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On isoperimetric problem in a complex plane

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Abstract
Classical isoperimetric inequality is shown in a complex plane. In a complex plane we can use effectively the complex Fourier expansion in the computations.

0 Introduction: isoperimetric problem and isoperimetric inequality in a plane

Let $C$ be a simply closed curve in a plane and $D$ be the domain enclosed by $C$. Let $l$ be the length of $C$ and $A$ be the area of $D$. Then the isoperimetric inequality is

$$A \leq \frac{l^2}{4\pi}.$$

The classical isoperimetric problem claims that for every simply closed curve $C$ in a plane the isoperimetric inequality holds and that its equality holds if and only if $C$ is a circle of radius $l/2\pi$.

Since the radius of a circle which has the length $l$ is $r = l/2\pi$ the circle has area $\pi(l/2\pi)^2 = l^2/4\pi$. The isoperimetric inequality thus shows that among all simply closed curves of length $l$, circles of radius $l/2\pi$ have the largest area $l^2/4\pi$ and the equality condition shows that the largest area is attained only by those circles.

We show the claim of isoperimetric problem in a complex plane $C$. The proof gets through along the classical line [1, 4]. The use of the complex Fourier series in a complex plane makes the reasoning a little straightforward.
1 A closed curve in $\mathbb{C}$ and its Fourier expansion

By similitude it suffices to consider curves of length $l = 2\pi$ and to show the isoperimetric inequality: $A \leq \pi$. Let $C$ be a simply closed curve of length $2\pi$ in a complex plane $\mathbb{C}$. We assume that $C$ is piecewise smooth and is parametrized by its arc length. Let

$$C : z(s) = x(s) + iy(s), \quad 0 \leq s \leq 2\pi, \quad z(0) = z(2\pi)$$

be the parametrization of a closed curve $z : [0, 2\pi] \rightarrow \mathbb{C}$. Then the tangent vector at $z(s)$ is $z'(s) = x'(s) + iy'(s)$. When the curve is parametrized by its arc length $s$, the length of the tangent vector is one: $|z'(s)| = 1$ (except finite points). And the total length of $C$ is

$$2\pi = \int_C |z'(s)| \, ds = \int_0^{2\pi} |z'(s)| \, ds.$$ 

Expand $z(s)$ into the complex Fourier series:

$$z(s) = \sum_{n \in \mathbb{Z}} c_n e^{ins}, \quad c_n = \int_0^{2\pi} z(s) e^{-ins} \frac{ds}{2\pi}.$$ 

By the term-by-term differentiation

$$z'(s) = \sum_{n=-\infty}^{\infty} ic_ne^{ins}.$$ 

The condition: $1 = |z'(s)|^2 = z'(s) \overline{z'(s)}$ thereby becomes

$$1 = \sum_{n=-\infty}^{\infty} ic_ne^{ins} \sum_{m=-\infty}^{\infty} -ic_me^{-ims} = \sum_{n,m=-\infty}^{\infty} c_n \overline{c_m} me^{i(n-m)s}.$$ 

Integrating $\int_0^{2\pi} * ds / 2\pi$ term by term, the only terms: $n = m$ remain,

$$1 = \sum_{n=-\infty}^{\infty} c_n \overline{c_n} n^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 n^2$$

(1)

since $\int_0^{2\pi} e^{i(n-m)s} ds / 2\pi = \delta_{nm}$. This is the condition of the curve length $l = 2\pi$. 


2 The Green formula and isoperimetric inequality

Let $D$ be a bounded domain in a plane with piecewise smooth boundary $\partial D$. Let $P(x, y)$ and $Q(x, y)$ be $C^1$-functions near $\bar{D}$. Then the Green formula is:

$$\iint_D \left( \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) \, dx \, dy = \oint_{\partial D} P \, dy + Q \, dx.$$  

The formula states a basic relation between integration in a region and integration over its boundary in a plane. So put $P = x, Q = -y$. Then if $A = \text{area}(D), \quad 2A = \oint_{\partial D} x \, dy - y \, dx.$

In $\mathbb{C}$ we have $xdy - ydx = (\bar{z}dz - zd\bar{z})/2i = \text{Im}(\bar{z}dz)$ ($dz = dx + idy, \ d\bar{z} = dx - idy$).

For a curve $C : z = z(s) \ (0 \leq s \leq 2\pi)$ and its enclosed region $D$ in $\mathbb{C}$ we have

$$2A = \text{Im} \oint_C \bar{z}dz = \text{Im} \int_0^{2\pi} \bar{z}(s)z'(s)ds.$$  

We calculate quantity $A/\pi = 2A/2\pi$.

$$\frac{1}{2\pi} \int_0^{2\pi} \bar{z}(s)z'(s)ds = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{-ins} \sum_{m=-\infty}^{\infty} ic_m m e^{ims} \frac{ds}{2\pi}$$

$$= i \sum_{n,m=-\infty}^{\infty} c_n c_m m \int_0^{2\pi} e^{i(m-n)s} \frac{ds}{2\pi} = i \sum_{n=-\infty}^{\infty} |c_n|^2 n.$$  

Hence we get

$$\frac{A}{\pi} = \frac{2A}{2\pi} = \frac{1}{2\pi} \text{Im} \int_0^{2\pi} \bar{z}(s)z'(s)ds = \sum_{n=-\infty}^{\infty} |c_n|^2 n.$$  

(2)

Subtract (2) from (1) we have

$$1 - \frac{A}{\pi} = \sum_{n=-\infty}^{\infty} |c_n|^2 n^2 - \sum_{n=-\infty}^{\infty} |c_n|^2 n = \sum_{n=-\infty}^{\infty} |c_n|^2 (n^2 - n)$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \left\{ \left( n - \frac{1}{2} \right)^2 - \frac{1}{4} \right\} \geq 0.$$
Hence we get
\[ A\pi = 2 \quad A^{2}\pi = 1 \]

\[
\int_{\pi}^{2\pi} z(s) z'(s) ds = \infty \sum_{n=-\infty}^{\infty} |c_n|^2 \]

(2)

Subtract (2) from (1) we have
\[
1 - A\pi = \infty \sum_{n=-\infty}^{\infty} |c_n|^2 - \infty \sum_{n=-\infty}^{\infty} |c_n|^2 = \infty \sum_{n=-\infty}^{\infty} |c_n|^2 \]

\[
\{ (n - 1/2)^2 - 1/4 \} \geq 0 \]

since \( n \in \mathbb{Z} \). This proves the isoperimetric inequality for the curve \( C \).

Because \( n^2 - n = n(n - 1) = 0 \) iff \( n = 0, 1 \), the equality above holds if and only if all \( c_n = 0 \) except \( n = 0, 1 \). In the case in which the equality holds the condition (1) becomes \( 1 = |c_1|^2 \) and the Fourier expansion of \( z(s) \) has the only two non-zero terms:

\[ z(s) = c_0 + c_1 e^{is}, \quad (0 \leq s \leq 2\pi). \]

Since \( |c_1| = 1 \) this is exactly the parametrization of a circle of radius one and of center \( c_0 \) in the complex plane \( \mathbb{C} \).

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References


