1. A real algebraic integer \( a (> 1) \) is said to be a PV-number if any conjugate \((\pm a)\) of \( a \) are in the unit circle \(|z| < 1\). Let \( S \) be the set of all PV-numbers and \( S' \) the set of all accumulation points of \( S \). Since \( S \) is closed (\([3]\)), we have \( S' \subset S \). Dufrenoy and Pisot found the least element of \( S' \) and proved that the elements of \( S' \) which are less than 1.8 are only two: \( \eta_1 = (1 + \sqrt{5})/2 = 1.618\ldots \), zero of \( 1 + z - z^2 \), and \( \eta_2 = 1.754\ldots \), zero of \( 1 - z + 2z^2 - z^3 \) (\([1]\)). In this paper we shall show that the number of the elements of \( S' \) which are less than 1.84 is at most five, and give the unique expansions of \( A(z)/Q(z) \) corresponding to these elements.

2. Let \( \theta \) be a PV-number. Let \( P(z) \) be the irreducible polynomial of \( \theta \):

\[
P(z) = p_0 + p_1 z + \cdots + p_{s-1} z^{s-1} + \varepsilon z^s \quad (p_0 > 0, \varepsilon = +1),
\]

and \( Q(z) \) the reciprocal polynomial:

\[
Q(z) = \varepsilon z^s P\left(\frac{1}{z}\right).
\]

For a number \( \theta \in S \), \( \theta \) belongs to \( S' \) if and only if there exists a polynomial \( A(z) \in \mathbb{Z}[z] \) such that

\[
|A(z)| \leq |P(z)| \quad \text{on} \quad |z| = 1,
\]

where the equality holds only at finite points on \(|z| = 1\) (\([3]\) Th. 1).

Assume that \( \theta \in S' \) and let \( P(z), Q(z) \) and \( A(z) \) be the polynomials corresponding to \( \theta \). We can assume that \( A(0) > 0 \). Then it is
easy to see that \( A(z)/Q(z) \) has a single pole of order 1 at \( 1/\theta \) in \( |z| \leq 1 \). Therefore we have the power series expansion of \( A(z)/Q(z) \) at zero:

\[
\frac{A(z)}{Q(z)} = v_0 + v_1z + \cdots + v_nz^n + \cdots
\]

We briefly denote this expansion by \((v_0, v_1, \cdots, v_n, \cdots)\).

The following was proved in [2]. For any natural number \( n \), there exist polynomials \( E_n(z) \) and \( E_n^*(z) \) (\( \in Q(z) \), of degree \( n \)) such that, if we put \( D_n(z) = -z^nE_n\left(\frac{1}{z}\right) \) and \( D_n^*(z) = z^nE_n^*\left(\frac{1}{z}\right) \), then the expansion of \( \frac{D_n(z)}{E_n(z)} \) at zero is \((v_0, v_1, \cdots, v_n-1, w_n, \cdots)\) and that of \( \frac{D_n^*(z)}{E_n^*(z)} \) is \((v_0, v_1, \cdots, v_n-1, w_n^*, \cdots)\), where \( w_n \) and \( w_n^* \) are uniquely determined. Then the following relations holds.

1. \( w_n + 1 \leq v_n \leq w_n^* - 1 \), except the case \((1, 1, 1, \cdots)\). \((n \geq 3)\).

2. \( D_{n+2}(z) = (1+z)D_{n+1}(z) - z\frac{v_{n+1} - w_{n+1}}{v_n - w_n}D_n(z) \) \((n \geq 1)\).

3. \( w_{n+1}^* - w_{n+1} = 2\frac{D_{n+1}(1)}{D_n(1)}(v_n - w_n) \) \((n \geq 1)\).

4. \( w_{n+1}^* - w_{n+1} = 4\frac{(w_n^* - v_n)(v_n - w_n)}{(w_n^* - w_n)} \) \((n \geq 3)\).

If \( \theta < 1.84 \), then

5. \( v_{n+1} - w_{n+1} < \frac{2.84}{1.84} \frac{D_{n+1}(1.84)}{D_n(1.84)}(v_n - w_n) \) \((n \geq 1)\).

The sequence \( w_n^* - w_n \) is monotone decreasing, i. e.,

6. \( w_{n+1}^* - w_{n+1} \leq w_n^* - w_n \).

Hence, if \( w_n^* - w_n < 3 \) then \( v_n, v_{n+1}, v_{n+2}, \cdots \) are determined from (1) without ambiguity.

3. Let \( \theta \) be any number of \( S^* \) which is less than 1.84. We shall give an expansion \((v_0, v_1, \cdots, v_n, \cdots)\) of \( A(z)/Q(z) \) correspond-
We have \( v_0 = 1, \ D_1 (z) = 1-z \ w_1 = 0 \) and \( v_1 = 1 \) or \( 2, \) in a similar way to \((1)\). Therefore we have only to consider two cases, i. e., the case \((1, 1, \ldots)\) and the case \((1, 2, \ldots)\).

\((1)\) The case \((1, 2, \ldots)\).

As it is known \((2) p 77, p 79)\)

\[(7)\quad D_2 (z) = v_0 + \frac{v_1}{1+v_0} \ z - z^2, \quad w_2 = v_0^2 - 1 + \frac{v_1^2}{1+v_0},\]

we have

\[D_2 (z) = 1+z-z^2, \quad w_2 = 2.\]

Hence, from \((5)\) it follows \( v_2 - w_2 < 2.005 \ldots, \) so \( v_2 \leq 4. \) On the other hand, since \( w_2 \leq v_2 \) \((1) p 56), \) we obtain

\[v_2 = 2 \text{ or } v_2 = 3 \text{ or } v_2 = 4.\]

The case \( v_2 = 2 \) corresponds to \( \eta_1; \) if \( v_2 = 3 \) there does not exist \( \theta \) which correspond to \((1, 2, 3, \ldots)\) \((1)\).

For the case \((1, 2, 4, \ldots)\), we have \( D_3(z) = 1+z+z^2 - z^3 \) from \((2)\). As \( E_3(z) = 1 - (z + z^2 + z^3), \) we have

\[
\frac{D_3(z)}{E_3(z)} = 1 + 2z + 4z^2 + 6z^3 + \ldots,
\]

so it follows \( w_3 = 6. \) On the other hand, the formula \((5)\) implies \( v_3 - w_3 < 0.02 \ldots, \) hence \( v_3 \leq 6. \) This contradicts to \((1)\). So there is no element in \( S' \) corresponding to \((1, 2, 4, \ldots)\).

\((II)\) The case \((1, 1, \ldots)\).

From \((7)\), we have

\[D_2 (z) = 1 + \frac{1}{2} z - z^2, \quad w_2 = \frac{1}{2}.\]

From \((5)\) it follows that \( v_2 - w_2 < 2.69 \ldots, \) so we have \( v_2 \leq 3, \) hence
\( v_2 = 1 \) or \( v_2 = 2 \) or \( v_2 = 3 \).

The expansions \((1, 1, 1, \ldots)\) and \((1, 1, 3, \ldots)\) correspond to \( \eta_1 \) and \( \eta_2 \), respectively \((1)\). Therefore we have only to deal with the case \((1, 1, 2, \ldots)\).

In this case, we obtain \( D_3(z) = 1 + z^2 - z^3 \) from \((2)\). As \( E_3(z) = 1 - z - z^3 \) and

\[
\frac{D_3(z)}{E_3(z)} = 1 + z + 2z^2 + 2z^3 + \ldots,
\]

we have \( w_3 = 2 \). It follows \( v_3 - w_3 < 2.91 \ldots \) from \((5)\), so \( v_3 \leq 4 \). From \((1)\) we have

\( v_3 = 3 \) or \( v_3 = 4 \).

For the case \( v_3 = 3 \), i.e., \((1, 1, 2, 3, \ldots)\), we only obtain \((1, 1, 2, 3, 5, 8, \ldots)\), \((1, 1, 2, 3, 5, 9, 16, \ldots)\) and \((1, 1, 2, 3, 6, \ldots)\) which correspond to \( \eta_1, \eta_2 \) and \( \eta_3 \), respectively, where \( \eta_3 = 1.839 \ldots \), zero of \( 1 + z + z^2 - z^3 \) \((1)\).

Therefore it is enough to deal with the case \((1, 1, 2, 4, \ldots)\).

From \((2)\), we have

\[
D_4(z) = 1 - \frac{1}{3}z + \frac{1}{3}z^2 + \frac{4}{3}z^3 - z^4, \quad w_4 = \frac{17}{3}.
\]

From \((3)\) it follows that \( w_4 - w_4 = \frac{16}{3} \), so \( w_4 = 11 \). Hence we obtain \( 7 \leq v_4 \leq 10 \) by \((1)\). As \( v_4 - w_4 < 2.73 \ldots \) from \((5)\), we have

\( v_4 = 7 \) or \( v_4 = 8 \).

If \( v_4 = 8 \), then we calculate \( D_5(z) \) by \((2)\), from which we have \( w_5 = 13.5 \) and \( w_5 = 18.75 : 15 \leq v_5 \leq 17 \). But we have \( v_5 = 15 \) using the formula \((5)\). Next we calculate \( D_6(z) \) by \((2)\), from which we have \( w_6 = 26.71 \ldots \) and \( w_6 = 30.99 \ldots : 28 \leq v_6 \leq 29 \). This contradicts \((5)\), hence \( v_4 = 7 \).
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From (2) and (4), we have

\[ D_5(z) = 1 + z^3 + z^4 - z^5, \quad w_5 = 11, \quad w_5^* = 15. \]

So it follows \( v_5 = 12 \) or \( v_5 = 13 \) or \( v_5 = 14 \) by (1). If \( v_5 = 12 \), then it corresponds to \( \eta_2 \) ( (1) ). If \( v_5 = 14 \), we have \( v_6 = 27 \) by (1), which contradicts (5).

Therefore we have only to deal with the case \((1, 1, 2, 7, 13, \ldots)\). Since

\[ D_6(z) = 1 - \frac{1}{2} z + \frac{1}{2} z^2 + \frac{1}{2} z^3 + \frac{3}{2} z^5 - z^6, \]

\[ w_6 = 22, \quad w_6^* = 26, \]

it follows \( 23 \leq v_6 \leq 25 \) from (1). On the other hand (5) implies \( v_6 = 23 \) or \( v_6 = 24 \). The number \( \theta \) corresponding to the case \( v_6 = 23 \) does not exist (cf. (1)). Hence we have \( v_6 = 24 \). Next we have

\[ D_7(z) = 1 - \frac{1}{2} z + z^3 - \frac{1}{2} z^4 + \frac{1}{2} z^5 + \frac{3}{2} z^6 - z^7, \]

\[ w_7 = 42.5, \quad w_7^* = 46.5, \]

so it follows \( v_7 = 44 \) or \( v_7 = 45 \). If \( v_7 = 45 \), then we have \( v_8 = 83 \) and \( v_9 = 154 \) which contradicts (5), so \( v_7 = 44 \).

For the expansion \((1, 1, 2, 4, 7, 13, 24, 44, \ldots)\) we have

\[ D_8(z) = 1 - \frac{1}{4} z - \frac{1}{8} z^2 + \frac{5}{8} z^3 + \frac{1}{8} z^4 + \frac{7}{8} z^6 + \frac{5}{4} z^7 - z^8, \]

\[ w_8 = 78.875, \quad w_8^* = 82.625, \]

which gives \( v_8 = 80 \) or \( v_8 = 81 \) by (1).

If \( v_8 = 81 \), in a similar way, we have a contradiction.

If \( v_8 = 80 \), then we calculate \( D_9(z) \), from which we have \( w_9 = 144.5 \), while (3) gives \( w_9^* = 147.65 \). Hence it follows \( v_9 = 146 \).

Furthermore we calculate \( D_{10}(z) \), from which \( w_{10} = 265 \) and \( w_{10}^* = 268.14 \ldots \), so we have \( v_{10} = 266 \) or \( v_{10} = 267 \).

For \( v_{10} = 266 \), it follows \( w_{11} - w_{11}^* > 2.72 \ldots \)
from (4), then \( v_{11}, v_{12}, \ldots \) are uniquely determined by (6).

For \( v_{10} = 267 \), it follows \( w_{11}^{1} - w_{11} < 2.90 \). .. from (4), then \( v_{11}, v_{12}, \ldots \) are uniquely determined.

Therefore we obtain that the unique expansions of \( A(z)/Q(z) \) corresponding to the numbers of \( S' \) which are less than 1.84 are at most the following:

1. \((1, 2, 2, \ldots ,)\),  
2. \((1, 1, 1, \ldots ,)\),  
3. \((1, 1, 3, \ldots ,)\),  
4. \((1, 1, 2, 3, 6, \ldots ,)\),  
5. \((1, 1, 2, 3, 5, 8, \ldots ,)\),  
6. \((1, 1, 2, 3, 5, 9, \ldots ,)\),  
7. \((1, 1, 2, 4, 7, 13, 24, 44, 46, 146, 266, \ldots ,)\),  
8. \((1, 1, 2, 4, 7, 13, 24, 44, 80, 146, 267, \ldots ,)\),

where ①, ② and ⑤ correspond to \( \eta_1 \), ③ and ⑥ to \( \eta_2 \), and ④ to \( \eta_3 \).

Finally, HITAC M-280H in Computer Centre University of Tokyo was used in order to determine \( w_n \) from \( D_n(z) \). The program and its application to the case \((1, 1, 2, 4, 7, 13, 24, 44, \ldots ,)\) will be shown. 'DN(Z)', 'C(Z)', 'E(Z)', 'W(Z)' and 'W8' in the list indicate \( D_n(z) \), \( 1 - E_n(z) \), \( 1/E_n(z) \), \( D_n(z)/E_n(z) \) and \( w_8 \), respectively. In this case,

\[
n = 8,
\]

\[
D_n(z) = 1 - \frac{1}{4} z - \frac{1}{8} z^2 + \frac{5}{8} z^3 + \frac{1}{8} z^4 + \frac{7}{8} z^6 + \frac{5}{4} z^7 - z^8,
\]

\[
1 - E_n(z) = \frac{5}{4} z + \frac{7}{8} z^2 + \frac{1}{8} z^3 + \frac{5}{8} z^5 - \frac{1}{8} z^6 - \frac{1}{4} z^7 + z^8,
\]

\[
\frac{1}{E_n(z)} = 1 + \frac{5}{4} z + \frac{39}{16} z^2 + \frac{265}{64} z^3 + \frac{1903}{256} z^4 + \ldots
\]

\[
+ \frac{5298671}{65536} z^8 + \ldots ,
\]
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\[
\frac{D_n(z)}{E_n(z)} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + \frac{631}{8}z^8 + \ldots ,
\]

\[w_8 = \frac{631}{8} .\]

1 C

PV-NUMBER

2 INTEGER*8 A(20), B(20), C(20), D(20), E(20), F(20)

3 INTEGER*8 CC(20), DD(20), AA(20), BB(20)

4 INTEGER*8 CI(20), DI(20)

5 READ(5,1000) N

6 READ(5,1100) (A(I), B(I), I=1,N+1)

7 WRITE(6,2000) N

8 WRITE(6,2500)

9 WRITE(6,3000)(A(I), B(I), I=1,N+1)

10 C(1)=0

11 D(1)=1

12 DO 10 I=2,N+1

13 C(I)=A(N-I+2)

14 D(I)=B(N-I+2)

15 10 CONTINUE

16 WRITE(6,1111)

17 WRITE(6,3000)(C(I), D(I), I=1,N+1)

18 E(1)=1

19 F(1)=1

20 DO 20 I=2,N+1

21 E(I)=0

22 F(I)=1

23 20 CONTINUE

24 CALL ADDPLY(N,E,F,C,D)

25 DO 25 I=1,N+1

26 CI(I)=C(I)

27 DI(I)=D(I)

28 25 CONTINUE

29 DO 30 JJ=2,N

30 CALL PRDPLY(N,C1,D1,C,DD)

31 CALL ADDPLY(N,E,F,CC,DD)

32 DO 30 I=1,N+1

33 CI(I)=CC(I)

34 DI(I)=DD(I)

35 30 CONTINUE

36 WRITE(6,1200)

37 WRITE(6,3000)(E(I), F(I), I=1,N+1)

38 CALL PRDPLY(N,A, B, E, F, AA, BB)

39 WRITE(6,1300)

40 WRITE(6,3000)(AA(I), BB(I), I=1,N+1)
SUBROUTINE PROPLY(N,A,P,C,I),E,F)
INTEGR*8 A(20),B(20),C(20),D(20),E(20),F(20)
INTEGER*8 EE,FF
INTEGER*8 LA,LB,LC,LD,LE,LF
DO 20 K=1,N+1
EE=0
FF=1
DO 30 J=1,K
LA=A(J)
LB=B(J)
LC=C(K-J+1)
LD=D(K-J+1)
CALL PRODQ(LA,LB,LC,LD,LE,LF)
CALL ADDQ(EE,FF,LE,LF)
CONTINUE
E(K)=EE
F(K)=FF
20 CONTINUE
RETURN
END
SUBROUTINE PROCTP(M,N,A,B,C,D,E,F)
INTEGR*8 A(20),B(20),C(20),D(20),E(20),F(20)
INTEGER*8 EE,FF
INTEGER*8 LA,LB,LC,LD,LE,LF
10 CONTINUE
I=M+2,N+1
A(I)=0
B(I)=1
DO 15 I=N+2,M+N+1
DO 22 I=1,M+1
CALL TSUBUN(A(I),B(I))
CONTINUE
DO 33 I=1,N+1
CALL TSUBUN(C(I),D(I))
33 CONTINUE
DO 20 K=1,M+N+1
ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

92    EE=0
93    FF=1
94    DO 30 J=1,K
95    LA=A(J)
96    LB=B(J)
97    LC=C(K-J+1)
98    LD=D(K-J+1)
99    CALL PRODQ(LA, LB, LC, LD, LE, LF)
100   CALL ADDQ(EE, FF, LE, LF)
101   30 CONTINUE
102    EE=EE
103    FF=FF
104   20 CONTINUE
105 END
106 END
107 SUBROUTINE ADDQ(A,B,C,D,G)
108     INTEGER*8 A(20),B(20),C(20),D(20),G(20)
109    DO 30 I=1,N+1
110    CALL ADDQ(A(I),B(I),C(I),D(I),G(I))
111   30 CONTINUE
112 END
113 END
114 SUBROUTINE PRODQ(LA, LB, LC, LD, LE, LF)
115     INTEGER*8 LA, LB, LC, LD, LE, LF
116     IF(LA.EQ.O .AND. LC.EQ.O) GOTO 9
117    CALL TSUBUN(LA, LB)
118    CALL TSUBUN(LC, LD)
119    CALL TSUBUN(LA, LD)
120    CALL TSUBUN(LC, LB)
121    LE=LA*LC
122    LF=LB*LD
123    RETURN
124    9 LE=0
125    LF=1
126    RETURN
127 END
128 SUBROUTINE ADDQ(A,B,C,D,E)
129     INTEGER*8 A, B, C, D, E
130    CALL EUCLID(F, E/LCOMM)
131    E=LF/LCOMM*E+FF/LCOMM*LE
132    FF=FF/LCOMM*LF
133    CALL TSUBUN(E, F)
134    RETURN
135 END
136 SUBROUTINE TSUBUN(LA, LB)
137     INTEGER*8 LA, LB
138     IF(LA.EQ.O) GOTO 99
139    CALL EUCLID(LA, LB, LC)
140    LA=LA/LC
141    LB=LB/LC
142    RETURN
143    99 LB=1
RETURN
END
SUBROUTINE EUCLID(A,B,C)
INTEGER*8 A,B,C
INTEGER*8 LA, LB, LQ, LR
LA=ABS(A)
LB=ABS(B)
1 LQ=LA/LB
LR=LA-LB*LQ
IF(LR.EQ.0) GOTO 9
LA=LB
LB=LR
GO TO 1
9 C=LB
RETURN
END

N = 8

DN(Z)=

/ 1
-1 / 4
-1 / 8
5 / 8
1 / 8
0 / 1
7 / 3
5 / 4
-1 / 1

C(Z)=

/ 0
5 / 4
7 / 8
0 / 1
1 / 8
5 / 8
1 / 8
-1 / 4
-1 / 4
1 / 1

E(Z)=

/ 1
5 / 4
39 / 16
265 / 64
1903 / 256
14025 / 1024
100703 / 4096
726649 / 16384
5298671 / 65536
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\[ w(z) = \]

\[
\begin{array}{c}
1/ \\
1/ \\
2/ \\
4/ \\
7/ \\
13/ \\
24/ \\
44/ \\
631/
\end{array}
\]

\[ w_8 = \]

\[
631/
\]

REFERENCES

