1. Introduction

In the preceding note [1], we dealt statistically with the problem of how to find (or construct) blocking coalitions when preference sets are strictly convex and these surfaces are smooth. In this note we deal statistically with it when preference sets are convex. The result of this note is a generalization of the result of the preceding note.

2. Theorems

In this note preference sets are convex. We deal with the problem in the space \( \{ x \mid x \leq 2s \} \cap \mathbb{R}^\ell_+ \). We assume \( |x(i)| < s \), and \( |\omega_i| < s \) for \( i \in I \). Let \( G(i) = \{ y \mid y \succ_i x(i) \} \) for \( i \in I \), and let \( G(K) = \frac{1}{\#K} \sum_{i \in K} y \mid y \in G(i) \), and \( |y| < 2s \) where \( K \subseteq I \).

Condition I: 1. Let \( K \subseteq I \). There exists a \( \rho < 0 \) and a point \( z \in \mathbb{R}^\ell \) such that
\[
\overline{G(K)} \supset c(z, \rho)
\]
where \( c(z, \rho) = \{ y \mid |y - z| < \rho \} \).

2. Let \( P(y, w, r) \) be a circular cone such that
\[
P(y, w, r) = \{ y + t (u - y) \mid u \in c(w, r), \text{ and } t > 0 \}.
\]

1) The author anticipates this term is not so severe.
Then, for \( z \) and \( \rho \) in 1, there exist innumerable\(^2\) collections of agencies \( I_1, I_2, \ldots \) and a \( c > 0 \) such that \( I_j \cap I_k = \emptyset \) for \( j + k, j, k = 1, 2, \ldots \),

\[
\sum_{i \in I_j} (\omega_i - x(i)) \in \mathbb{P}(0, z - \bar{x}, \rho/4)
\]  

(2)

where \( \bar{x} = \frac{1}{\# K} \sum_{i \in K} x(i) \), \( |\sum_{i \in I_j} (\omega_i - x(i))| < cs \) for \( j = 1, 2, \ldots \), and \( |\sum_{i \in I_0} (\omega_i - x(i))| \) is sufficiently large where \( I_0 = I_1 \cup I_2 \cup \cdots \).

The Condition II is a modification of the Condition I.

Condition II:

Modification:

1. Let \( K = I \) in Condition I.
2. Change (1) for
   \[ \text{Int} \ G(K) \supset \text{Cl} \ C(z, \rho) \]
   where \( \text{Int} A \) is the interior of \( A \), and \( \text{Cl} B \) is the closure of \( B \).
3. Change (2) for
   \[ \sum_{i \in I_j} (\omega_i - x(i)) \in \mathbb{P}(0, z - \bar{x}, \rho/6). \]

Assume Condition I. Then

\[ | y / | y - z | = \rho/2 | \cap \mathbb{P}(\bar{x}, z, \rho/4) \]

is composed of two surfaces \( A \) and \( B \) such that \( A \) is farther away from \( \bar{x} \) than \( B \) (see Figure 1).

Then the distance between \( A \) and \( B \) is greater than \( \rho/2 \), because, for a triangle, the sum of two sides is greater than the other (see Figure 2).

Therefore, since \( |\sum_{i \in I_j} (\omega_i - x(i))| < cs \) for \( j = 1, 2, \ldots \) (Condition I, 2), if
there exists a set of indices \( J \) such that

\[
\bar{x} + \frac{1}{\#K} \sum_{j \in J} \sum_{i \in I_j} (\omega_i - x(i)) \in y \mid y - z < \rho/2 \cap \Pr(\bar{x}, z, \rho/4) \subseteq \mid y - z < \rho/2 \mid \tag{4}
\]

Let \( K \) be a set of agencies picked up at random. Let \( 1 > \gamma > 0 \) be a probability, Then

\[
\Pr \left[ \left| \frac{1}{\#K} \sum_{i \in k} (\omega_i - x(i)) \right| < \rho/2 \right] \geq 1 - 2l \cdot \exp \left[ -2 \cdot \#K \cdot \frac{\rho^2}{16l^2} \right] \tag{5}
\]

Therefore if \( \#K \geq \frac{8l^2}{\rho^2} \log \frac{2l}{1 - \gamma} \),

\[
1 - 2l \cdot \exp \left[ -2 \cdot \#K \cdot \frac{\rho^2}{16l^2} \right] \geq \gamma. \text{ Hence, from (5),}
\]

\[
\Pr \left[ \left| \frac{1}{\#K} \sum_{i \in k} (\omega_i - x(i)) \right| < \rho/2 \right] \geq \gamma \tag{6}
\]

Assume

\[
\#K \geq \text{Max} \left( \frac{2c_\delta / \rho}{\rho^2} \cdot \frac{8l^2}{\rho^2} \log \frac{2l}{1 - \gamma} \right) \tag{7}
\]

By Condition I, there exists a set of indices \( J \) such that \( K \cap I_j = \phi \) for \( j \in J \) and \( J \) fits (4).

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2), 3) Now the innumerability of the set \{ I_1, I_2, \cdots \} can be put concretely as follows.

There exists a \( \epsilon > 0 \) and a set \( \{ I_i \} \) such that

\[
\sum_{j \in I_i} (\omega_j - x(i)) > \epsilon \quad \text{for } I_i \in \{ I_i \},
\]

and

\[
\# \{ I_i \} > \#K \quad z - \bar{x} / \epsilon + \#K.
\]
Then

\[ x + \frac{1}{\#K} \sum_{i \in J} \sum_{d \in D} (\omega_i - x(i)) + \frac{1}{\#K} \sum_{i \in K} (\omega_i - x(i)) \]

\[ \epsilon \{ y / |y - z| < \rho \} \subseteq \overline{G(K)} \]

under the probability \( \gamma \).

Since

\[ \# \overline{G(K)} = \sum_{i \in K} G(i) \subseteq \sum_{i \in K} G(i) + \sum_{j \in J} \sum_{d \in D} (G(i) - x(i)), \]

\[ \sum_{i \in K \cup (\bigcup_{i \in J} I_i)} \omega_i \in \sum_{i \in K \cup (\bigcup_{j \in J} I_j)} G(i) \]

under the probability \( \gamma \).

Therefore

\[ K \cup (\bigcup_{j \in J} I_j) \]

is a blocking coalition under the probability \( \gamma \).

Now we can solve statistically the problem of how to find (or construct) blocking coalitions by the following theorem.

Theorem 1. Let \( 1 > \gamma > 0 \) be a probability. Let \( K \) be a set of agencies picked up at random. Then if Condition I presents itself and (7) holds, the coalition in (9) is a blocking coalition under the probability \( \gamma \).

Usually \( \overline{G(I)} \) is unknown. Though in the following, we deal statistically with the problem of how to find (or construct) blocking coalitions when \( \overline{G(I)} \) is known.

We assume Condition II. Let

\( U(x) = \{ y / y \in \mathbb{R}^\ell, x \in \text{Bd}(\overline{G(I)}) \}, \)

and \( |y - x| < |x - z| - \rho \),

and let

\( V(x) = \{ y / y \in \mathbb{R}^\ell, x \in \text{Bd}(\overline{G(I)}) \}, \)

and \( |y - x| < \frac{1}{3} |x - z| - \frac{1}{3} \rho \)

where \( \text{Bd}(A) \) is the boundary of a set \( A \). Since \( \text{Bd}(\overline{G(I)}) \) is compact, there exist finite number of points \( x_1, \ldots, x_m \in \text{Bd}(\overline{G(I)}) \)

such that \( \bigcup_{i=1}^m V(x_i) \supseteq \text{Bd}(\overline{G(I)}) \) (see Figure 3).
Lemma 1. Assume Condition II. Assume that $H$ is a convex set and there exist points $d_1, \cdots, d_m$, $x_1, \cdots, x_m$ such that $d_i \in \text{Bd}(H)$, $d_i \in \vee(x_i)$ for $i = 1, \cdots, m$, and $\text{Bd}(\overline{G(I)}) \subseteq \bigcup_{i=1}^{m} \vee(x_i)$. Then $H \supseteq c(z, \rho)$.

Proof. Suppose, contrary to the lemma, that there exists a point $d$ such that $d \in \text{Bd}(H)$, and $d \in c(z, \rho)$.

Then, by the supporting hyperplane theorem, there exists a vector $p$ such that $p \cdot (y - d) \leq 0$ for $y \in \text{Cl}(H)$ (see Figure 4). For $p$, there exists a point $d'$ such that $d' \in \text{Cl}(c(z, \rho))$, and $p \cdot (y - d') \leq 0$ for $y \in c(z, \rho)$.

Therefore

$$p \cdot (d_i - d') \leq 0 \text{ for } i = 1, \cdots, m \quad (10).$$

Then there exists a point $d''$ and a $t > 0$ such that $d'' \in \text{Bd}(\overline{G(I)})$ and $(d' - d') = t \cdot p$. Therefore $p \cdot (y - d') > 0$ for $y \in \text{U}(d'')$, There exists a $i$ such
that $V(x_i) \ni d'$. Since $U(d') \supset V(x_i)$, $p \cdot (d_i - d') < 0$, contrary to (10). This completes the proof of the lemma.

Let $x_i \in \overline{Bd(G(I))}$ for $i = 1, \ldots, m$, and let $\bigcup_{i=1}^m V(x_i) \supset \overline{Bd(G(I))}$.

Let $\sigma_i$ be the radius of $V(x_i)$, i.e., $\sigma_i = \frac{1}{3} | x_i - z | - \rho/3$ for $i = 1, \ldots, m$. For $x_i$, $i = 1, \ldots, m$, there exist $a_{ij}, j \in I$, such that $a_{ij} \in y \in \text{Cl}(G(j))$, and $| y | \leq 2s$ for $j \in I$, and $\frac{1}{\# I} \sum_{j \in I} a_{ij} = x_i$.

Let $K$ be a set of agencies picked up, and let $\frac{1}{\# K} \sum_{j \in K} a_{ij} = \overline{a}_i$ for $i = 1, \ldots, m$. Then

$$\Pr[| \overline{a}_i - x_i | < \sigma_i, i = 1, \ldots, m] \geq 1 - \sum_{i=1}^m 2 \exp \left[ -2 \cdot \# K \cdot \frac{\sigma_i^2}{4ls^2} \right]$$ (11).

Let $1 > \gamma > 0$ be a probability. Then, by (11), there exists a $M$ such that if

$$\# K \geq M$$ (12).

$$\Pr[| \overline{a}_i - x_i | < \sigma_i, i = 1, \ldots, m] > \gamma.$$ Similarly, since

$$\Pr\left[ \left| \frac{1}{\# K} \sum_{i \in K} x(i) - \frac{1}{\# I} \sum_{i \in I} x(i) \right| < \rho/3 \right] \geq 1 - 2l \cdot \exp \left[ -2 \cdot \# K \cdot \frac{\rho^2}{9ls^2} \right],$$

if

$$\# K \geq \frac{9ls^2}{2\rho^2} \log \frac{2l}{1 - \gamma}$$ (13).

$$\Pr\left[ \left| \frac{1}{\# K} \sum_{i \in K} x(i) - \frac{1}{\# I} \sum_{i \in I} x(i) \right| < \rho/3 \right] \geq \gamma$$ (14).

From Lemma 1, (12), and (13), we obtain the following lemma.

Lemma 2. Assume Condition II. Let $1 > \gamma > 0$ be a probability. Let $K$ be a set of agencies picked up at random. Then, if

$$\# K \geq \max \left( M, \frac{9ls^2}{2\rho^2} \log \frac{2l}{1 - \gamma} \right)$$ (15).
where M is as in (12), and $\gamma_0 = \frac{1}{2} (1 + \gamma)$. Condition I presents itself under the probability $\gamma$.

Proof. Assume that $| \bar{x}_i - x_i | < \sigma_i$ for $i = 1, \ldots, m$, hold. Then, by Lemma 1, $G(K) \supset c(z, \phi)$. Further assume

$$\frac{1}{\#K} \sum_{i \in K} x(i) - \frac{1}{\#I} \sum_{i \in I} x(i)$$

holds. Then see Figure 5 where $z' = z + \frac{1}{\#K} \sum_{i \in K} x(i) - \frac{1}{\#I} \sum_{i \in I} x(i)$. Then we know Condition II presents itself.

From (12), (13), (16), and (14), the lemma holds.

By Lemma 2, if we assume Condition II, Condition I presents itself under a probability. Then, by Theorem 1, we can find (or construct) a blocking coalition under a probability. Hence we obtain the following theorem.

Theorem 2. Assume Condition II. Let $1 > \gamma_1 > 0$ be a probability. Let K be a set of agencies picked up at random. Then, if $\#K \geq \text{Max} \left( \frac{18ls^2}{\rho^2} \log \frac{2l}{1 - \gamma}, M, \frac{9ls^2}{2\rho^2} \log \frac{2l}{1 - \gamma_0} \right)$

where $\gamma = \frac{1}{2} (1 + \gamma_1)$, $\gamma_0 = \frac{1}{2} (1 + \gamma) = \frac{3}{4} + \frac{\gamma_1}{4}$, and M is as in (12), the coalition given in (9) is a blocking coalition under the probability $\gamma_1$.

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