<table>
<thead>
<tr>
<th>Title</th>
<th>PERFECT ONE-FACTORIZATIONS OF THE COMPLETE GRAPH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kobayashi, Midori</td>
</tr>
<tr>
<td>Citation</td>
<td>経済学部研究年報, 4, pp.85-90; 1988</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1988-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/26123">http://hdl.handle.net/10069/26123</a></td>
</tr>
</tbody>
</table>

NAOSITE: Nagasaki University’s Academic Output SITE

http://naosite.lb.nagasaki-u.ac.jp
PERFECT ONE-FACTORIZATIONS OF
THE COMPLETE GRAPH

MIDORI KOBAYASHI

1. Introduction

We denote by $K_{2n}=(V, E)$ the complete graph with $2n$ vertices, where $V$ is the set of $2n$ vertices and $E$ is the set of $n(2n-1)$ edges. A 1-factor of $K_{2n}$ is a set of pairwise disjoint edges that partition the set of vertices $V$. A 1-factorization of $K_{2n}$ is a set of 1-factors that partition the set of edges $E$. A 1-factorization is called perfect if the union of every pair of distinct 1-factors is a Hamiltonian circuit. Two 1-factorizations $F$ and $F'$ are isomorphic if there exists a permutation of $V$ which sends each member of $F$ into a member of $F'$.

The existence of a perfect 1-factorization of the complete graph $K_{2n}$ for all $n \geq 2$ is conjectured, and the problem is settled only for $2n=p+1, 2p$ ($p$ is prime), and $2n=16, 28, 244, 344$. In this paper, these perfect 1-factorizations are explicitly shown.

Perfect 1-factorizations of $K_{36}, K_{1332}$ and $K_{6860}$ have recently been found ([4, 5]). The papers are in submission.

2. Perfect 1-factorization of $K_{p+1}$ ($p$ is an odd prime)

$GF(p)$ denotes the Galois field with $p$ elements. We put

$$V = GF(p) \cup \{\infty\}$$

and

$$F_0 = \left\{(i,j) \mid i+j=0, \ i,j \in GF(p) - \{0\}\right\} \cup \{(0,\infty)\}$$

$F_0$ is called a starter 1-factor and

$$F_g = F_0 + g$$

$$= \left\{(i+g,j+g) \mid i+j=0, \ i,j \in GF(p) - \{0\}\right\} \cup \{(g,\infty)\}.$$ 

is an induced 1-factor, where $g$ is an element of $GF(p)$. We obtain a perfect 1-factorization $GK_{p+1}$.
\[ GK_{p+1} = \left\{ F_g \middle| g \in GF(p) \right\}. \]

For example, a starter of \( GK_{12} \) is shown in FIGURE 1.

3. Perfect 1-factorization of \( K_{2p} \) (\( p \) is an odd prime)

Let

\[ V = \{ w_0, w_1, \ldots, w_{p+1}, w_0^*, w_1^*, \ldots, w_{p-1}^* \}. \]

For mathematical simplicity, we use \( w_{i+kp} \) and \( w_{i+kp}^* \) instead of \( w_i \) and \( w_i^* \), respectively, where \( k \) is an integer.

For an integer \( s \) with \( 0 \leq s \leq p-1 \), we put

\[
OG_s = \left\{ \{ w_i, w_j \} \middle| i + j = s, \ i \equiv j \ (\text{mod } p) \right\}
\]

\[
\cup \left\{ \{ w_i^*, w_j^* \} \middle| i + j = p - 2 - s, \ i \equiv j \ (\text{mod } p) \right\}
\]

\[
\cup \left\{ \{ w_{s/2}, w_{(p-2-s)/2}^* \} \right\},
\]

where \( 1/2 \) means \( 2^{-1} \ (\text{mod } p) \). For an integer \( s \) with \( 0 \leq s \leq p-2 \), we put

\[
IG_s = \left\{ \{ w_i, w_j^* \} \middle| i + j = s \ (\text{mod } p) \right\}.
\]
Then
\[ GA_{2p} = \left\{ OG_s \mid s = 0, 1, \ldots, p - 1 \right\} \cup \left\{ IG_s \mid s = 0, 1, \ldots, p - 2 \right\} \]
is a perfect 1-factorization of \( K_{2p} ([2]) \). For example, \( GA_{10} \) is shown in FIGURE 2.

Let
\[ V' = \left\{ v_0, v_1, \ldots, v_{2p-1} \right\}, \]
\[ E' = \left\{ \{ v_i, v_j \} \mid 0 \leq i \leq 2p - 1, 0 \leq j \leq 2p - 1, i \neq j \right\}. \]
For mathematical simplicity, we use \( v_{i+2pk} \) instead of \( v_i \), where \( k \) is an integer.

For any integer \( s \) with \( 0 \leq s \leq 2p - 1 \) and \( s \neq p \), we define \( G_s (\subseteq E) \) as follows:
If \( s \) is even, then
\[ G_s = \left\{ \{ v_i, v_j \} \mid i + j \equiv s, i \equiv j \ (\text{mod} \ 2p) \right\} \cup \left\{ \{ v_{si2}, v_{si2+p} \} \right\} \]
If \( s \) is odd and \( s \neq p \), then
\[ G_s = \left\{ \{ v_i, v_j \} \mid i: \text{odd}, i - j \equiv s \ (\text{mod} \ 2p) \right\}. \]
\( G_s \) is a 1-factor of \( K_{2p} \) and the set of \( G_s \) denoted by
\[ GN_{2p} = \left\{ G_s \mid 0 \leq s \leq 2p - 1, s \neq p \right\} \]
is a perfect 1-factorization of \( K_{2p} ([7]) \). For example, \( GN_{10} \) is shown in FIGURE 3.
GA_{2p} and GN_{2p} are isomorphic perfect 1-factorizations ([3]).

4. Perfect 1-factorization of K_{16}

A 1-factorization $F$ is called factor-1-rotational if $F$ has an automorphism fixing two vertices (and one 1-factor), and permuting the remaining $2n-2$ vertices (and $2n-2$ 1-factors) in a single cycle. It has a convenient geometric representation. One takes the vertices of the regular polygon with $2n-2$ vertices and labels them with elements of $Z_{2n-2}$; the other two vertices is labeled with $\infty_1$, $\infty_2$, where $Z_{2n-2}$ denotes the residue class group modulo $2n-2$. Let $F_1$ be a starter 1-factor of a factor-1-rotational 1-factorization $F$. The $2n-2$ 1-factors are obtained by rotating the figure successively through an angle $2\pi/(2n-2)$. $F$ consists of these $2n-2$ 1-factors and the fixed 1-factor $F^*$:

$$F^* = \left\{ (i, j) \mid i - j \equiv n - 1 \pmod{2n-2} \right\} \cup \left\{ (\infty_1, \infty_2) \right\}$$

A starter 1-factor of a factor-1-rotational, perfect 1-factorization of $K_{16}$ is shown in FIGURE 4.
5. Perfect 1-factorizations of $K_{2n}$ for $2n = 28, 244, 344$

Let $p$ be a prime number and $m$ be a natural number such that $p^m \equiv 3 \pmod{4}$. We put $q = p^m$, $s = (q - 1)/2$ and $2n = q + 1$. $GF(q)$ denotes the Galois field with $q$ elements. $K_{2n} = (V, E)$ denotes the complete graph with $2n$ vertices, and

$$V = GF(q) \cup \{\infty\}.$$  

Let $\omega$ be a primitive element of $GF(q)$. We define a starter 1-factor $F_0$:

$$F_0 = \left\{ \{\omega^{2i}, \omega^{2i+1}\} \mid i = 0, 1, 2, \ldots, s - 1 \right\} \cup \{\{0, \infty\}\}$$

For any $g \in GF(q)$,

$$F_g = F_0 + g = \left\{ \{\omega^{2i} + g, \omega^{2i+1} + g\} \mid i = 0, 1, 2, \ldots, s - 1 \right\} \cup \{\{g, \infty\}\}$$

is a 1-factor which is induced by the starter $F_0$. Then

$$F(\omega) = \left\{ F_g \mid g \in GF(q) \right\}$$

is a 1-factorization of $K_{2n}$. It is proved that $F(\omega)$ is semi-regular ([1]). By suitable selections of the semi-regulars, we may construct perfect 1-factorizations.
In case $p=3$ and $m=3$, let $\omega$ be a primitive element of $GF(3^3)$ with a minimal polynomial $x^3 + 2x^2 + 1$. Then $F(\omega)$ is a perfect 1-factorization of $K_{28}$.

In case $p=3$ and $m=5$, let $\omega$ be a primitive element of $GF(3^5)$ with a minimal polynomial $x^5 + x^4 + x^3 + x + 1$. Then $F(\omega^5)$ is a perfect 1-factorization of $K_{244}$.

In case $p=7$ and $m=3$, let $\omega$ be a primitive element of $GF(7^3)$ with a minimal polynomial $x^3 + x^2 + x + 2$. Then $F(\omega^{37})$ is a perfect 1-factorization of $K_{344}$.

Acknowledgment. The author would like to express her thanks to Professor Z. Kiyasu and Professor G. Nakamura for their helpful advice.

REFERENCES


[5] M. Kobayashi and Kiyasu-Zen’iti, Semi-regular one-factorizations of the complete graph $K_{pm+1}$, in submission.

