EQUIVALENCE THEOREMS FOR OLIGOPOLISTIC MARKETS AND OLIGOPOLISTIC MIXED MARKETS

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First we prove the equivalence theorem for oligopolistic markets without a continuum of traders. It has been obtained by improving the method of proof by 'splitting the atoms into a continuum of traders' in Greenberg and Shitovitz (1986). Second, using this method, we prove the equivalence theorem for oligopolistic mixed markets. By splitting the atoms, Greenberg and Shitovitz (1986) proves the theorem under the condition that there exists a minimal atom $\overline{A}$ such that $x(A) \geq_A x(\overline{A})$ for any atom $A$. But we generally prove the theorem without the above condition.

1. Shitovitz's paper (1973) is interesting and important. However his proof is complicated and long. Greenberg and Shitovitz (1986) very simply proves it, by splitting the atoms into a continuum of traders, under the condition that there exists a minimal atom $\overline{A}$ such that $x(A) \geq_A x(\overline{A})$ for any atom $A$. It states that 'Choose an atom $\overline{A} \in T_1$ such that $x(A) \geq_A x(\overline{A})$ for all $A \in T_1$, ⋮' (Greenberg and Shitovitz (1986, p. 81)). But I have never known that there generally exists such an atom. I think that it may be a hypothesis for the convenience of proof. By improving the method of proof by 'splitting the atoms', we can generally prove the theorem without the condition.

By examining the proof, we can find out that the continuum of traders is essentially unnecessary for the proof. Therefore we can prove the
equivalence theorem for oligopolistic markets without a continuum of traders.

The focuses of this paper are in the proof of Theorem $A$, (ii) and Theorem $C$, (ii) in the following section.

Unlike Greenberg and Shitovitz, we do not use Lyapunov’s theorem on the atoms splitted into a continuum of traders. Rather, we use some one to one transformation in measure theory. By this, ‘splitting the atoms’ is successful. This is the key of this paper.

2. Following the well-known market model, let $(T, \Sigma, \mu)$ be a measure space of economic agents, where $T$ denotes the set of traders, $\Sigma$ denotes a $\sigma$-field of subsets of $T$ (the family of all possible coalitions) and $\mu$ denotes a positive measure. Denote by $T_0 \subset T$ the atomless part. In Theorem $A$ and Theorem $B$ we assume $T_0$ is empty set and in Theorem $C$ and Theorem $D$ we assume $T_0$ is not $\mu$-null. The set of atoms is, therefore, $T_1 = T \setminus T_0$.

Let $A_1, A_2, \ldots$ be an enumeration of all atoms, i.e., let $T_1 = \{A_i / i \in I\}$. Assume that $I$ is at least two and countable set. Let $a_0 < a_1 < a_2 < \cdots$ where $a_0 = 0$ if $T_0$ is empty set, and $a_0 = 1$ if $T_0$ is not $\mu$–null and without loss of generality assume $\mu(T_0) = 1$ and assume $T_0$ is $[0, 1)$. Assume $a_i - a_{i-1} = \mu(A_i)$ and assume the atom $A_i$ is $[a_{i-1}, a_i)$ for $i \in I$.

An assignment is an integrable function from $T$ to $\Omega$–the consumption set of each trader— which for simplicity is assumed to coincide with the non–negative orthant of $\mathbb{R}^n$. Each trader $t \in T$ is endowed with an initial endowment $i(t) \in \Omega$, and has a preference ordering $>_t$ over $\Omega$ which is measurable, continuous and strictly monotonic. Moreover, it is assumed that all atoms are of the same type. That is, their initial endowment density is the same and all atoms have the same quasi–concave utility function over densities in $\Omega$. Note, however, that atoms in $T_1$ may well have different measures.

An allocation is an assignment $x$ for which $\int_T x = \int_T i$. The core is the set
of all allocations for which there exist no coalition $S \in \Sigma$, $\mu(S) > 0$, and an assignment $y$ such that $\int_S y = \int_S i$ and $y(t) > x(t) \text{ a.e. in } S$. A price vector $p$ is an $n$-tuple of non-negative real numbers, not all of which vanish. A competitive equilibrium (c.e.) is a pair $(p, x)$ consisting of a price vector $p$ and an allocation $x$, such that for almost all traders $t$, $x(t)$ is maximal with respect to $>_i$ in $t'$s budget set $B_p(t) = \{x : p \cdot x \leq p \cdot i(t)\}$. A competitive allocation $x$ is an allocation for which there exists a price vector $p$ such that $(p, x)$ is a competitive equilibrium.

We transform the original market to a market with an atomless set of traders $(T^*, \Sigma^*, \mu^*)$, where $T^* = \{0, a_0 + \mu(T_1)\}$ and when $T_0$ is not $\mu$-null, $\Sigma^*$ and $\mu^*$ are obtained by the direct sum of $\Sigma$ and $\mu$ restricted to $T_0$ and the Lebesgue atomless measure space over $[a_0, a_0 + \mu(T_1))$. Define the split of an atom $A_i$, denoted $A_i^*$, to be a continuum of small traders such that $\mu(A_i) = \mu^*(A_i^*)$ and every trader $t \in A_i^*$ is of the same type as $A_i$. Then $T^*$ denotes that market obtained from $T$ when all atoms in $T$ have been split. Let $T_1^* = T^* \setminus T_0 = \cup_{i \in I} [a_i, a_0 + \mu(T_1)) = \cup_{i \in I} [a_i-1, a_i]$.

In order to formally state our main results, namely that the cores of the two economies $(T, \Sigma, \mu)$ and $(T^*, \Sigma^*, \mu^*)$ are 'equivalent', we need to associate with each allocation in one economy, an allocation in the other economy. So, let $x^*$ be an allocation in $T^*$. We define the allocation $x = \phi(x^*)$ in $T$ by

$$x(t) = x^*(t) \quad \forall t \in T_0,$$

$$x(A_i) = \frac{\int_{A_i} x^* d\mu^*}{\mu^*(A_i)} \quad \forall A_i \in T_1.$$

Similarly, for the opposite direction, let $x$ be an allocation in $T$. We define the allocation $x^* = \phi(x)$ in $T^*$ by

$$x^*(t) = x(t) \quad \forall t \in T_0,$$

$$x^*(t) = x(A_i) \quad \forall t \in A_i^* \text{ where } A \in T_1.$$

The following lemma is the key of this paper. Unlike Greenberg and
Shitovitz (1986), we do not use Lyapunov's theorem on the atoms splitted in the proof of Theorem A, (ii), and Theorem C, (ii). Rather we use the following lemma.

Lemma. Let \( M^* \) and \( N^* \) be the coalitions such that \( \mu^*(M^*) > 0 \) and \( \mu^*(N^*) > 0 \). Then there exists the one to one transformation \( V \) from \( M^* \) onto \( N^* \) such that

\[ V(E^*) \text{ is a coalition} \]

and

\[ \mu^*(V(E^*)) = \mu^*(N^*) \cdot \mu^*(E^*) / \mu^*(M^*) \]

whenever \( E^* \) is a coalition and \( E^* \subseteq M^* \).

Proof. There exists the one to one isomorphic transformation \( V_1 \) from \( M^* \) onto \([0, \mu^*(M^*)]\). Let \( V_2 \) be the one to one transformation from \([0, \mu^*(M^*)]\) onto \([\mu^*(M^*), \mu^*(M^*) + \mu^*(N^*)]\) such that \( V_2(t) = \mu^*(N^*) \cdot t / \mu^*(M^*) + \mu^*(M^*) \) for \( t \in [0, \mu^*(M^*)] \). There exists the one to one isomorphic transformation \( V_3 \) from \( N^* \) onto \([\mu^*(M^*), \mu^*(M^*) + \mu^*(N^*)]\). Let \( V = V_3 \circ V_2 \circ V_1 \). This completes the proof. Q. E. D.

Theorem A. Assume \( T_0 \) is empty set. Then \( \text{Core}(T^*) \) is equivalent to \( \text{Core}(T) \). That is,

(i) \( x^* \in \text{Core}(T^*) \) implies \( x = \phi(x^*) \in \text{Core}(T) \),

(ii) \( x \in \text{Core}(T) \) implies \( x^* = \phi(x) \in \text{Core}(T^*) \).

Proof of (i). (Greenberg and Shitovitz (1986)).

Assume, in negation, that there exists \( x^* \in \text{Core}(T^*) \) such that \( x = \phi(x^*) \notin \text{Core}(T) \). Thus, there exists an allocation \( y \) and a coalition \( S \subseteq T \) in \( \Sigma \) that block \( x \). But then the coalition \( S^* \subseteq T^* \) in \( \Sigma^* \), derived from \( S \) by splitting the atoms, blocks \( x^* \) via the allocation \( y^* = \phi(y) \). This contradicts \( x^* \in \text{Core}(T^*) \).

Hence, (i) holds. Q. E. D.

The following proof is the essence of this paper.

Proof of (ii). Assume, in negation, that there exists \( x \in \text{Core}(T) \) such that \( x^* = \phi(x) \notin \text{Core}(T^*) \). Choose any two atoms \( A_{i_1} \) and \( A_{i_2} \), and assume
that \( x(A_{i2}) \geq A_{i1}x(A_{i1}) \). Let

\[
J = \{ i / x(A_{i1}) > A_{i1}x(A_{i1}) \text{ for } i \in I \} \cup \{ i_1 \}.
\]

Then \( J \subset I \) and \( J \) is a countable set. Let \( \bar{A}^* = \bigcup_{j \in J} A_j^* = \bigcup_{j \in J} (a_{j-1}, a_j) \) and \( \alpha = \mu^*(\bar{A}^*) \).

Since \( x^* \in \text{Core}(T^*) \) and \( \mu^*(T^*) > \alpha \), by Vind (1972), there exists a coalition \( S^* \subset T^* \) in \( \Sigma^* \), with \( \mu^*(S^*) = \alpha \), that blocks \( x^* \) via the allocation \( y^* \). For each \( j \in J \), by Lemma, there exists the one to one isomorphic transformation \( V_j \) from \( S^* \cap [a_{j-1}, a_j] \) onto \( [a_{j-1}, b_j] \) such that \( V_j(S^* \cap [a_{j-1}, a_j]) = [a_{j-1}, b_j] \) and \( \mu^*(V_j(E^*)) = \mu^*(E^*) \) for any coalition \( E^* \subseteq S^* \cap [a_{j-1}, a_j] \) where \( b_j = a_{j-1} + \mu^*(S^* \cap [a_{j-1}, a_j]) \), and define \( y^{**}(V_j(t)) = y^*(t) \) for \( t \in S^* \cap [a_{j-1}, a_j] \). Then

\[
\int_{[a_{j-1}, b_j]} y^{**} = \int_{S^* \cap [a_{j-1}, a_j]} y^* \quad (1)
\]

\[
\int_{[a_{j-1}, b_j]} i = \int_{S^* \cap [a_{j-1}, a_j]} i \quad (2)
\]

By Lemma, there exists the one to one isomorphic transformation \( V \) from \( S^* \setminus \bar{A}^* \) onto \( \bigcup_{j \notin J} (b_j, a_j) \) such that \( \mu^*(V(E^*)) = \mu^*(E^*) \) for any coalition \( E^* \subseteq S^* \setminus \bar{A}^* \), and define \( y^{**}(V(t)) = y^*(t) \) for \( t \in S^* \setminus \bar{A}^* \). Then

\[
\int_{\bigcup_{j \notin J} (b_j, a_j)} y^{**} = \int_{S^* \setminus \bar{A}^*} y^* \quad (3)
\]

\[
\int_{\bigcup_{j \notin J} (b_j, a_j)} i = \int_{S^* \setminus \bar{A}^*} i \quad (4)
\]

By (1), (2), (3) and (4),

\[
\int_{\bigcup_{j \notin J} A_j^*} y^{**} = \int_{S^*} y^* = \int_{S^*} i = \int_{\bigcup_{j \notin J} A_j^*} i \quad (5)
\]

Define

\[
y(A_j) = \int_{[a_{j-1}, a_j]} y^{**} \frac{d\mu^*(t)}{(a_j-a_{j-1})_1} \quad \text{for } j \in J.
\]

Since \( x^*(t) = x(A_j) \forall t \in A_j^* \), for \( j \in J \), and the preferences of all atoms are convex, by the choice of \( A_j \) and (5), it follows that coalition \( \{ A_j : j \in J \} \) in \( (T, \Sigma, \mu) \) blocks \( x \) via \( y \). But this contradicts the assumption that \( x \in \text{Core}(T) \).
Theorem B. Assume $T_o$ is empty set. Then the core coincides with the set of competitive allocations.

Proof (Greenberg and Shitovtitz (1986)). By Aumann (1964), the set of Walrasian allocations for $T^*$, $W(T^*)$, coincides with Core$(T^*)$. Now, $x^* \in W(T^*)$ implies that a.e. in $T^*$, $x^*(t)$ is a maximal element (w.r.t. t's preference ordering) in t's budget set. Moreover, since allocations in $T_1$ have the same quasi-concave utility function, it follows that $(1 / \mu^*(A_i^*)) \cdot \int_{A_i^*} x^*(t)$ is a maximal element in $A_i^*$'s budget set, for all $A_i^*$, $i \in I$.

Therefore, the set of Walrasian allocations in T, $W(T)$, is equivalent to $W(T^*)$, in the sense that

$$x^* \in W(T^*) \implies x = \phi(x^*) \in W(T),$$

and

$$x \in W(T) \implies x^* = \Psi(x) \in W(T^*).$$

By Theorem A, Core$(T)$ coincides with $W(T)$. Q.E.D.

Theorem C. Assume $T_o$ is $[0, 1)$. Then Core$(T^*)$ is equivalent to Core$(T)$. That is,

(i) $x^* \in$ Core$(T^*) \implies x = \phi(x^*) \in$ Core$(T)$,

(ii) $x \in$ Core$(T) \implies x^* = \Psi(x) \in$ Core$(T^*)$.

Proof of (i). Similar to the proof of Theorem A, (i).

The following proof is the essence of this paper.

Proof of (ii). Assume, in negation, that there exists $x \in$ Core$(T)$ such that $x^* = \Psi(x) \notin$ Core$(T^*)$. Choose any two atoms $A_{i1}$ and $A_{i2}$, and assume that $x(A_{i2}) \geq \alpha = \mu^*(A^*)$. Choose any two atoms $A_{i1}$ and $A_{i2}$, and assume that $x(A_{i2}) \geq \alpha = \mu^*(A^*)$. Let $J = \{i / x(A_{i1}) > x(A_i) \text{ for } i \in I \} \cup \{i_1\}$.

Then $J \subseteq I$ and J is a countable set. Let $\bar{A}^* = \bigcup_{j \in J} A_{i1}^* = \bigcup_{j \in J} (a_{i1} - 1, a_j)$, and $\alpha = \mu^*(\bar{A}^*)$. Since $x^* \notin$ Core$(T^*)$ and $\mu^*(T^*) > 1 + \alpha$, by Vind (1972), there exists a coalition $S^* \subseteq T^*$ in $\Sigma^*$, with $\mu^*(S^*) = 1 + \alpha$, that blocks $x^*$ via the allocation $y^*$. Then, $\mu^*(S^* \cap T_o) \leq \mu(T_o) = 1$ and therefore $\mu^*(S^* \cap T_1) \leq \alpha$. For
each \( j \in J \), by Lemma, there exists the one to one transformation \( V_j \) from \( S^* \cap (a_{j-1}, a_j) \) onto \( (a_{j-1}, b_j) \) such that \( V_j(S^* \cap (a_{j-1}, a_j)) = (a_{j-1}, b_j) \) and 
\[
\mu^*(V_j(E^*)) = (\alpha / \mu^*(S^* \cap T_1)) \cdot \mu^*(E^*) \quad \text{for all coalition } E^* \subseteq S^* \cap (a_{j-1}, a_j)
\]
where \( b_j = a_{j-1} + (\alpha / \mu^*(S^* \cap T_1)) \).

\( \mu^*(S^* \cap (a_{j-1}, a_j)) \), and define \( y^{**}(V_j(t)) = y^*(t) \) for \( t \in S^* \cap (a_{j-1}, a_j) \).

Then
\[
\int_{(a_{j-1}, b_j)} y^{**} = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S^* \cap (a_{j-1}, a_j)} y^*
\]  
and
\[
\int_{(a_{j-1}, b_j)} i = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S^* \cap (a_{j-1}, a_j)} i
\]

By Lemma, there exists the one to one transformation \( V \) from \( (S^* \cap T_1) \setminus A^* \) onto \( \bigcup_{j \in J} (b_j, a_j) \) such that \( \mu^*(V(E^*)) = (\alpha / \mu^*(S^* \cap T_1)) \cdot \mu^*(E^*) \) for any coalition \( E^* \subseteq (S^* \cap T_1) \setminus A^* \), and define \( y^{**}(V(t)) = y^*(t) \) for \( t \in (S^* \cap T_1) \setminus A^* \).

Then
\[
\int_{\bigcup_{j \in J} (b_j, a_j)} y^{**} = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{(S^* \cap T_1) \setminus A^*} y^*
\]
and
\[
\int_{\bigcup_{j \in J} (b_j, a_j)} i = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{(S^* \cap T_1) \setminus A^*} i
\]

Define a non-atomic vector measure on \( S^* \cap T_o \) by
\[
m(S) = \left( \int_S i, \int_S y^* \right)
\]
for any coalition \( S \subseteq S^* \cap T_o \). By Lyapunov’s theorem, there exists a coalition \( S_o \subseteq S^* \cap T_o \) such that
\[
m(S_o) = (\alpha / \mu^*(S^* \cap T_1)) \cdot m(S^* \cap T_o).
\]

Define \( y^{**}(t) = y^*(t) \) for \( t \in S_o \). Then
\[
\int_{S_o} y^{**} = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S^* \cap T_o} y^*
\]
and
\[
\int_{S_o} \gamma^*(i) = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S \cap T_o} \gamma^*
\]  

(11)

By (6), (7), (8), (9), (10) and (11),
\[
\int_{\bigcup_{j \in J} A_j^* \cup S_o} \gamma^* = (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S^*} \gamma^*
\]
\[
= (\alpha / \mu^*(S^* \cap T_1)) \cdot \int_{S^*} \int_{\bigcup_{j \in J} A_j^* \cup S_o} \gamma^*
\]  

(12)

Define
\[y(t) = y^*(t) \quad \text{for } t \in S_o,\]
and
\[y(A_j) = \int_{(a_j, a_j)} y^* d\mu^*(t)/(a_j - a_{j-1}) \quad \text{for } j \in J.\]

Since \(x^*(t) = x(A_j) \quad \forall t \in A_j^*\), for \(j \in J\), and the preferences of all atoms are convex, by the choice of \(A_j\) and (12) it follows that coalition \(\{A_j \mid j \in J\} \cup S_o\) in \((T, \Sigma, \mu)\) blocks \(x\) via \(y\). But this contradicts the assumption that \(x \in \text{Core}(T)\).

Q. E. D.

Theorem D. Assume \(T_o = (0, 1)\). Then the core coincides with the set of competitive allocations.

Proof. Similar to the proof of Theorem B.

Q. E. D.

References.

