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長い表記を踏襲したため、コピペで直接入力するのではなく、手入力で入力してください。
VALID INEQUALITIES FOR MIXED INTEGER PROGRAMMING PROBLEMS

Midori Kobayashi
Consider the mixed integer programming problem \((P_M)\)

\[
\begin{align*}
\text{minimize} \quad & z = c^T x \\
\text{subject to} \quad & Ax = b \\
& 0 \leq x \in R^n \\
& x_j \in Z \quad (1 \leq j \leq n_1)
\end{align*}
\]

where \(A\) is an \(R\)-component \(m \times n\) matrix, \(b\) is a vector in \(R^m\), \(c\) is a vector in \(R^n\) and \(n\) and \(n_1\) are integers with \(0 < n_1 < n\). \(R\) and \(Z\) denote the set of all real numbers and the set of all integers, respectively.

Let \(X\) be the set of all feasible solutions:

\[
X = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n ; \ Ax = b, \ x \geq 0, \ x_j \in Z \ (1 \leq j \leq n_1) \right\}.
\]

Let \(a_1, \ldots, a_n\) be the column vectors of \(A\): \(A = (a_1 \ \ldots \ a_n)\), where \(a_j = (a_{1j} \ \ldots \ a_{nj})^T\). We denote by \(V\) the abelian group generated by \(a_1, \ldots, a_{n_1}, b\), i.e.,

\[
V = a_1Z + \ldots + a_{n_1}Z + bZ \subseteq R^n.
\]

Let \(f\) be a real valued function defined on \(V\), satisfying

\[
f(v_1) + f(v_2) \geq f(v_1 + v_2)
\]

for any \(v_1, v_2 \in V\), then \(f\) is called a subadditive function on \(V\).

Let \(\pi\) be a vector in \(R^n\) and \(\pi_0\) be a real number. An inequality \(\pi^T x \geq \pi_0\) is called a valid inequality for \(X\) if every \(x \in X\) satisfies \(\pi^T x \geq \pi_0\).

**Theorem 1.** Let \(f\) be a subadditive function on \(V\) satisfying \(f(0) = 0\). Then, for any \(x \in X\),

\[
\sum_{j=1}^{n_1} f(a_j)x_j + f\left( \sum_{j=n_1+1}^{n} a_jx_j \right) \geq f(b).
\]

Proof. For any \(z_1, \ldots, z_{n_1} \in Z\), we have
\[
\sum_{j=1}^{n_1} f(a_j) Z_j \geq f\left(\sum_{j=1}^{n_1} a_j Z_j\right)
\]

(See the proof of Theorem 1 in [4]).

For any \( x \in X \), we have \( Ax = b \), i.e.,

\[
\sum_{j=1}^{n_1} a_j x_j + \sum_{j=n_1+1}^{n} a_j x_j = b,
\]

so

\[
f\left(\sum_{j=1}^{n_1} a_j x_j + \sum_{j=n_1+1}^{n} a_j x_j\right)
\]

\[
\leq f\left(\sum_{j=1}^{n_1} a_j x_j\right) + f\left(\sum_{j=n_1+1}^{n} a_j x_j\right)
\]

\[
\leq \sum_{j=1}^{n_1} f(a_j) x_j + f\left(\sum_{j=n_1+1}^{n} a_j x_j\right),
\]

hence

\[
\sum_{j=1}^{n_1} f(a_j) x_j + f\left(\sum_{j=n_1+1}^{n} a_j x_j\right) \geq f(b).
\]

For an integer \( l(>1) \), we denote by \( Z_l \) a complete residue system modulus \( l : Z_l = \{0, 1, \ldots, l-1\} \). We define the function \( f_i : Z \rightarrow Z_l \) as follows: for any \( a \in Z \), there exists the integer \( b \) such that \( a \equiv b \) (mod \( l \)) and \( b \in Z_l \); we define \( f_i(a) = b \). Then \( f_i \) is a subadditive function on \( Z \) and on any subset of \( Z \).

Further we define the function \( p_i : Z^l \rightarrow Z \) as follows: for any \( (a_j)_{1 \leq j \leq l} \in Z^l \), \( p_i((a_j)) = a_i \). We put \( f_i = f_i \circ p_i \); then \( f_i \) is a subadditive function on \( Z^l \) and on any subset of \( Z^l \).

Now, we consider the mixed integer programming problem \((P_M)\). We assume all of the components of \( A \) and \( b \) are integers. This is actually equivalent to assuming the components rational.
**Theorem 2.** Let $i$ be an integer with $1 \leq i \leq m$. If $a_{ij} \geq 0$ for any $j(n_1 + 1 \leq j \leq n)$, then

$$
\sum_{j=1}^{n_1} f_i(a_{ij})x_j + \sum_{j=n_1+1}^{n} a_{ij}x_j \geq f_i(b_i)
$$

is a valid inequality for $X$.

**Proof.** We should notice that $V = a_1Z + \cdots + a_nZ + bZ \subseteq Z^n$. By Theorem 1, we have for any $x \in X$,

$$
\sum_{j=1}^{n_1} f_i'(a_j)x_j + f_i'(\sum_{j=n_1+1}^{n} a_{ij}x_j) \geq f_i'(b_i),
$$

since $f_i'$ is a subadditive function on $V$. So it follows

$$
\sum_{j=1}^{n_1} f_i(a_{ij})x_j + f_i(\sum_{j=n_1+1}^{n} a_{ij}x_j) \geq f_i(b_i).
$$

We have

$$
f_i(\sum_{j=n_1+1}^{n} a_{ij}x_j) \leq \sum_{j=n_1+1}^{n} a_{ij}x_j
$$

by our assumption. Therefore

$$
\sum_{j=1}^{n_1} f_i(a_{ij})x_j + \sum_{j=n_1+1}^{n} a_{ij}x_j \geq f_i(b_i)
$$

is a valid inequality for $X$.

**Example.**

- **minimize** $z = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$
- **subject to** $3x_1 + x_2 + 5x_3 - 2x_4 - 4x_5 = 15$
  $5x_1 + x_2 - x_3 + 4x_4 - 5x_5 = 10$
  $0 \leq x_j \in Z$ (1 $\leq j \leq 4$)
  $0 \leq x_5 \in R$

The solution $x_1 = \frac{65}{28}, x_2 = 0, x_3 = \frac{45}{28}, x_4 = 0, x_5 = 0$ with $z = \frac{50}{7}$ is optimal for the associated linear programming problem without the
integrality restrictions.

Put \( A = \begin{pmatrix} 3 & 1 & 5 & -2 & -4 \\ 5 & 1 & -1 & 4 & -5 \end{pmatrix}, \quad b = \begin{pmatrix} 15 \\ 10 \end{pmatrix} \) and \( x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \).

We multiply both sides of the equation \( Ax = b \) on the left by the matrix \( \begin{pmatrix} 3 & 1 \\ 5 & 1 \end{pmatrix}^{-1} \). Then we have

\[
\frac{1}{2} \begin{pmatrix} -2 & 0 & 6 & -6 & 1 \\ 0 & -2 & -28 & 22 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 5 \\ -45 \end{pmatrix},
\]

so

\[
-2x_1 + 6x_3 - 6x_4 + x_5 = 5
\]

\[
-2x_2 - 28x_3 + 22x_4 + 5x_5 = -45.
\]

The coefficients of \( x_5 \) are non-negative, so we obtain the following valid inequalities by Theorem 2:

- for \( i=1 \) and \( l=2 \) \( x_5 \geq 1 \)
- for \( i=1 \) and \( l=3 \) \( x_1 + x_5 \geq 2 \)
- for \( i=1 \) and \( l=4 \) \( 2x_1 + 2x_3 + 2x_4 + x_5 \geq 1 \)
- for \( i=1 \) and \( l=6 \) \( 4x_1 + x_5 \geq 5 \)
- for \( i=2 \) and \( l=2 \) \( 5x_5 \geq 1 \)
- for \( i=2 \) and \( l=4 \) \( 2x_2 + 2x_4 + 5x_5 \geq 3 \)
- for \( i=2 \) and \( l=6 \) \( 4x_2 + 2x_3 + 4x_4 + 5x_5 \geq 3 \)
- for \( i=2 \) and \( l=7 \) \( 5x_2 + x_4 + 5x_5 \geq 4 \)

The first, 5th, 6th and the last of these inequalities are cuts.
REFERENCES


