and higher harmonic response, and 0.5% in the case of the subharmonic response.

The width of the higher harmonic response near $\omega = 0.3$ is narrow as shown in Fig.4, the amplitude becomes considerably small. As damping is added, the response curve of the
harmonic motion in the neighborhood of $\omega = 1.0$ is round off in the vicinity of the curve for $\rho = 0$ at $A = 4.3$, and the response curve is bented to the right. There are two vertical tangents as shown in Fig.4. The curve which is surrounded by these two vertical tangents seems to be unstable. The present solution for the lower transmitting point from stable amplitude to unstable one agrees with the lower point of the vertical tangent. However the result of the stability analysis in the neighborhood of the upper vertical tangent does not coincide with the point of the vertical tangent when the damping constant is smaller than 10%. The accuracy of the stability analysis is not improved even if we adopt further terms of the harmonic balance method. This reason may be caused by the fact that the stability boundary corresponding to $C_{12}$ is not affected by the numbers of the term as shown in Fig.2.

The subharmonic response with damping is also presented in Fig.4. In this case, the response curves have the upper and lower round off points. The region where the subharmonic response occurs is considerably limited.

(2) Two Degree-of-Freedom System

If more than one mode is considered in the analysis, then a coupled set of nonlinear differential equations results. For example, the following set of equations is given for two symmetric modes of the hinged-hinged beam

$$
\dot{T}_1 + 2hT_1 + T_1 + 0.25(T_1^2 + 9T_1 T_2) = \frac{4}{3}\pi b\cos \omega \tau, \tag{26}
$$

$$
\dot{T}_3 + 2hT_3 + 81T_3 + 2.25(T_1^2 T_3 + 9T_3^2) = \frac{4}{3}\pi b\cos \omega \tau. \tag{27}
$$

To apply the harmonic balance method, let

$$
T_1 = a_1 \cos \omega \tau + b_1 \sin \omega \tau, \tag{28}
$$

$$
T_3 = a_3 \cos \omega \tau + b_3 \sin \omega \tau \tag{29}.
$$

Substituting Eq.(27) into Eq.(26) and taking only the first terms of the perturbation yields

$$
E\ddot{T} + 2H\dot{T} + A\ddot{T} + (B + C \cos 2\omega \tau +
D\sin 2\omega \tau)T = 0. \tag{30}
$$

where

$$
E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 81 \end{bmatrix},
$$

$$
B = \begin{bmatrix} q_0 & 0 \\ 0 & q_0 \end{bmatrix}, \quad C = \begin{bmatrix} q_2 & q_1 \\ q_1 & q_2 \end{bmatrix}, \quad D = \begin{bmatrix} q_5 & q_4 \\ q_4 & q_5 \end{bmatrix},
$$

$$
\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \quad \rho = (0, 0) \tag{31}.
$$

Let introduce $\tilde{\delta} = e^{-i\alpha T} \delta$, then we obtain

$$
E\ddot{\delta} + (A + B - H^2 + C \cos 2\omega \tau +
D\sin 2\omega \tau)\delta = 0. \tag{32}
$$

We assume

$$
\tilde{\delta} = e^{i\omega}(\frac{1}{2}b_0 + \sum_{k=2,3} a_k \sin 2k\omega \tau +
\tilde{b}_k \cos 2k\omega \tau), \tag{33}
$$

where $b_0 = (b_0^1, b_0^2)^T$, $b_k = (b_k^1, b_k^2)^T$, $a_k = (a_k^1, a_k^2)^T$.

Substituting Eq.(33) into Eq.(32) and equating the coefficients of $\cos 2\omega \tau$ and $\sin 2\omega \tau$, we obtain a set of homogeneous algebraic equations

$$
\begin{bmatrix}
A + B - H^2 + C \bigg(2(\omega + \lambda)^2 - 4\omega^2\bigg) \\
2(\omega + \lambda)^2 - 4\omega^2
\end{bmatrix} \begin{bmatrix}
b_0 \\
b_1
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}. \tag{34}
$$

This results in an eigenvalue problem whose determinant is required to vanish.

Amplitude frequency curves of the hinged-hinged beam for the first and the second modes are shown in Fig.5. These response curves are obtained by using damping constants equal to 0% and 5%.
from the stability analysis, it will be seen that a similar result is obtained in the cases of the first and second natural frequency region as well as the result of the single degree-of-freedom system. Consequently, the present method of stability analysis can be easily applied to multiple degree-of-freedom systems and can be treated as well as single degree-of-freedom system.

7. Conclusions
The results of the numerical examples indicate that the present method of solution for the stability question gives a excellent approximate solution for a Mathieu equation and can be applied to investigate the stability analysis of steady state response of nonlinear vibrations.

The method only analyzes one point on the response curve at a time so it is necessary to determine the stability point of several response in order to construct regions of instability. Therefore, in the problem in which the nonlinear coupling is weak, it is possible to use the instability region in the Mathieu diagram maps or to obtain the position of the vertical tangent when the amplitude frequency curve is obtained. However, in the case when the nonlinear coupling appears or when the amplitude frequency curve can be not obtained easily, the proposed method of solution will be useful and play a significant role.

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