Let \( n \) be any triangular number: \( n = 1 + 2 + \cdots + k \), where \( k \) is a natural number.

Form a pile of \( n \) cards, then divide it into arbitrary piles with an arbitrary number of cards in each pile. Take one card from each pile and with them make a new pile. Keep repeating the procedure. It is conjectured that regardless of the initial state you will reach the consecutive state, i.e., \((1, 2, 3, \ldots, k)\) in finite steps: the game must end because the consecutive state cannot change. This game is called Bulgarian Solitaire.

For example, in case \( k = 3 \), \( n = 6 \),

\[
(1, 1, 4) \rightarrow (3, 3) \rightarrow (2, 2, 2) \rightarrow (1, 1, 1, 3) \rightarrow (2, 4) \rightarrow (1, 2, 3),
\]

\[
(6) \rightarrow (1, 5) \rightarrow (2, 4) \rightarrow (1, 2, 3),
\]

and in case \( k = 4 \), \( n = 10 \),

\[
(1, 1, 3, 5) \rightarrow (2, 4, 4) \rightarrow (1, 3, 3, 3) \rightarrow (2, 2, 2, 4) \rightarrow (1, 1, 1, 3, 4) \rightarrow (2, 3, 5) \rightarrow (1, 2, 3, 4).
\]

The above games end with the consecutive state in 5, 3 and 6 steps, respectively.

It is conjectured that for \( n = 1 + 2 + \cdots + k \), any game must end in no more than \( k(k - 1) \) steps, and in 1982 Donald E. Knuth and his students of Stanford University confirmed it for \( k \leq 10 \) by computer.

In this paper we shall show that the above conjecture cannot be

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made better in a sense, that is, we shall prove the following:

Let $k$ be any natural number ($\geq 3$). Put $n=1+2+\cdots+k$. The partition of $n,$ $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$ reaches the consecutive state by Bulgarian operation in $k(k-1)$ steps.

The partition $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$ is called the top of the main trunk of Bulgarian tree by Gardner. \(^{(3)}\)

Now we shall prove the above theorem for $k \geq 6$; it is easily checked for $k \leq 5$.

The initial state is $(1, 1, 2, 3, \cdots, k-2, k-1, k-1)$, so we have $(1, 2, 3, \cdots, k-2, k-2, k+1)$ after the 1st step, $(1, 2, \cdots, k-3, k-3, k, k)$ after the 2nd step, $(1, 2, \cdots, k-4, k-4, k-1, k-1, k)$ after the 3rd step, and so on, $(1, 1, 4, 4, 5, \cdots, k)$ after the $(k-2)$th step and $(3, 3, 4, 5, \cdots, k-1, k)$ after the $(k-1)$th step. Hence we have $(2, 2, 3, 4, \cdots, k-2, k-1, k-1)$ after the $k$th step.

Let $2 \leq l \leq k-3$. We shall show by induction on $l$ that we have $(1, 2, \cdots, l-1, l+1, l+1, l+2, \cdots, k-1, k-1)$ after the $lk$th step.

If $l=2$, it is easily checked that we have $(1, 1, 3, 5, 5, 6, \cdots, k)$ after the $(2k-2)$th step, so $(1, 3, 3, 4, 5, \cdots, k-2, k-1, k-1)$ after the $2k$th step.

If $l=3$, we have $(2, 2, 3, 4, \cdots, k-3, k-2, k-2, k)$ after the $(2k+1)$th step and $(1, 1, 3, 4, 6, 6, 7, \cdots, k)$ after the $(3k-2)$th step, so we have $(1, 2, 4, 4, 5, \cdots, k-1, k-1)$ after the $3k$th step.

Suppose, then, that $4 \leq l \leq k-3$. By induction we may have $(1, 2, \cdots, l-2, l, l, l+1, \cdots, k-1, k-1)$ after the $(l-1)k$th step. Then we have $(1, 2, \cdots, l-3, l-1, l-1, l, \cdots, k-2, k-2, k)$ after the $((l-1)k+1)$th step.

step, and so on, \((1, 3, 3, 4, \cdots, k-l+2, k-l+2, k-l+4, \cdots, k-1, k)\) after the \(((l-1)k+l-3)\)th step. So we get \((1, 2, \cdots, k-l-2, k-l, \cdots, k-2, k, k)\) after the \(((l-1)k+l+1)\)th step. And we have \((1, 1, 3, 4, \cdots, l+1, l+3, l+3, l+4, \cdots, k)\) after the \(((l-1)k+k-2)\)th step, so \((2, 3, \cdots, l, l+2, l+2, l+3, \cdots, k-1, k)\) after the \(((l-1)k+k-1)\)th step, hence we obtain \((1, 2, \cdots, l-1, l+1, l+1, l+2, \cdots, k-1, k-1)\) after the \(((l-1)k+k) = lk\)th step.

Therefore, putting \(l = k-3\), we have \((1, 2, \cdots, k-4, k-2, k-2, k-1, k-1)\) after the \((k-3)k\)th step. So we get \((1, 2, \cdots, k-4, k-3, k-1, k-1, k-1)\) after the \((k-2)k\)th step. Further we have \((1, 2, \cdots, k-4, k-2, k-2, k-2, k)\) after the \(((k-2)k+1)\)th step, \((1, 3, 3, 3, 5, \cdots, k)\) after the \(((k-2)k+k-4)\)th step, hence \((1, 2, \cdots, k-2, k-1, k)\) after the \((k-1)k\)th step. This completes the proof.

We shall next show that for any triangular number \(n = 1+2+\cdots+k\), the partition \((n)\) reaches the consecutive state in \((n-k)\)th steps, where \((n)\) is the next state of the partition \((1, 1, \cdots, 1)\) by Bulgarian operation.

Put \(S_m = 1+2+\cdots+m\) with \(1 \leq m \leq k-1\). We shall show by induction on \(m\) the state after the \(S_m\)th step is \((1, 2, \cdots, m, n-S_m)\).

The state after the 1st step is \((1, n-1)\) and the state after the 3rd step is \((1, 2, n-3)\), so the assertion holds for \(m = 1, 2\).

Suppose \(3 \leq m \leq k-1\). By induction we may have \((1, 2, \cdots, m-1, n-S_{m-1})\) after the \(S_{m-1}\)th step. Then we have \((1, 2, \cdots, m-2, m, n-S_{m-1}-1)\) after the \((S_{m-1}+1)\)th step, \((1, 3, 4, \cdots, m, n-S_m+2)\) after the \((S_{m-1}+m-2)\)th step, so \((1, 2, \cdots, m, n-S_m)\) after the \((S_{m-1}+m) = S_m\)th step.

Hence, putting \(m = k-1\), we have \((1, 2, \cdots, k)\) after the \((n-k)\)th step because \(S_{k-1} = n-k\).

For \(t < S_{k-1}\), the state after the \(t\)th step cannot be consecutive con-
sidering \( n-t > k \). Therefore for the first time we reach the consecutive state after the \((n-k)\)th step.