<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>作者</td>
<td>Kobayashi, Midori</td>
</tr>
<tr>
<td>註明</td>
<td>明治大学経済学部経済学研究科経済学専攻 教授</td>
</tr>
<tr>
<td>暦名</td>
<td>経営と経済</td>
</tr>
<tr>
<td>番号</td>
<td>62(4), pp.91-101; 1983</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/28182">http://hdl.handle.net/10069/28182</a></td>
</tr>
</tbody>
</table>

NAOSITE: Nagasaki University’s Academic Output SITE

http://naosite.lb.nagasaki-u.ac.jp
ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

BY MIDORI KOBAYASHI

1. A real algebraic integer \( \alpha (> 1) \) is said to be a PV-number if any conjugate \( (\pm \alpha) \) of \( \alpha \) are in the unit circle \( |z| < 1 \). Let \( S \) be the set of all PV-numbers and \( S' \) the set of all accumulation points of \( S \). Since \( S \) is closed ( [3] ), we have \( S' \subset S \). Dufrenoy and Pisot found the least element of \( S' \) and proved that the elements of \( S' \) which are less than 1.8 are only two: \( \eta_1 = (1 + \sqrt{5})/2 = 1.618 \cdots \), zero of \( 1 + z - z^2 \), and \( \eta_2 = 1.754 \cdots \), zero of \( 1 - z + 2z^2 - z^3 \) ( [1] ).

In this paper we shall show that the number of the elements of \( S' \) which are less than 1.84 is at most five, and give the unique expansions of \( A(z)/Q(z) \) corresponding to these elements.

2. Let \( \theta \) be a PV-number. Let \( P(z) \) be the irreducible polynomial of \( \theta \):

\[
P(z) = p_0 + p_1 z + \cdots + p_{s-1} z^{s-1} + \varepsilon z^s \quad (p_0 > 0, \varepsilon = +1),
\]

and \( Q(z) \) the reciprocal polynomial:

\[
Q(z) = \varepsilon z^s P(\frac{1}{z}).
\]

For a number \( \theta \in S \), \( \theta \) belongs to \( S' \) if and only if there exists a polynomial \( A(z) \in \mathbb{Z}[z] \) such that

\[
|A(z)| \leq |P(z)| \quad \text{on} \quad |z| = 1,
\]

where the equality holds only at finite points on \( |z| = 1 \) ( [3] Th. 1).

Assume that \( \theta \in S' \) and let \( P(z) \), \( Q(z) \) and \( A(z) \) be the polynomials corresponding to \( \theta \). We can assume that \( A(0) > 0 \). Then it is
easy to see that $A(z)/Q(z)$ has a single pole of order 1 at $1/\theta$ in $|z| \leq 1$. Therefore we have the power series expansion of $A(z)/Q(z)$ at zero:

$$\frac{A(z)}{Q(z)} = v_0 + v_1z + \cdots + v_nz^n + \cdots$$

We briefly denote this expansion by $(v_0, v_1, \ldots, v_n, \ldots)$.

The following was proved in (2). For any natural number $n$, there exist polynomials $E_n(z)$ and $E_n^*(z)$ ($\in \mathbb{Q}(z)$, of degree $n$) such that, if we put $D_n(z) = -z^nE_n(-\frac{1}{z})$ and $D_n^*(z) = z^nE_n^*(-\frac{1}{z})$, then the expansion of $\frac{D_n(z)}{E_n(z)}$ at zero is $(v_0, v_1, \ldots, v_{n-1}, w_n, \ldots)$ and that of $\frac{D_n^*(z)}{E_n^*(z)}$ is $(v_0, v_1, \ldots, v_{n-1}, w_n^*, \ldots)$, where $w_n$ and $w_n^*$ are uniquely determined. Then the following relations holds.

1. $w_n + 1 \leq v_n \leq w_n^* - 1$, except the case $(1, 1, 1, \ldots)$. ($n \geq 3$).
2. $D_{n+2}(z) = (1+z)D_{n+1}(z) - z \frac{v_{n+1} - w_{n+1}}{v_n - w_n} D_n(z)$ ($n \geq 1$).
3. $w_{n+1}^* - w_n = 2 \frac{D_{n+1}(1)}{D_n(1)} (v_n^* - v_n)$ ($n \geq 1$).
4. $w_{n+1}^* - w_{n+1} = 4 \frac{(w_n^* - v_n) (v_n - w_n)}{(w_{n+1}^* - w_n)}$ ($n \geq 3$).

If $\theta < 1.84$, then

5. $v_{n+1} - w_{n+1} < \frac{2.84}{1.84} \frac{D_{n+1}(1.84)}{D_n(1.84)} (v_n - w_n)$ ($n \geq 1$).

The sequence $w_n^* - w_n$ is monotone decreasing, i. e.,

6. $w_{n+1}^* - w_{n+1} \leq w_n^* - w_n$.

Hence, if $w_n^* - w_n < 3$ then $v_n, v_{n+1}, v_{n+2}, \ldots$ are determined from (1) without ambiguity.

3. Let $\theta$ be any number of $S'$ which is less than 1.84. We shall give an expansion $(v_0, v_1, \ldots, v_n, \cdots)$ of $A(z)/Q(z)$ correspond-
We have \( v_0 = 1 \), \( D_1(z) = 1 - z \) \( w_1 = 0 \) and \( v_1 = 1 \) or 2, in a similar way to (1). Therefore we have only to consider two cases, i.e., the case (1, 1, . . . ) and the case (1, 2, . . . ).

(I) The case (1, 2, . . . ).

As it is known ([2] p77, p79)

\[
D_2(z) = v_0 + \frac{v_1}{1 + v_0} z - z^2, \quad w_2 = v_0^2 - 1 + \frac{v_1}{1 + v_0},
\]

we have

\[
D_2(z) = 1 + z - z^2, \quad w_2 = 2.
\]

Hence, from (5) it follows \( v_2 - w_2 < 2.005 \ldots \), so \( v_2 \leq 4 \). On the other hand, since \( w_2 \leq v_2 \) ([1] p56), we obtain

\[
v_2 = 2 \quad \text{or} \quad v_2 = 3 \quad \text{or} \quad v_2 = 4.
\]

The case \( v_2 = 2 \) corresponds to \( \eta_1 \); if \( v_2 = 3 \) there does not exist \( \theta \) which correspond to (1, 2, 3, . . . ) ([1]).

For the case (1, 2, 4, . . . ), we have \( D_3(z) = 1 + z + z^2 - z^3 \)

from (2). As \( E_3(z) = 1 - (z + z^2 + z^3) \), we have

\[
\frac{D_3(z)}{E_3(z)} = 1 + 2z + 4z^2 + 6z^3 + \ldots,
\]

so it follows \( w_3 = 6 \). On the other hand, the formula (5) implies \( v_3 - w_3 < 0.02 \ldots \), hence \( v_3 \leq 6 \). This contradicts to (1). So there is no element in \( S' \) corresponding to (1, 2, 4, . . . ).

(II) The case (1, 1, . . . ).

From (7), we have

\[
D_2(z) = 1 + \frac{1}{2} z - z^2, \quad w_2 = \frac{1}{2}.
\]

From (5) it follows that \( v_2 - w_2 < 2.69 \ldots \), so we have \( v_2 \leq 3 \), hence
\[ v_2 = 1 \text{ or } v_2 = 2 \text{ or } v_2 = 3. \]

The expansions \((1, 1, 1, \ldots)\) and \((1, 1, 3, \ldots)\) correspond to \(\eta_1\) and \(\eta_2\), respectively \(([1])\). Therefore we have only to deal with the case \((1, 1, 2, \ldots)\).

In this case, we obtain \(D_3(z) = 1 + z^2 - z^3\) from (2). As \(E_3(z) = 1 - z - z^3\) and

\[
\frac{D_3(z)}{E_3(z)} = 1 + z + 2z^2 + 2z^3 + \ldots,
\]

we have \(w_3 = 2\). It follows \(v_3 - w_3 < 2.91\ldots\) from (5), so \(v_3 \leq 4\). From (1) we have

\[ v_3 = 3 \text{ or } v_3 = 4. \]

For the case \(v_3 = 3\), i.e., \((1, 1, 2, 3, \ldots)\), we only obtain \((1, 1, 2, 3, 5, 8, \ldots)\), \((1, 1, 2, 3, 5, 9, 16, \ldots)\) and \((1, 1, 2, 3, 6, \ldots)\) which correspond to \(\eta_1\), \(\eta_2\) and \(\eta_3\), respectively, where \(\eta_3 = 1.839\ldots\), zero of \(1 + z + z^2 - z^3\) \(([1])\).

Therefore it is enough to deal with the case \((1, 1, 2, 4, \ldots)\).

From (2), we have

\[ D_4(z) = 1 - \frac{1}{3}z + \frac{1}{3}z^2 + \frac{4}{3}z^3 - z^4, \quad w_4 = \frac{17}{3}. \]

From (3) it follows that \(w_i - w_4 = \frac{16}{3}\), so \(w_i = 11\). Hence we obtain \(7 \leq v_4 \leq 10\) by (1). As \(v_4 - w_4 < 2.73\ldots\) from (5), we have

\[ v_4 = 7 \text{ or } v_4 = 8. \]

If \(v_4 = 8\), then we calculate \(D_5(z)\) by (2), from which we have \(w_5 = 13.5\) and \(w_5 = 18.75 : 15 \leq v_5 \leq 17\). But we have \(v_5 = 15\) using the formula (5). Next we calculate \(D_6(z)\) by (2), from which we have \(w_6 = 26.71\ldots\) and \(w_6 = 30.99\ldots: 28 \leq v_6 \leq 29\). This contradicts (5), hence \(v_4 = 7\).
ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

From (2) and (4), we have
\[ D_5(z) = 1 + z^3 + z^4 - z^5, \quad w_5 = 11, \quad w_5^* = 15. \]

So it follows \( v_5 = 12 \) or \( v_5 = 13 \) or \( v_5 = 14 \) by (1). If \( v_5 = 12 \), then it corresponds to \( \eta_2 \) ( (1) ). If \( v_5 = 14 \), we have \( v_6 = 27 \) by (1), which contradicts (5).

Therefore we have only to deal with the case \((1, 1, 2, 7, 13, \ldots)\). Since
\[ D_6(z) = 1 - \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \frac{3}{2}z^5 - z^6, \]
\[ w_6 = 22, \quad w_6^* = 26, \]
it follows \( 23 \leq v_6 \leq 25 \) from (1). On the other hand (5) implies \( v_6 = 23 \) or \( v_6 = 24 \). The number \( \theta \) corresponding to the case \( v_6 = 23 \) does not exist (cf. (1) ). Hence we have \( v_6 = 24 \). Next we have
\[ D_7(z) = 1 - \frac{1}{2}z + z^3 - \frac{1}{2}z^4 + \frac{1}{2}z^5 + \frac{3}{2}z^6 - z^7, \]
\[ w_7 = 42.5, \quad w_7^* = 46.5, \]
so it follows \( v_7 = 44 \) or \( v_7 = 45 \). If \( v_7 = 45 \), then we have \( v_8 = 83 \) and \( v_9 = 154 \) which contradicts (5), so \( v_7 = 44 \).

For the expansion \((1, 1, 2, 4, 7, 13, 24, 44, \ldots)\) we have
\[ D_8(z) = 1 - \frac{1}{4}z - \frac{1}{8}z^2 + \frac{5}{8}z^3 + \frac{1}{8}z^4 + \frac{7}{8}z^6 + \frac{5}{4}z^7 - z^8, \]
\[ w_8 = 78.875, \quad w_8^* = 82.625, \]
which gives \( v_8 = 80 \) or \( v_8 = 81 \) by (1).

If \( v_8 = 81 \), in a similar way, we have a contradiction.

If \( v_8 = 80 \), then we calculate \( D_9(z) \), from which we have \( w_9 = 144.5 \), while (3) gives \( w_9^* = 147.65 \). Hence it follows \( v_9 = 146 \). Furthermore we calculate \( D_{10}(z) \), from which \( w_{10} = 265 \) and \( w_{10}^* = 268.14 \ldots \), so we have \( v_{10} = 266 \) or \( v_{10} = 267 \).

For \( v_{10} = 266 \), it follows \( w_{11} - w_{11} > 2.72 \ldots \)
from (4), then \( v_{11}, v_{12}, \ldots \) are uniquely determined by (6).

For \( v_{10} = 267 \), it follows \( w_{11}^1 - w_{11} < 2.90 \). from (4), then \( v_{11}, v_{12}, \ldots \) are uniquely determined.

Therefore we obtain that the unique expansions of \( A(z)/Q(z) \) corresponding to the numbers of \( S \) which are less than 1.84 are at most the following:

1. \((1, 2, 2, \ldots )\),
2. \((1, 1, 1, \ldots )\),
3. \((1, 1, 3, \ldots )\),
4. \((1, 1, 2, 3, 6, \ldots )\),
5. \((1, 1, 2, 3, 5, 8, \ldots )\),
6. \((1, 1, 2, 3, 5, 9, \ldots )\),
7. \((1, 1, 2, 4, 7, 13, 24, 44, 80, 146, 266, \ldots )\),
8. \((1, 1, 2, 4, 7, 13, 24, 44, 80, 146, 267, \ldots )\),

where 1, 2 and 5 correspond to \( \eta_1 \), 3 and 6 to \( \eta_2 \), and 4 to \( \eta_3 \).

Finally, HITAC M-280H in Computer Centre University of Tokyo was used in order to determine \( w_n \) from \( D_n(z) \). The program and its application to the case \((1, 1, 2, 4, 7, 13, 24, 44, \ldots )\) will be shown. 'DN(Z)', 'C(Z)', 'E(Z)', 'W(Z)' and 'W8' in the list indicate \( D_n(z) \), \( 1 - \text{E}_n(z) \), \( 1/\text{E}_n(z) \), \( D_n(z)/\text{E}_n(z) \) and \( w_8 \), respectively.

In this case,

\[ n = 8, \]

\[ D_n(z) = 1 - \frac{1}{4}z - \frac{1}{8}z^2 + \frac{5}{8}z^3 + \frac{1}{8}z^4 + \frac{7}{8}z^5 + \frac{5}{4}z^6 - z^8, \]

\[ 1 - \text{E}_n(z) = \frac{5}{4}z + \frac{7}{8}z^2 + \frac{1}{8}z^3 + \frac{5}{8}z^5 - \frac{1}{8}z^6 - \frac{1}{4}z^7 + z^8, \]

\[ -\frac{1}{\text{E}_n(z)} = 1 + \frac{5}{4}z + \frac{39}{16}z^2 + \frac{265}{64}z^3 + \frac{1903}{256}z^4 + \ldots \]

\[ + \frac{5298671}{65536}z^8 + \ldots, \]
\[
\frac{D_n(z)}{E_n(z)} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + \frac{631}{8}z^8 + \ldots ,
\]

\[\omega_n = \frac{631}{8}.\]
SUBROUTINE PRPLY(N,A,P,C,E,F)
INTEGER*8 A(20),B(20),C(20),D(20),E(20),F(20)
INTEGER*8 EE,FF
INTEGER*8 LA,LB,LC,LD,LE,LF
DO 20 K=1,N+1
EE=0
FF=1
DO 30 J=1,K
LA=A(J)
LB=B(J)
LC=C(K-J+1)
LD=D(K-J+1)
CALL PRODQ(LA,LB,LC,LD,LE,LF)
CALL ADDQ(EE,FF,LE,LF)
CONTINUE
E(K)=EE
F(K)=FF
20 CONTINUE
RETURN
END

SUBROUTINE PROCTP(M,N,A,B,C,D,E,F)
INTEGER*8 A(20),B(20),C(20),D(20),E(20),F(20)
INTEGER*8 EE,FF
INTEGER*8 LA,LB,LC,LD,LE,LF
DO 10 I=M+2,M+N+1
A(I)=0
B(I)=1
CONTINUE
DO 15 I=N+2,M+N+1
C(I)=0
D(I)=1
CONTINUE
DO 22 I=1,M+1
CALL TSBUN(A(I),B(I))
CONTINUE
DO 33 I=1,N+1
CALL TSBUN(C(I),D(I))
CONTINUE
DO 20 K=1,M+N+1
ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

92    EE=0
93    FF=1
94    DO 30 J=1,K
95    LA=A(J)
96    LB=B(J)
97    LC=C(K-J+1)
98    LD=D(K-J+1)
99    CALL PRODQ(LA, LB, LC, LD, LE, LF)
100   CALL ADDQ(EE, FF, LE, LF)
101   30 CONTINUE
102   E(K)=EE
103   F(K)=FF
104   20 CONTINUE
105   END
106   SUBROUTINE ADDQ(E, F, L, F)
107       IN INTEGER*8 E, F, L, F
108       DO 30 I=1,N+1
109           CALL AOOQ(A(I), B(I), C(I), D(I))
110       30 CONTINUE
111       RETURN
112   END
113   SUBROUTINE PRODQ(LA, LB, LC, LD, LE, LF)
114       IN INTEGER*8 LA, LB, LC, LD, LE, LF
115       IF(LA.EQ.0, LC.EQ.0) GOTO 9
116           CALL T5UBUN(LA, LB)
117           CALL T5UBUN(LC, LD)
118           CALL T5UBUN(LA, LD)
119           CALL T5UBUN(LC, LB)
120           LE=LA*LC
121           LF=LB*LD
122           RETURN
123   9 LE=0
124       LF=1
125       RETURN
126       END
127   SUBROUTINE ADDQ(E, F, L, F)
128       IN INTEGER*8 E, F, L, F
129           CALL EUCLID(F, F, L, LCOMM)
130           EE=LF/LCOMM*E+FF/LCOMM*L
131           FF=FF/LCOMM*LF
132           FF=FF/LCOMM*LF
133           CALL T5UBUN(EE, FF)
134       RETURN
135       END
136   SUBROUTINE T5UBUN(LA, LB)
137       IN INTEGER*8 LA, LB
138           IF(LA.EQ.0) GOTO 99
139           CALL EUCLID(LA, LB, LC)
140           LA=LA/LC
141           LB=LB/LC
142       RETURN
143   99 LB=1
144    RETURN
145    END
146    SUBROUTINE EUCLID(A,B,C)
147    INTEGER*8 A,B,C
148    INTEGER*8 LA, LB, LQ, LR
149    LA=ABS(A)
150    LB=ABS(B)
151    1 LQ=LA/LB
152    LR=LA-LB*LQ
153    IF(LR.EQ.0) GOTO 9
154    LA=LB
155    LB=LR
156    GO TO 1
157    9 C=LB
158    RETURN
159    END

N = 8

DN(Z)=
   1/ 1
   -1/ 4
   -1/ 8
    5/ 8
    1/ 8
    0/ 1
    7/ 3
    5/ 4
   -1/ 1

C(Z)=
   0/ 1
    5/ 4
    7/ 8
    0/ 1
    1/ 8
    5/ 8
    1/ 8
   -1/ 4
   -1/ 1
    1/ 1

E(Z)=
   1/ 1
    5/ 4
   39/ 16
  265/ 64
 1903/ 256
14025/ 1024
100703/ 4096
 726649/ 16384
 5298671/ 65536
ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

\[ W(2) = \]

\[
\begin{array}{c}
1/ \\
1/ \\
2/ \\
4/ \\
7/ \\
13/ \\
24/ \\
44/ \\
631/ \\
\end{array}
\]

\[ W(8) = \]

\[
\begin{array}{c}
631/ \\
\end{array}
\]

REFERENCES

