Infinite 2-class field towers of some imaginary quadratic number fields

by

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Abstract. By the results of Golod-Shafarevich and Vinberg-Gaschütz, the 2-class field tower of an imaginary quadratic number field is infinite if the 2-rank of the ideal class group is greater than or equal to 5. In this paper, we study the case where the 2-class rank is equal to 4 and the field has only one negative prime discriminant. We prove that the 2-class field tower of such a field is infinite, except one type of Rédei matrix.

1. Introduction. Let \( K \) be an imaginary quadratic number field with discriminant \( d \), and \( C_K \) denote the ideal class group of \( K \). We mean by the 2-class field tower of \( K \) the sequence of fields \( K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots \), where \( K_{i+1} \) is the Hilbert 2-class field \( (i.e. \) the maximal unramified abelian 2-extension) of \( K_i \). If \( K_{i+1} \neq K_i \) for all \( i \), then we say that the 2-class field tower of \( K \) is infinite.

By the results of Golod-Shafarevich[3] and Vinberg-Gaschütz[12, 15], the 2-class field tower of \( K \) is infinite if 2-rank \( C_K \geq 5 \). On the other hand, Koch[6] and Hajir[4, 5] proved that the 2-class field tower of \( K \) is infinite if 4-rank \( C_K \geq 3 \). When 2-rank \( C_K = 3 \) and 4-rank \( C_K = 0 \), there are some examples of infinite families of \( K \) with infinite (resp. finite) 2-class field towers[4, 7]. However, when 2-rank \( C_K = 4 \), no example of \( K \) with finite 2-class field tower has ever been known. It has been conjectured[9] that the 2-class field tower of such a \( K \) is always infinite. In this direction, Benjamin[1, 2] proved that the 2-class field tower of \( K \) is infinite if 2-rank \( C_K = 4 \) and 4-rank \( C_K = 2 \), except some type of Rédei matrix of \( K \).

In this paper, we study the case where 2-rank \( C_K = 4 \) and exactly one negative prime discriminant divides \( d \), and prove that the 2-class field tower of such a \( K \) is infinite, except one type of Rédei matrix of \( K \).

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To prove our theorem, we use Martinet’s inequalities[9] and their corollaries. We also use some properties of Rédei matrices[10, 11, 13, 14]. A similar problem for real quadratic number fields is treated by Maire[8], by a different method.

2. Martinet’s inequalities and their corollaries. Let $K$ be an imaginary quadratic number field.

Martinet’s inequality (general case)[9]. Let $E/F$ be a quadratic extension of number fields. We denote by $r_1$ (resp. $r_2$) the number of real (resp. imaginary) places of $F$. We also denote by $t$ (resp. $\rho$) the number of finite (resp. infinite) places of $F$ which ramify in $E$. If the inequality

$$t \geq r_1 + r_2 - \rho + 3 + 2\sqrt{2(r_1 + r_2)} - \rho + 1$$

holds, then the 2-class field tower of $E$ is infinite.

Martinet’s inequality I. Let $F$ be a totally real number field of degree $n$, and $E$ a totally imaginary quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If the inequality

$$t \geq 3 + 2\sqrt{n + 1}$$

holds, then the 2-class field tower of $E$ is infinite.

Proof. Since $r_1 = \rho = n$ and $r_2 = 0$, the assertion follows from the general case of Martinet’s inequality.

Corollary 1. Let $F$ be a real quadratic number field. Suppose that four rational primes split in $F$ and ramify in $K$, or that a rational prime remains prime in $F$ and three other rational primes split in $F$ and these four rational primes ramify in $K$, then the 2-class field tower of $E = FK$ is infinite.

Proof. Since $n = 2$ and $t \geq 7 \geq 3 + 2\sqrt{2 + 1} = 6.464\cdots$ in these cases, the 2-class field tower of $E = FK$ is infinite by Martinet’s inequality I.

Corollary 2. Let $F$ be a totally real number field of degree 4. Suppose that two rational primes split completely in $F$ and ramify in $K$, or that
a rational prime splits completely in $F$ and two other rational primes are unramified and split into at least two primes in $F$ and these three rational primes ramify in $K$, then the 2-class field tower of $E = FK$ is infinite.

Proof. Since $n = 4$ and $t \geq 8 \geq 3 + 2\sqrt{4 + 1} = 7.472 \cdots$ in these cases, the 2-class field tower of $E = FK$ is infinite by Martinet’s inequality I.

Martinet’s inequality II. Let $F$ be a totally imaginary number field of degree $n$, and $E$ a quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If the inequality

$$t \geq \frac{n}{2} + 3 + 2\sqrt{n + 1}$$

holds, then the 2-class field tower of $E$ is infinite.

Proof. Since $r_1 = \rho = 0$ and $r_2 = \frac{n}{2}$, the assertion follows from the general case of Martinet’s inequality.

**Corollary 3.** Let $F$ be an imaginary quadratic number field. Suppose that four rational primes split in $F$ and ramify in $K$, then the 2-class field tower of $E = FK$ is infinite.

Proof. Since $n = 2$ and $t \geq 8 \geq \frac{2}{2} + 3 + 2\sqrt{2 + 1} = 7.464 \cdots$, the 2-class field tower of $E = FK$ is infinite by Martinet’s inequality II.

3. The case with one negative prime discriminant. Let $K$ be an imaginary quadratic number field with discriminant $d$. First we recall some properties of Rédei matrices of quadratic number fields[10, 11, 13, 14].

A rational integer is called a discriminant if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a prime discriminant. Prime discriminants are denoted by $p^* = (-1)^{\frac{d-1}{2}}p$ (if $p$ is an odd prime), or $p^* = -4$, 8 or $-8$ (if $p$ is equal to 2). Let $d = p_1^*p_2^*\cdots p_t^*$ be the unique factorization of $d$ into a product of prime discriminants. By genus theory, we have 2-rank $C_K = t - 1$.

Using Kronecker symbols $(\frac{D}{p})$, where $D$ is a discriminant and $p$ is a prime number satisfying $p \nmid D$, we define the Rédei matrix $R_K = (a_{ij}) \in M_{4t}({\mathbb{Z}}/2{\mathbb{Z}})$ of $K$ by

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By the definition of the Kronecker symbol, we have \( a_{ij} = 0 \) if and only if the rational prime \( p_j \) splits in \( Q(\sqrt{p_i^*}) \). Note that the sum of all row vectors of \( R_K \) is equal to the zero vector \( 0 \) in \( (\mathbb{Z}/2\mathbb{Z})^t \) so that rank \( R_K \leq t - 1 \) and the solution space \( X \) of the linear equations \( xR_K = 0 \) \( (x \in (\mathbb{Z}/2\mathbb{Z})^t) \) contains the vector \( 1 = (1, 1, \cdots, 1) \). By the results of Rédei and Rédei-Reichardt, we have 4-rank \( C_K = t - 1 - \text{rank} \ R_K \).

In the case where \( p_i^* \neq -4 \) and \( p_j^* \neq -4 \), we have \( a_{ij} = a_{ji} \) if and only if \( p_i^* > 0 \) or \( p_j^* > 0 \), by the quadratic reciprocity law. Therefore, if exactly one negative prime discriminant \( (\neq 4) \) divides \( d \), then \( R_K \) is a symmetric matrix.

**Theorem.** Let \( K \) be an imaginary quadratic number field with discriminant \( d \). Suppose that 2-rank \( C_K = 4 \) and exactly one negative prime discriminant divides \( d \). Let \( d = p_1^* p_2^* p_3^* p_4^* p_5^* \) \( (p_i^* < 0) \) be the unique factorization of \( d \) into a product of prime discriminants. Then the 2-class field tower of \( K \) is infinite, except the case where

\[
R_K = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

by changing the order of \( p_i \)'s \( (2 \leq i \leq 5) \). In the exceptional case, \( p_1^* \neq -4 \) and the 4-rank of \( C_K \) is equal to 0.

**Proof.** First, suppose that \( p_i^* = 4 \), then we have \( p_j \equiv 1 \pmod{4} \) for any \( j \) \( (2 \leq j \leq 5) \). Put \( F = Q(\sqrt{p_i^*}) = Q(\sqrt{-T}) \), then four rational primes \( p_j \) \( (2 \leq j \leq 5) \) split in \( F \) and ramify in \( K \). Hence, the 2-class field tower of \( E = FK = K(\sqrt{-T}) \) is infinite by Corollary 3. Since \( E/K \) is an unramified 2-extension, the 2-class field tower of \( K \) is also infinite.
In the following, we assume that $p_1^* 
eq -4$. Therefore, $R_K$ is a symmetric matrix. For each Rédei matrix $R_K$, if we could find a subfield $F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j})$, $\mathbb{Q}(\sqrt{p_i}, \sqrt{p_j\sqrt{p_k}})$ or $\mathbb{Q}(\sqrt{p_i\sqrt{p_j}}, \sqrt{p_j\sqrt{p_k}})$ ($i, j, k \in \{2, 3, 4, 5\}$) of the genus field $\mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_5})$ of $K$ which satisfies the condition of Corollary 2, then the 2-class field tower of $E = FK$ would be infinite. Since $E/K$ is an unramified 2-extension, we conclude that the 2-class field tower of $K$ is also infinite, in those cases.

First, suppose that there exists a column vector $a_j = (a_{ij})$ ($1 \leq j \leq 5$) of $R_K$ for which at least two of $a_{ij}$’s ($2 \leq i \leq 5$, $i \neq j$) are 0, then, assuming that $a_{ij} = a_{kj} = 0$ ($i \neq j$, $k \neq j$), we put $F = \mathbb{Q}(\sqrt{p_i}, \sqrt{p_j})$. Since the rational prime $p_j$ splits completely in $F$ and ramifies in $K$, and two rational primes $p_i, p_m$ ($\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$) are unramified and split into at least two primes in $F$ and ramify in $K$, the 2-class field tower of $E = FK$ is infinite by Corollary 2. Hence, the 2-class field tower of $K$ is also infinite.

In the following, we assume that at most one of $a_{ij}$’s ($2 \leq i \leq 5$, $i \neq j$) is 0 for each column vector $a_j = (a_{ij})$ ($1 \leq j \leq 5$) of $R_K$.

(i) The case where one of $a_{i1}$’s ($2 \leq i \leq 5$) is 0: In this case, we may assume that $a_{21} = 0$ and $a_{31} = a_{41} = a_{51} = 1$ without loss of generality. If $a_{32} = a_{42} = a_{52} = 1$, then we put $F = \mathbb{Q}(\sqrt{p_3p_4}, \sqrt{p_3p_5})$. Since $\left(\frac{p_3p_4}{p_j}\right) = (-1)(-1) = 1$ for each $j \in \{1, 2\}$ and $i, k \in \{3, 4, 5\}$, two rational primes $p_1$ and $p_3$ split completely in $F$ and ramify in $K$. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of $K$ is also infinite. On the other hand, if one of $a_{i2}$’s ($3 \leq i \leq 5$) is 0, then we may assume that $a_{32} = 0$ and $a_{42} = a_{52} = 1$ without loss of generality. So, we have $a_{23} = 0$ and $a_{43} = a_{53} = 1$ by our assumption and

$$R_K = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & *
\end{pmatrix},$$

where the asterisks “*” mean 0 or 1. We put $F = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_4p_5})$. Since $\left(\frac{p_2}{p_j}\right) = 1$ and $\left(\frac{p_4p_5}{p_j}\right) = (-1)(-1) = 1$ for $j \in \{1, 3\}$, two rational primes $p_1$ and $p_3$ split completely in $F$ and ramify in $K$. Therefore, the 2-class field tower of $E = FK$ is infinite by Corollary 2, and the 2-class field tower of $K$
is also infinite.

(ii) The case where \( a_{21} = a_{31} = a_{41} = a_{51} = 1 \): If there exists a column vector \( a_j = (a_{ij}) \) (\( 2 \leq j \leq 5 \)) of \( R_K \) satisfying \( a_{ij} = 1 \) for all \( i \) (\( 2 \leq i \leq 5, \ i \neq j \)), then we put \( F = \mathbb{Q}(\sqrt{p_k p_l}, \sqrt{p_k p_m}) \) \( \{j, k, l, m\} = \{2, 3, 4, 5\} \). In this case, as in the first half of the case (i), we see that two rational primes \( p_1 \) and \( p_j \) split completely in \( F \) and ramify in \( K \). Therefore, the 2-class field tower of \( E = FK \) is infinite by Corollary 2, and the 2-class field tower of \( K \) is also infinite. However, if there exists no such column vector \( a_j \) (\( 2 \leq j \leq 5 \)) of \( R_K \), then we cannot find an appropriate field \( F \) which satisfies the condition of Martinet’s inequality. In this case, we have \( a_{23} = a_{32} = a_{45} = a_{54} = 0 \), by changing the order of \( p_i \)’s. So, \( R_K \) is as described in the assertion of our Theorem. This completes the proof of Theorem.

Remark 1. In Theorem 1 of [1], Benjamin classified the case with only one negative prime discriminant \( \neq -4 \) and 2-rank \( C_K = 4 \) into 32 types, by using “Kronecker symbol configurations”. Among them, the infiniteness of the 2-class field tower remained unsettled for 5 types. Actually, there are two more Kronecker symbol configurations \( (\frac{p_1}{p_3}) = (\frac{p_4}{p_3}) = (\frac{p_2}{p_3}) = 1 \) with 4-rank \( C_K = 0 \), and \( (\frac{p_2}{p_3}) = (\frac{p_4}{p_3}) = (\frac{p_3}{p_5}) = (\frac{p_4}{p_5}) = -1 \) with 4-rank \( C_K = 2 \). The numbers of Rédei matrices (= the numbers of Kronecker symbol configurations) with given 4-rank are as follows:

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{4-rank } C_K & 4 & 3 & 2 & 1 & 0 \\
\text{total} & & & & & \\
\hline
\text{\# of Rédei matrices} & 1 & 2 & 8 & 10 & 13 \\
\hline
\end{array}
\]

In our Theorem, we showed the infiniteness of the 2-class field tower for 33 types, except the third case of Theorem 1(C) in [1] where \( p_4 \) is negative \( \neq -4 \) and \( (\frac{p_2}{p_5}) = (\frac{p_4}{p_5}) = 1 \).

Examples. The following are some examples of imaginary quadratic number fields with only one negative prime discriminant \( \neq -4 \), 2-rank \( C_K = 4 \) and the Rédei matrix of exceptional type:

\[
\mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 29 \cdot 8 \cdot 17}), \quad \mathbb{Q}(\sqrt{-7 \cdot 5 \cdot 41 \cdot 13 \cdot 17}),
\]

\[
\mathbb{Q}(\sqrt{-3 \cdot 5 \cdot 41 \cdot 17 \cdot 53}), \quad \mathbb{Q}(\sqrt{-8 \cdot 5 \cdot 61 \cdot 37 \cdot 53}).
\]
Remark 2. In Theorems 1 and 2 of [2], Benjamin proved the infiniteness of the 2-class field tower of an imaginary quadratic number field $K$ with 2-rank $C_K = 4$ and 4-rank $C_K = 2$, in the case where $R_K$ is not of the type

\[
\begin{pmatrix}
* & 1 & 1 & 0 & 0 \\
* & 1 & 1 & 1 & 1 \\
* & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & *
\end{pmatrix}
\]

with $p_1^* = -4$, $p_2^* < 0$, $p_3^* < 0$ and $p_4^* > 0$, $p_5^* > 0$. With the methods above one can prove the Theorems of Benjamin and Koch-Hajir as well.

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References


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