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PERFECT ONE-FACTORIZATIONS OF
THE COMPLETE GRAPH

MIDORI KOBAYASHI

1. Introduction

We denote by $K_{2n}=(V, E)$ the complete graph with $2n$ vertices, where $V$ is the set of $2n$ vertices and $E$ is the set of $n(2n-1)$ edges. A 1-factor of $K_{2n}$ is a set of pairwise disjoint edges that partition the set of vertices $V$. A 1-factorization of $K_{2n}$ is a set of 1-factors that partition the set of edges $E$. A 1-factorization is called perfect if the union of every pair of distinct 1-factors is a Hamiltonian circuit. Two 1-factorizations $F$ and $F'$ are isomorphic if there exists a permutation of $V$ which sends each member of $F$ into a member of $F'$.

The existence of a perfect 1-factorization of the complete graph $K_{2n}$ for all $n \geq 2$ is conjectured, and the problem is settled only for $2n=p+1, 2p$ ($p$ is prime), and $2n=16, 28, 244, 344$. In this paper, these perfect 1-factorizations are explicitly shown.

Perfect 1-factorizations of $K_{36}, K_{1332}$ and $K_{6860}$ have recently been found ([4, 5]). The papers are in submission.

2. Perfect 1-factorization of $K_{p+1}$ ($p$ is an odd prime)

$GF(p)$ denotes the Galois field with $p$ elements. We put

$$V = GF(p) \cup \{\infty\}$$

and

$$F_0 = \{\{i, j\} \mid i + j = 0, i, j \in GF(p) - \{0\}\} \cup \{(0, \infty)\}$$

$F_0$ is called a starter 1-factor and

$$F_\sigma = F_0 + g$$

$$= \{\{i + g, j + g\} \mid i + j = 0, i, j \in GF(p) - \{0\}\} \cup \{(g, \infty)\}.$$ is an induced 1-factor, where $g$ is an element of $GF(p)$. We obtain a perfect 1-factorization $GK_{p+1}.$
For example, a starter of $GK_{12}$ is shown in FIGURE 1.

![Diagram of $GK_{12}$]

3. Perfect 1-factorization of $K_{2p}$ ($p$ is an odd prime)

Let

$$V = \{ w_0, w_1, \ldots, w_{p+1}, w_0^*, w_1^*, \ldots, w_{p-1}^* \}.$$ 

For mathematical simplicity, we use $w_{i+kp}$ and $w_{i+kp}^*$ instead of $w_i$ and $w_i^*$ respectively, where $k$ is an integer.

For an integer $s$ with $0 \leq s \leq p-1$, we put

$$OG_s = \left\{ \{w_i, w_j\} \mid i+j = s, \ i \equiv j \pmod{p} \right\}$$

$$\cup \left\{ \{w_i^*, w_j^*\} \mid i+j = p-2-s, \ i \equiv j \pmod{p} \right\}$$

$$\cup \left\{ \{w_{is/2}, w_{(p-2-s)/2}^*\} \right\},$$

where $1/2$ means $2^{-1} \pmod{p}$. For an integer $s$ with $0 \leq s \leq p-2$, we put

$$IG_s = \left\{ \{w_i, w_j^*\} \mid i+j \equiv s \pmod{p} \right\}.$$
Then
\[ GA_{2p} = \left\{ O_{G_s} \mid s = 0, 1, \ldots, p - 1 \right\} \cup \left\{ I_{G_s} \mid s = 0, 1, \ldots, p - 2 \right\} \]
is a perfect 1-factorization of \( K_{2p}([2]) \). For example, \( GA_{10} \) is shown in FIGURE 2.

![Diagram](image)

**FIGURE 2**

Let
\[ V' = \left\{ v_0, v_1, \ldots, v_{2p-1} \right\}, \]
\[ E' = \left\{ \{v_i, v_j\} \mid 0 \leq i \leq 2p - 1, 0 \leq j \leq 2p - 1, i \neq j \right\}. \]

For mathematical simplicity, we use \( v_{i+2pk} \) instead of \( v_i \), where \( k \) is an integer.

For any integer \( s \) with \( 0 \leq s \leq 2p - 1 \) and \( s \neq p \), we define \( G_s(\subseteq E) \) as follows:

If \( s \) is even, then
\[ G_s = \left\{ \{v_i, v_j\} \mid i + j \equiv s, i \equiv j \pmod{2p} \right\} \cup \left\{ v_{si2}, v_{si2+p} \right\} \]
If \( s \) is odd and \( s \neq p \), then
\[ G_s = \left\{ \{v_i, v_j\} \mid i: \text{odd}, i - j \equiv s \pmod{2p} \right\}. \]

\( G_s \) is a 1-factor of \( K_{2p} \) and the set of \( G_s \) denoted by
\[ GN_{2p} = \left\{ G_s \mid 0 \leq s \leq 2p - 1, s \neq p \right\} \]
is a perfect 1-factorization of \( K_{2p}([7]) \). For example, \( GN_{10} \) is shown in FIGURE 3.
GA_{2p} and GN_{2p} are isomorphic perfect 1-factorizations (\((3)\)).

4. Perfect 1-factorization of \(K_{16}\)

A 1-factorization \(F\) is called factor-1-rotational if \(F\) has an automorphism fixing two vertices (and one 1-factor), and permuting the remaining \(2n-2\) vertices (and \(2n-2\) 1-factors) in a single cycle. It has a convenient geometric representation. One takes the vertices of the regular polygon with \(2n-2\) vertices and labels them with elements of \(\mathbb{Z}_{2n-2}\); the other two vertices is labeled with \(\infty_1, \infty_2\), where \(\mathbb{Z}_{2n-2}\) denotes the residue class group modulo \(2n-2\). Let \(F_1\) be a starter 1-factor of a factor-1-rotational 1-factorization \(F\). The \(2n-2\) 1-factors are obtained by rotating the figure successively through an angle \(2\pi/(2n-2)\). \(F\) consists of these \(2n-2\) 1-factors and the fixed 1-factor \(F^*\):

\[
F^* = \{ \{i, j\} \mid i - j \equiv n - 1 \pmod{2n-2} \} \cup \{ \{\infty_1, \infty_2\} \}
\]

A starter 1-factor of a factor-1-rotational, perfect 1-factorization of \(K_{16}\) is shown in FIGURE 4.
5. Perfect 1-factorizations of $K_{2n}$ for $2n = 28, 244, 344$

Let $p$ be a prime number and $m$ be a natural number such that $p^m \equiv 3 \pmod{4}$. We put $q = p^m$, $s = (q - 1)/2$ and $2n = q + 1$. $GF(q)$ denotes the Galois field with $q$ elements. $K_{2n} = (V, E)$ denotes the complete graph with $2n$ vertices, and

$$V = GF(q) \cup \{\infty\}.$$ 

Let $\omega$ be a primitive element of $GF(q)$. We define a starter 1-factor $F_0$:

$$F_0 = \left\{ \{ \omega^{2i}, \omega^{2i+1} \mid i = 0, 1, 2, \ldots, s - 1 \} \cup \{0, \infty\} \right\}$$

For any $g \in GF(q)$,

$$F_g = F_0 + g$$

$$= \left\{ \{ \omega^{2i} + g, \omega^{2i+1} + g \mid i = 0, 1, 2, \ldots, s - 1 \} \cup \{g, \infty\} \right\}$$

is a 1-factor which is induced by the starter $F_0$. Then

$$F(\omega) = \left\{ F_g \mid g \in GF(q) \right\}$$

is a 1-factorization of $K_{2n}$. It is proved that $F(\omega)$ is semi-regular (1). By suitable selections of the semi-regulars, we may construct perfect 1-factorizations.
In case $p = 3$ and $m = 3$, let $\omega$ be a primitive element of $GF(3^3)$ with a minimal polynomial $x^3 + 2x^2 + 1$. Then $F(\omega)$ is a perfect 1-factorization of $K_{28}$.

In case $p = 3$ and $m = 5$, let $\omega$ be a primitive element of $GF(3^5)$ with an minimal polynomial $x^5 + x^4 + x^2 + 1$. Then $F(\omega^5)$ is a perfect 1-factorization of $K_{244}$.

In case $p = 7$ and $m = 3$, let $\omega$ be a primitive element of $GF(7^3)$ with a minimal polynomial $x^3 + x^2 + x + 2$. Then $F(\omega^{37})$ is a perfect 1-factorization of $K_{344}$.

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**REFERENCES**


