This document is downloaded at: 2019-07-31T22:26:14Z
1. Introduction

The author thinks that large markets are intrinsically statistical. Therefore it is meaningful to study the statistical properties of large markets. In this note we deal statistically with the problem of how to find (or construct) the blocking coalitions, i.e., when a Pareto optimal allocation and an arbitrary confidence level are given, how to find (or construct) a blocking coalition under this probability (statistical estimation). Several authors have dealt with this problem definitely (for examples [2], [6]).

The problem of this note can be interpreted as: "How many blocking coalitions are there among the coalitions that are constructed by a given procedure?" Therefore this is a variety of the problem "How many blocking coalitions are there?" (where no procedure is given) [4]. The theorem of this note is a version of the Core Equivalence Theorem.

In this note, we deal with a special case where the preference sets are convex and smooth. In the succeeding note [5], we deal with a general case where it is possible that the preference sets are not convex, nor smooth.

2. Model and result

The consumption set is $P = R_{++}^T$. The space of convex and monotone $C^2$ preference relation on $P$ satisfying the boundary condition, "for every $x \in P$, $\{y \in P : y \succeq x\}$ is closed in $R^T$", is denoted $S$. 
We assume that there exists \( \rho > 0 \) such that for every point \( x \in \mathbb{R}_{++}^\ell \), there is a function \( Q(x) \in \mathbb{R}_{++}^\ell \) such that \( |Q(x) - x| = \rho \) and for \( y : |y - Q(x)| \leq \rho \), \( y \geq x \) [1].

An economy \( \varepsilon \) is a map \( \varepsilon : I \to \mathcal{S} \times P \) where \( I \) is a finite indexing set : \( \varepsilon (i) \) is denoted \((\geq_1, \omega_i)\). Given an economy \( \varepsilon \) an allocation is a map \( x : I \to P \) with \( \sum_{i \in I} x(i) \leq \sum_{i \in I} \omega_i \). Given an allocation \( x \) for \( \varepsilon \), one says that \( x \) is blocked by coalition \( C \subset I \) if \( C \neq \emptyset \) and there is \( x' : C \to P \) such that \( \sum_{i \in C} x'(i) \leq \sum_{i \in C} \omega_i \) and \( x'(i) >_1 x(i) \) for every \( i \in C \): \( x \) is Pareto optimal if it is not blocked by \( I \).

For given \( \varepsilon \) let \( x : I \to P \) be Pareto optimal. Then there is a price vector \( p \) such that \( p > 0, |p| = 1 \) and \( p \cdot (y - x(i)) \geq 0 \) for \( y : y >_1 x(i) \) and \( i \in I \), and if each indifference hypersurface \( \{y : y \sim x(i)\} \) has a tangent sphere at \( x(i) \), there exists only one price vector.

We use the following assumptions.

Assumptions:
1. There exists \( s > 0 \) such that \( s > |\omega_i| \) and \( s > |x(i)|, i \in I \).
2. \( x : I \to P \) is Pareto optimal.
3. Let \( I_0 = \{i \in I : p \cdot (\omega_i - x(i)) > 0\} \). \#I and \#I_0 are sufficiently large.
4. In some way \( c = \sum_{i \in I_0} p \cdot (\omega_i - x(i)) / \#I_0 \) is known.

Theorem: Under Assumptions 1–4, \( 1 > \gamma > 0 \) is given. Then there exist integer \( M, N, n \) and a procedure such that,

Procedure:
Step 1: pick up \( N \) agencies from \( I \) at random.
\( \bar{N} \) is the set of these agencies. If \( \#(\bar{N} \cap I_0) \geq n \), this step holds.
Then pick up \( n \) agencies from \( \bar{N} \cap I_0 \) at random. \( \bar{n} \) is the set of these agencies. If \( \#(\bar{N} \cap I_0) < n \), this step does not hold.
Step 2: pick up \( M \) agencies from \( I - \bar{N} \) at random.
The probability that the step 1 holds and \( \bar{M} \cup \bar{n} \) is a blocking coali-
tion is greater than or equal to $\gamma$.

3. Proof of the theorem

3.1. Lemma: If $-1 < a_i < 1$, $i \in I$ and $\sum_{i \in I} a_i = 0$, then for $0 < t < 1$

$$P_r \left| a \right| > t \leq 2e^{-2mt^2}$$

where $\bar{a} = \frac{\sum_{i \in I} a_i}{n}$ and $n$ is a set of $n$ integers picked up from $I$ at random.

Proof: See [3], th. 1.

3.2. Proof of the theorem: Let $\gamma_0 = (1 - (1 - \gamma))/3$. Then

$$1 - \gamma = 3(1 - \gamma_0).$$

(1) Let $n_1$ be an integer such that

$$\# I_0 > n_1 \geq (8s^2/c^2) \log \frac{2}{1 - \gamma_0}.$$  

From Assumption 3, there exists such integer.

Then

$$1 - 2e^{-2n_1 \left( \frac{c}{2s^2} \right)^2} \geq \gamma_0$$

$n_1$ is the set of $n_1$ agencies picked up from $I_0$ at random. Then from

Assumption 1, 4 and the lemma, $P_r[3c/4s > \frac{1}{2n_1s} \sum_{i \in n_1} p \cdot (\omega_i - x(i)) > c/4s] \geq 1 - 2e^{-2n_1 \left( \frac{c}{2s^2} \right)^2} > \gamma_0$

(II) Let

$$t = \max(3c/2, \rho[\frac{s^2 + (c/2)^2}{\rho c}] + k\rho)$$

where $[\ ]$ is the Gauss' notation and $k$ is a positive integer.

---

* There is more precise inequalities than this.

Nevertheless we use this inequality for simplicity.
Let \( a = \frac{1}{2}(t - \sqrt{s^2 + (t - c/2)^2}) \) Then \( a > 0 \), (See Figure)

In Figure, \( \overline{OA} = \overline{OE} = t, \overline{AB} = c/2, \overline{BC} = c, \overline{BD} = s, \overline{DE} = 2a \) where \( \overline{FG} \) is the distance between points F and G.

Let \( n_2 \) be an integer such that

\[
n_2 \geq \frac{2s^2t}{\rho a^2} \log \frac{2\ell}{1 - \gamma_0}
\]  

Then

\[
1 - 2\ell \exp \left\{-2n_2 \left( \frac{t}{\rho} \right) \left( \frac{a/2s}{(t/\rho)\sqrt{\ell}} \right)^2 \right\} \geq \gamma_0
\]  

Let \( I_1 \) is a subset of \( I \) with sufficiently many elements. From Assumption 3, there exists such subset. Let \( \overline{N}_2 \) be the set of \( n_2 \left( \frac{t}{\rho} \right) \) integers picked up from \( I \) at random. Let \( \overline{x}_{j} = \frac{\sum_{i \in \overline{N}_2} x_{ij}}{\# \overline{N}_2} \) where \( x_i = \omega i - x(i), i \in I_1 \) and \( x_{ij} = j \)-component of \( x_i \), and let \( \mu_j = \frac{1}{\# I_1} \sum_{i \in I_1} x_{ij}, j = 1, ..., \ell. \)

Then from the lemma and (6),

\[
\Pr \left\{ \frac{\overline{x}_{j} - \mu_j}{2s} \leq \frac{a/2s}{\sqrt{\ell} \left( \frac{t}{\rho} \right)} \right\} \geq \gamma_0
\]

Let

\[
n = \max(n_1, n_2)
\]

and

\[
M = \left( \frac{t}{\rho} \right).
\]

\textbf{(III)} Let \( N \) be an integer such that

\[
\sum_{u=0}^{N} \frac{u}{\# I_0} \left( \frac{\# (I - I_0)}{N - u} \right) \geq \gamma_0
\]

From Assumption 3, there exists such integer.
(IV) From Assumption 3, we can assume that
\[ \#I > N + \frac{2N \cdot s}{a/2} \]
Hence
\[ N \cdot s / (\#I - N) < a/2 \]  \hspace{1cm} (11)

(V) Now \( \bar{N} \) be the set of any \( N \) agencies picked up from \( I \) at random, and set \( I_1 = I - \bar{N} \). Then, by (11),
\[ |\mu| < a/2 \]  \hspace{1cm} (12)
where \( \mu = \frac{\sum_{i \in I_1} x_i}{\#I_1} \), i.e., \( \mu = (\mu_1, ..., \mu_\ell) \).

(VI) From Assumption 3, the lemma, (2), (3), (4), and (8), we can substitute \( n \) for \( n_1 \) in (4). Similarly, from Assumption 3, the lemma, (5), (6), (7) and (8), we can substitute \( n \) for \( n_2 \) in (7). Therefore, from (1), (9), (10) and (12), the theorem holds.

College of Commerce, Nagasaki University
Katafuchi, Nagasaki, Japan

References.