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<td>引用</td>
<td>経営と経済, 56(4), pp.123-132; 1977</td>
</tr>
<tr>
<td>発行日</td>
<td>1977-03-31</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/28022">http://hdl.handle.net/10069/28022</a></td>
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A Limit Theorem of the Core of an Economy
An Application of the Gross Competition

Hidesuke Okuda

1. Introduction

In this paper we prove the Limit Theorem on the core of an economy by using the notion of the "Gross Competition" which was defined in [1]. By this method, there is obtained a straightforward and visible proof of the theorem.

2. The Mathematical Model

Let \( m \) be the number of "traders". The "commodity space" is the closed positive orthant \( \Omega \) of \( \mathbb{R}^n \), where \( n \) denotes the number of commodities. \( w \) is the "initial endowment". We assume \( \sum_{i=1}^{m} w(i) \).

\[ \sum_{i=1}^{m} w(i) \]

For \( x \) and \( y \) in \( \mathbb{R}^n \), we write \( x \gtrdot y \) to mean \( x^j \geq y^j \) for all \( j \) and \( x \succ y \) to mean \( x^j > y^j \) for all \( j \) with \( x^k > y^k \) for some \( k \).

An "allocation" is \( x \) such that \( \sum_{i=1}^{m} [x(i) - w(i)] = 0 \). For each trade \( i \), a "preference relation" \( \succ_i \) is defined on \( \Omega \). We define the following assumption on \( \succ_i \):

Assumption 1 — \textit{Strong monotonicity}; \( x \succ y \Leftrightarrow x \succ_i y \), for \( i = 1, \ldots, m \).

Assumption 2 : \( \succ_i \) is \textit{irreflexive} and \textit{transitive}, for \( i = 1, \ldots, m \).

Assumption 3: The sets \( \{ y \mid y \succ_i x \} \) and \( \{ y \mid x \succ_i y \} \) are open, for
i=1,⋯,m.

A "coalition" is a subset of the set of all traders. A coalition $L$ "blocks" an allocation $x$ if there exists $y$ such that $y(i) \succ_i x(i)$ for $i \in L$ and $\sum_{i \in L} [y(i) - w(i)] = 0$. The "core" is the set of allocations which are not blocked by any coalition.

A "price vector" is a non null element of $\Omega$. An assignment $x$ is "competitive" and a price vector $p$ is competitive price vector" if $p\cdot x(i) = p\cdot w(i)$ and $y \succ_i x(i) \iff p\cdot y \succ p\cdot w(i)$ for $i = 1,\cdots,m$.

We define $G(i) = \{ y | y + x(i) \succ_i x(i) \}$, $G^* = \sum_{i=1}^{m} G(i)$, and $M = \{1, 2, \cdots, m\}$. $M^*$ is the set of all coalitions, and $Y^* = \{ y | y = \sum_{i \in L} [w(i) - x(i)] \}$, $L \in M^*$).

3. Preliminaries

We defined "gross competition" in [1] as follows.

Definition: If $G^* \cap Y^* = \emptyset$, then we call that $x$ is "gross competitive".

The following lemma is the main method of this thesis.

Lemma 3.1: Let $M = \{1,\cdots,m\}$ and $M' = \{1,\cdots,m', m'\}$ be the two sets of traders such that $M \subseteq M'$. And let $G^* = \sum_{i=1}^{m} G(i)$, and $G'^* = \sum_{i=1}^{m'} G(i)$. Then $G^* \subseteq G'^*$.

Proof: Let $a \in G^*$, then there exists an allocation $x$ such that $a = \sum_{i=1}^{m} x(i)$, and $x(i) \in G(i)$, for $i = 1,\cdots,m$. By Assumption 3, $G(1)$ is open. Therefore there is a positive number $\varepsilon$ such that $[x(1) - \varepsilon] \in G(1)$. 
Let $y(1) = x(1) - \varepsilon$, $y(i) = x(i)$ for $i = 2, \cdots, m$, and $y(i) = \varepsilon/(m' - m)$ for $i = m + 1, \cdots, m'$. Then $y$ is an allocation, and $y(i) \in G(i)$ for $i = 1, \cdots, m'$, and $a = \sum_{i=1}^{m'} G(i)$. Therefore $a \in G^\ast$, this completes the proof of the lemma.

The following two lemmas are another expressions that $x$ is in the core and that $x$ is competitive. But we only demonstrate necessity for the former and sufficiency for the latter.

**Lemma 3.2:** If $x$ is in the core, then for any coalition $L$, \[
\sum_{i \in L} [w(i) - x(i)] + \sum_{i \in M \setminus L} G(i) \geq 0.\]

**Proof:** Suppose, contrary to the lemma, that there exists a coalition $L$ such that
\[
\sum_{i \in L} [w(i) - x(i)] + \sum_{i \in M \setminus L} G(i) < 0. \tag{3.1}\]

Then there exists $h$ such that
\[
[h(i) - x(i)] \in G(i), \text{ for } i \in M \setminus L, \sum_{i \in L} [w(i) - x(i)] + \sum_{i \in M \setminus L} [h(i) - x(i)] = 0. \tag{3.2}\]

So,
\[
\sum_{i \in L} w(i) - \sum_{i \in M} x(i) + \sum_{i \in M \setminus L} h(i) = 0.\]

Hence
\[
\sum_{i \in M \setminus L} h(i) = \sum_{i \in M \setminus L} w(i). \tag{3.3}\]

Therefore, by (3.2) and (3.3), $x$ is blocked by $M \setminus L$. This contradicts to the lemma.

**Lemma 3.3:** If there exists a price vector $p$ such that $p \cdot G^\ast \geq 0$, $p$ is a competitive price vector.

**Proof:** Suppose, contrary to the lemma, that $p$ is not a competitive price vector. Then there exists an integer $i$ and a vector $a$
such that $1 \leq i \leq m$, $a \in G(i)$ and $p \cdot a < 0$.

Then there exists a positive number $\varepsilon$ such that $[a - \varepsilon \cdot e] \in G(i)$, since $G(i)$ is open. Let

$$y(j) = \begin{cases} a - \varepsilon \cdot e & \text{for } j = i, \\ \frac{[\varepsilon/(m-1)] \cdot e}{m-1} & \text{for } j \neq i, \end{cases}$$

then $y(j) \in G(j)$ for $j = 1, \ldots, m$, and $a = \sum_{i \in M} y(j)$. Therefore $a \in G^*$. And since $p \cdot a < 0$, $p \cdot G^* \geq 0$ yields a contradiction. This completes the proof of the lemma.

Let

$$G(i, j) = G(i) \quad \text{for } i = 1, \ldots, m, j = 1, \ldots, r,$$

$$G^* = \sum_{i=1}^{m} \sum_{j=1}^{r} G(i, j)$$

$$\tilde{G}^r = \frac{1}{r} \tilde{G}^r$$

Lemma 3.4: If $r'$ is a multiple of $r$, $\tilde{G}^r \subseteq \tilde{G}^{r'}$.

Proof: There exists a positive integer $k$ such that $r' = kr$. Let $G(i, j, k) = G(i)$ for $i = 1, \ldots, m, j = 1, \ldots, r, k = 1, \ldots, l$.

Then $G^{r'} = \sum_{i, j, k} G(i, j, k)$, and $G^r = \frac{1}{r} \tilde{G}^r$. Suppose that $a \in \tilde{G}^r$, then there exist $a(i, j)$ such that

$$a(i, j) \in G(i) \text{ for } i = 1, \ldots, m, i = 1, \ldots, r,$$

and

$$a = \frac{1}{r} \sum_{i, j}^{m, r} a(i, j).$$

Let

$$a(i, j, k) = a(i, j) \text{ for } k = 1, \ldots, l,$$

then $a(i, j, k) \in G(i, j, k)$. Therefore
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\[ a = \frac{1}{r} \sum_{i,j,k}^{m,r,k} a(i,j,k) \in \tilde{G}r'. \]

This completes the proof of the lemma.

Lemma 3.5: Let \( U = \bigcup_{r=1}^{\infty} \tilde{G}r \), then \( U \) is a convex set.

Proof: We demonstrate that if \( a \in U \) and \( b \in U \),

\[ a\alpha + \beta b \in U, \]

where \( \alpha \) and \( \beta \) are real numbers such that

\[ \alpha \geq 0, \ \beta \geq 0, \text{ and } \alpha + \beta = 1. \]

First suppose that \( \alpha \) and \( \beta \) are rational numbers. Since \( a \in U \) and \( b \in U \), there exist two positive integers \( r_a \) and \( r_b \) such that \( a \in \tilde{G}r_a \) and \( b \in \tilde{G}r_b \). Suppose that \( r_e \) is a multiple of \( r_a \) and \( r_b \).

Then, by Lemma 3.4,

\[ a \in \tilde{G}r_e, \text{ and } b \in \tilde{G}r_e. \]

Since \( \alpha \) and \( \beta \) are rational numbers, there exist four positive integers \( \alpha_1, \ \alpha_2, \ \beta_1, \text{ and } \beta_2 \) such that \( \alpha = \alpha_2/\alpha_1, \text{ and } \beta = \beta_2/\beta_1 \).

Then

\[ 1 = \alpha + \beta = \alpha_2/\alpha_1 + \beta_2/\beta_1 = (\alpha_2 \beta_1 + \alpha_1 \beta_2)/\alpha_1 \beta_1. \]

Therefore \( \alpha_1 \beta_1 = \alpha_2 \beta_1 + \alpha_1 \beta_2 \).

Let \( r = r_e \alpha_1 \beta_1 \). By (3.5), there exist \( a(i) \) and \( b(i), \ i = 1, \ldots, r_e \), such that

\[ r_e \cdot a = \sum_{i=1}^{r_e} a(i), \text{ and } r_e \cdot b = \sum_{i=1}^{r_e} b(i). \]

Let

\[ c(i,j) = \begin{cases} a(i) \text{ for } i = 1, \ldots, r_e, j = 1, \ldots, \alpha_2 \beta_1, \\ b(i) \text{ for } i = 1, \ldots, r_e, j = \alpha_2 \beta_1 + 1, \ldots, \alpha_1 \beta_1. \end{cases} \]
Then
\[
\tilde{G}_r \equiv \frac{1}{r} \sum_{i,j} c(i,j)
\]
\[
= \frac{1}{r} \left[ a_2 \beta_1 \sum_{i=1}^{r_e} a(i) + a_1 \beta_2 \sum_{i=1}^{r_e} b(i) \right]
\]
\[
= \frac{1}{r_e} \left( \frac{a_2}{a_1} \cdot r_e a + \frac{\beta_2}{\beta_1} \cdot r_e b \right)
\]
\[
= a_1 a + \beta b.
\]
Therefore
\[
(3.7) \quad U \equiv a_1 a + \beta b.
\]
For the case such that \( \alpha \) and \( \beta \) are irrational numbers, we demonstrate (3.7)

By (3.6), \( a(i) \in G(i) \) and \( b(i) \in G(i) \), for \( i = 1, \ldots, r_e \). Since \( G(i) \) are open, there exists a real positive number \( \varepsilon \) such that \( U(b(i), \varepsilon) \subseteq G(i) \) where \( U(b(i), \varepsilon) \) are \( \varepsilon \)-neighbourhood of \( b(i) \).

Then there exists a real rational number \( \alpha' \) such that
\[
\alpha/\beta = (1 - \beta)/\beta = 1/\beta - 1 < \alpha' < (1 + \varepsilon)/\beta - 1.
\]
Let
\[
b'(i) = a(i) + (\alpha' + \beta)[b(i) - a(i)],
\]
\[
\hat{\alpha} = \alpha'/(\alpha' + \beta),
\]
and
\[
\hat{\beta} = \beta/(\alpha' + \beta).
\]
Then, if we substitute \( b'(i) \) for \( b(i) \), \( \hat{\alpha} \) for \( \alpha \), and \( \hat{\beta} \) for \( \beta \) in the above proof of (3.7), similarly we can prove (3.7) for this case.

This completes the proof of the lemma.

The following lemma is a generalization of the supporting hyperplane theorem.

Lemma 3.6: Let \( U \) be a convex set such that \( o \in \text{int} \ U \), and let
L be a linear subspace of \( \mathbb{R}^n \) such that \( o \in L \) and \((U \setminus \{o\}) \cap L = \emptyset\). Then there exists a supporting hyperplane \( H \) of \( U \) such that \( L \subseteq H \).

Proof: Let \( r \) be a positive integer such that \( r = \text{dim } L \). If \( r = n - 1 \), or \( r = 0 \), the lemma is trivial. Therefore we suppose that \( 1 \leq r \leq n - 2 \).

Let \( e = (e_1, \ldots, e_n) \) be an orthogonal coordinate of \( \mathbb{R}^n \) such that \( e_1, i = 1, \ldots, n \), are coordinates of \( e \), and \( e_1, \ldots, e_r \) span \( L \). Let \( K \) be a linear subspace of \( \mathbb{R}^n \) which is spanned by \( e_{r+1}, \ldots, e_n \). For a subset \( S \) in \( \mathbb{R}^n \), we write \( \text{Pr}(S) \) to mean the parallel projection of \( S \) on \( K \) which is parallel to \( e_1, \ldots, e_r \). Then \( \text{Pr}(L) = 0 \), and \( \text{Pr}(U) \) is convex in \( K \).

Therefore, by the supporting hyperplane theorem on \( K \), there exists a \((n-r-1)\)-dimensional linear subspace \( A \) of \( K \) such that \( A \) is a supporting hyperplane of \( \text{Pr}(U) \) in \( K \).

Then \( H \) is a supporting hyperplane of \( U \) in \( \mathbb{R}^n \) where \( H \) is a linear subset of \( \mathbb{R}^n \) which is spanned by \( A \) and \( L \). This completes the proof of the lemma.

4. Proof of the Limit Theorem

Now we prove the limit theorem of the core in the similar way to \([2]\), i.e., we prove that if first there are \( m \) traders, and if the numbers of traders increase by \( r \) times where the additional traders have the similar preferences and the similar initial endowments to the first traders, and if \( r \) increases infinitely, then in this market the core coincides with the set of the competitive equilibriums.

Let \( Imr \) be the market with the \( m \times r \) traders as is stated above, and let \( c(mr) \) be the core in this market. Then the following two lemmas are proved.
Lemma 4.1 (Dedreu and Scard [2]): If \( \{x^q\} \in c(m_r) \) where \( i=1, \ldots, m, \ q=1, \ldots, r, \) then \( x^{q_1}=\cdots=x^{q_r}, \ i=1, \ldots, m. \)

Lemma 4.2 (Dedreu and scarf [2]): If \( r_1 \leq r_2, \ c(m_r_1) \supseteq c(m_r_2). \)

Therefore, by Lemma 4.1, we can write \( x^{q_i} \) as \( x^i, \) for \( i=1, \ldots, m, \ q=1, \ldots, r. \) And we write \( c^*(r) = \{(x^1, \ldots, x^m)/\{x^{q_i}\} \in c(m_r)\}. \)

Theorem 4.1: If \( (x^1, \ldots, x^m) \in \bigcap c^*(r), \ \{x^{q_i}\}, i=1, \ldots, m, \ q=1, 2, \ldots \) is competitive.

Proof: By Lemma 3.3, in order to prove the theorem, it is sufficient only to prove that there exists a price vector \( p \) such that
\[
\sum_{i=1}^{m} p_i U \leq 0, \text{ and } \sum_{i=1}^{m} p_i [w^i - x^i] = 0, \ \text{for } i=1, \ldots, m.
\]

Now, on the contrary, suppose that there exists no price vector which suffice (4.1), then, by Lemma 3.6, \( K \cap (U \setminus \{0\}) = \emptyset \) where \( K \) is a linear subspace of \( \mathbb{R}^n \) which is spanned by \( \left\{ (w^i - x^i)/i=1, \ldots, m \right\}. \)

Therefore there exist real numbers \( \alpha_1, \ldots, \alpha_m \) such that
\[
\sum_{i=1}^{m} \alpha_i [w^i - x^i] \in U.
\]

Since
\[
(w^i - x^i) = - \sum_{j \neq i} (w^j - x^j),
\]

if \( \alpha_i < 0, \) we substitute (4.3) for (4.2), and by arranging for \( (w^i - x^i), \) we can rewrite (4.2) as follows
\[
\sum_{i=1}^{m} \alpha_i (w^i - x^i) = \sum_{i=1}^{m} \beta_i (w^i - x^i)
\]
where \( \beta_i, i=1, \ldots, m, \) are real numbers such that \( \beta_i \geq 0. \)

By (4.2) and (4.4), \( \sum_{i=1}^{m} \beta_i (w^i - x^i) \in U. \)

Therefore
\[
r \sum_{i=1}^{m} \beta_i (w^i - x^i) \in Gr^*.
\]
Since $G^*$ is open, replacing $r\beta_i, i=1, \ldots, m$, by non negative rational numbers $r_i$ which is sufficiently near to $r\beta_i$, we obtain

$$\sum_{i=1}^{m} r_i (w^i - x^i) \in Gr^*.$$  

Since $r_i$ are non negative rational numbers, there exist non negative integers $\delta_i^1, \delta_i^2, i=1, \ldots, m$, such that $r_i = \delta_i^1 / \delta_i^2$. Therefore, by (4.5),

$$\sum_{i=1}^{m} \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2 (w^i - x^i) \in \prod_{j \in M} \delta_i^2 \cdot Gr^*.$$  

Therefore, by Lemma 3.1,

$$\sum_{i=1}^{m} \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2 (w^i - x^i) \in Gr^*,$$

where $r' = \prod_{j \in M} \delta_i^1 \delta_i^2 r$.

(4.6) implies that \{x^{i,q}\} is not gross competitive.

By (4.6),

$$\sum_{i=1}^{m} (r' - \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2) (w^i - x^i) + Gr^* + \sum_{i=1}^{m} \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2 G(i) \geq 0.$$  

Therefore, by Lemma 3.1,

$$\sum_{i=1}^{m} (r' - \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2) (w^i - x^i) + Gr^* + \sum_{i=1}^{m} \delta_i^1 \prod_{j \in M \setminus \{i\}} \delta_i^2 G(i) \geq 0.$$  

Therefore, by Lemma 3.2, \(x^{i,1} \cdots x^m \notin c(2r')\). This completes the proof of the lemma.
References
