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PERFECT ONE-FACTORIZATIONS OF
THE COMPLETE GRAPH

MIDORI KOBAYASHI

1. Introduction

We denote by $K_{2n} = (V, E)$ the complete graph with $2n$ vertices, where $V$ is the set of $2n$ vertices and $E$ is the set of $n(2n-1)$ edges. A 1-factor of $K_{2n}$ is a set of pairwise disjoint edges that partition the set of vertices $V$. A 1-factorization of $K_{2n}$ is a set of 1-factors that partition the set of edges $E$. A 1-factorization is called perfect if the union of every pair of distinct 1-factors is a Hamiltonian circuit. Two 1-factorizations $F$ and $F'$ are isomorphic if there exists a permutation of $V$ which sends each member of $F$ into a member of $F'$.

The existence of a perfect 1-factorization of the complete graph $K_{2n}$ for all $n \geq 2$ is conjectured, and the problem is settled only for $2n = p+1, 2p$ ($p$ is prime), and $2n = 16, 28, 244, 344$. In this paper, these perfect 1-factorizations are explicitly shown.

Perfect 1-factorizations of $K_{36}, K_{1332}$ and $K_{6860}$ have recently been found ([4, 5]). The papers are in submission.

2. Perfect 1-factorization of $K_{p+1}$ ($p$ is an odd prime)

$GF(p)$ denotes the Galois field with $p$ elements. We put

$$V = GF(p) \cup \{\infty\}$$

and

$$F_0 = \left\{ \{i, j\} \mid i + j = 0, \ i, j \in GF(p) - \{0\} \right\} \cup \{(0, \infty)\}$$

$F_0$ is called a starter 1-factor and

$$F_g = F_0 + g$$

$$= \left\{ \{i + g, j + g\} \mid i + j = 0, \ i, j \in GF(p) - \{0\} \right\} \cup \{(g, \infty)\}.$$ 

is an induced 1-factor, where $g$ is an element of $GF(p)$. We obtain a perfect 1-factorization $GK_{p+1}:$
\[ \text{For example, a starter of } GK_{12} \text{ is shown in FIGURE 1.} \]

\begin{figure}
\centering
\begin{tikzpicture}
    \node[circle,draw,fill=black] (A) at (0,0) {$0$};
    \node[circle,draw,fill=black] (B) at (1,0) {$1$};
    \node[circle,draw,fill=black] (C) at (2,0) {$2$};
    \node[circle,draw,fill=black] (D) at (3,0) {$3$};
    \node[circle,draw,fill=black] (E) at (4,0) {$4$};
    \node[circle,draw,fill=black] (F) at (5,0) {$5$};
    \node[circle,draw,fill=black] (G) at (6,0) {$\infty$};
    \draw (A) -- (B);
    \draw (B) -- (C);
    \draw (C) -- (D);
    \draw (D) -- (E);
    \draw (E) -- (F);
    \draw (F) -- (G);
    \draw (G) -- (A);
\end{tikzpicture}
\caption{A starter of } GK_{12} \caption{FIGURE 1}
\end{figure}

3. Perfect 1-factorization of \( K_{2p} \) \((p \text{ is an odd prime})\)

Let
\[ V = \{ w_0, w_1, \ldots, w_{p+1}, w_0^*, w_1^*, \ldots, w_{p-1}^* \}. \]

For mathematical simplicity, we use \( w_{i+kp} \) and \( w_{i+kp}^* \) instead of \( w_i \) and \( w_i^* \), respectively, where \( k \) is an integer.

For an integer \( s \) with \( 0 \leq s \leq p-1 \), we put
\[ OG_s = \left\{ \{ w_i, w_j \} \mid i+j \equiv s, \; i \equiv j \pmod{p} \right\} \]
\[ \quad \cup \left\{ \{ w_i^*, w_j^* \} \mid i+j \equiv p-2-s, \; i \equiv j \pmod{p} \right\} \]
\[ \quad \cup \left\{ \{ w_{si/2}, w_{(p-2-s)i/2}^* \} \right\}, \]
where \( 1/2 \) means \( 2^{-1} \pmod{p} \).

For an integer \( s \) with \( 0 \leq s \leq p-2 \), we put
\[ IG_s = \left\{ \{ w_i, w_j^* \} \mid i+j \equiv s \pmod{p} \right\}. \]
Then
\[ GA_{2p} = \left\{ OG_{s} \mid s = 0, 1, \ldots, p - 1 \right\} \cup \left\{ IG_{s} \mid s = 0, 1, \ldots, p - 2 \right\} \]
is a perfect 1-factorization of \( K_{2p} \). For example, \( GA_{10} \) is shown in FIGURE 2.

\[ GA_{10} \]
FIGURE 2

Let
\[ V' = \left\{ v_0, v_1, \ldots, v_{2p-1} \right\}, \]
\[ E' = \left\{ \left\{ v_i, v_j \right\} \mid 0 \leq i \leq 2p - 1, 0 \leq j \leq 2p - 1, i \neq j \right\}. \]
For mathematical simplicity, we use \( v_{i+2pk} \) instead of \( v_i \), where \( k \) is an integer.

For any integer \( s \) with \( 0 \leq s \leq 2p - 1 \) and \( s \neq p \), we define \( G_{s}(\subseteq E) \) as follows:

If \( s \) is even, then
\[ G_s = \left\{ \left\{ v_i, v_j \right\} \mid \begin{cases} i + j \equiv s \pmod{2p} \\ i \neq j \end{cases}, \right\} \cup \left\{ v_{s/2}, v_{s/2+p} \right\} \]
If \( s \) is odd and \( s \neq p \), then
\[ G_s = \left\{ \left\{ v_i, v_j \right\} \mid \begin{cases} i \text{ odd}, i - j \equiv s \pmod{2p} \end{cases} \right\}. \]
\( G_s \) is a 1-factor of \( K_{2p} \) and the set of \( G_s \) denoted by
\[ GN_{2p} = \left\{ G_{s} \mid 0 \leq s \leq 2p - 1, s \neq p \right\} \]
is a perfect 1-factorization of \( K_{2p} \). For example, \( GN_{10} \) is shown in FIGURE 3.
GA_{2p} and GN_{2p} are isomorphic perfect 1-factorizations ([3]).

4. Perfect 1-factorization of K_{16}

A 1-factorization F is called factor-1-rotational if F has an automorphism fixing two vertices (and one 1-factor), and permuting the remaining 2n-2 vertices (and 2n-2 1-factors) in a single cycle. It has a convenient geometric representation. One takes the vertices of the regular polygon with 2n-2 vertices and labels them with elements of \( Z_{2n-2} \); the other two vertices is labeled with \( \infty_1, \infty_2 \), where \( Z_{2n-2} \) denotes the residue class group modulo 2n-2. Let \( F_1 \) be a starter 1-factor of a factor-1-rotational 1-factorization \( F \). The 2n-2 1-factors are obtained by rotating the figure successively through an angle \( 2\pi/(2n-2) \). \( F \) consists of these 2n-2 1-factors and the fixed 1-factor \( F^* \):

\[
F^* = \left\{ (i, j) \mid i - j \equiv n - 1 \pmod{2n-2} \right\} \cup \left\{ (\infty_1, \infty_2) \right\}
\]

A starter 1-factor of a factor-1-rotational, perfect 1-factorization of \( K_{16} \) is shown in FIGURE 4.
5. Perfect 1-factorizations of $K_{2n}$ for $2n = 28, 244, 344$

Let $p$ be a prime number and $m$ be a natural number such that $p^m \equiv 3 \pmod{4}$. We put $q = p^m$, $s = (q - 1)/2$ and $2n = q + 1$. $GF(q)$ denotes the Galois field with $q$ elements. $K_{2n} = (V, E)$ denotes the complete graph with $2n$ vertices, and

$$V = GF(q) \cup \{\infty\}.$$ 

Let $\omega$ be a primitive element of $GF(q)$. We define a starter 1-factor $F_0$:

$$F_0 = \left\{ \omega^{2i}, \omega^{2i+1} \mid i = 0, 1, 2, \ldots, s - 1 \right\} \cup \{0, \infty\}$$

For any $g \in GF(q)$,

$$F_g = F_0 + g$$

$$= \left\{ \omega^{2i} + g, \omega^{2i+1} + g \mid i = 0, 1, 2, \ldots, s - 1 \right\} \cup \{g, \infty\}$$

is a 1-factor which is induced by the starter $F_0$. Then

$$F(\omega) = \left\{ F_g \mid g \in GF(q) \right\}$$

is a 1-factorization of $K_{2n}$. It is proved that $F(\omega)$ is semi-regular ((1)). By suitable selections of the semi-regulars, we may construct perfect 1-factorizations.
In case \( p = 3 \) and \( m = 3 \), let \( \omega \) be a primitive element of \( GF(3^3) \) with a minimal polynomial \( x^3 + 2x^2 + 1 \). Then \( F(\omega) \) is a perfect 1-factorization of \( K_{28} \).

In case \( p = 3 \) and \( m = 5 \), let \( \omega \) be a primitive element of \( GF(3^5) \) with a minimal polynomial \( x^5 + x^4 + x^2 + 1 \). Then \( F(\omega^5) \) is a perfect 1-factorization of \( K_{244} \).

In case \( p = 7 \) and \( m = 3 \), let \( \omega \) be a primitive element of \( GF(7^3) \) with a minimal polynomial \( x^3 + x^2 + x + 2 \). Then \( F(\omega^{37}) \) is a perfect 1-factorization of \( K_{344} \).

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