ON 2-CLASS FIELD TOWERS OF IMAGINARY
QUADRATIC NUMBER FIELDS

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Abstract

By the results of Golod-Shafarevich and Vinberg-Gaschütz, the 2-class field tower of an imaginary quadratic number field $K$ is infinite if the 2-rank of the ideal class group of $K$ is greater than or equal to 5. In our earlier paper, we examined the case where the 2-class rank of $K$ is equal to 4, and proved that the 2-class field tower of $K$ is infinite if $K$ has only one negative prime discriminant, except for one type of Rédei matrix of $K$. In this paper, we investigate the case where all the prime discriminants of $K$ are negative, by classifying the Rédei matrices of $K$.

1. Introduction

Let $K$ be an imaginary quadratic number field with discriminant $d$, and $C_K$ denote the ideal class group of $K$. We mean by the 2-class field tower of $K$ the sequence of fields

$$K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots,$$

where $K_{i+1}$ is the Hilbert 2-class field (i.e. the maximal unramified abelian 2-extension) of $K_i$ for $i \geq 0$. If $K_{i+1} \neq K_i$ for all $i$, then we say that the 2-class field tower of $K$ is infinite.

By the results of Golod-Shafarevich [3] and Vinberg-Gaschütz [12, 16], the 2-class field tower of $K$ is infinite if $2$-rank $C_K := \dim_{F_2} C_K/C_K^2 \geq 5$, where $F_2$ is the finite field with two elements and we consider the elementary abelian 2-group $C_K/C_K^2$ as a vector space over $F_2$. When $2$-rank $C_K = 3$ and $4$-rank $C_K := \dim_{F_2} C_K^2/C_K^4 = 0$, there are some examples of infinite families of $K$ with infinite (resp. finite) 2-class field towers [4, 7]. However, when

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2-rank $C_K = 4$, no example of $K$ with finite 2-class field tower has ever been known.

Concerning the estimate of the upper bound of the inferior limit of the root discriminants of imaginary number fields, Martinet [8] has conjectured that the 2-class field tower of $K$ with 2-rank $C_K = 4$ is always infinite. In this direction, Koch [6] and Hajir [4, 5] proved that the 2-class field tower of $K$ is infinite if 4-rank $C_K \geq 3$. Further, Benjamin [1, 2] proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and 4-rank $C_K = 2$, except for some types of Rédei matrices of $K$.

In our earlier paper [15], we proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and exactly one negative prime discriminant divides $d$, except for one type of Rédei matrix of $K$. In this paper, we investigate the case where 2-rank $C_K = 4$ and all the prime discriminants of $d$ are negative, by classifying the Rédei matrices of $K$.

2. Martinet’s inequalities and Rédei matrices

Let $K$ be an imaginary quadratic number field with discriminant $d$. To prove that the 2-class field tower of $K$ is infinite, we use Martinet’s inequalities and their corollaries [8, 15], as in [1, 2, 4, 5, 15].

**Martinet’s inequality (general case)** [8]. Let $E/F$ be a quadratic extension of number fields. Denote by $r_1$ (resp. $r_2$) the number of real (resp. imaginary) places of $F$. Also denote by $t$ (resp. $\rho$) the number of finite (resp. infinite) places of $F$ which ramify in $E$. If

$$t \geq r_1 + r_2 - \rho + 3 + 2\sqrt{2(r_1 + r_2) - \rho + 1},$$

then the 2-class field tower of $E$ is infinite.

The method to prove that the 2-class field tower of $K$ is infinite is as follows. Let $F$ be a subfield of the genus field of $K$, then $F/Q$ is a 2-extension and the Galois group of $F/Q$ is an elementary abelian 2-group. Put $E = FK$. If the extension $E/F$ satisfies Martinet’s inequality, then the 2-class field tower of $E$ is infinite. Since $E/K$ is an unramified 2-extension, the 2-class field tower of $K$ would also be infinite.
Martinet’s inequalities (special cases). (i) Let $F$ be a totally real number field of degree $n$, and $E$ a totally imaginary quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If
\[ t \geq 3 + 2\sqrt{n + 1}, \]
then the 2-class field tower of $E$ is infinite.

(ii) Let $F$ be a totally imaginary number field of degree $n$, and $E$ a quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If
\[ t \geq \frac{n}{2} + 3 + 2\sqrt{n + 1}, \]
then the 2-class field tower of $E$ is infinite.

**Proof.** (i) Since $r_1 = \rho = n$ and $r_2 = 0$, the assertion follows from the general case of Martinet’s inequality.

(ii) Since $r_1 = \rho = 0$ and $r_2 = \frac{n}{2}$, the assertion follows from the general case of Martinet’s inequality.

**Proposition.** Let $F$ be a subfield of the genus field of $K$.

(i) Suppose that $F/\mathbb{Q}$ is a real quadratic extension. If three rational primes split in $F$ and another rational prime is unramified in $F$ and these four rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.

(ii) Suppose that $F/\mathbb{Q}$ is a totally real biquadratic extension. If two rational primes split completely in $F$ and ramify in $K$, or if a rational prime splits completely in $F$ and two other rational primes are unramified in $F$ and these three rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.

(iii) Suppose that $F/\mathbb{Q}$ is an imaginary quadratic extension. If four rational primes split in $F$ and ramify in $K$, then the 2-class field tower of $K$ is infinite.

(iv) Suppose that $F/\mathbb{Q}$ is a totally imaginary biquadratic extension. If two rational primes split completely in $F$ and another rational prime is unramified in $F$ and these three rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.
Proof. Put \( E = FK \). Since \( F/Q \) is a 2-extension contained in the genus field of \( K \) in all cases, it is sufficient to prove that the 2-class filed tower of \( E \) is infinite. Note also that if \( F/Q \) is a biquadratic extension and a rational prime \( p \) is unramified in \( F \), then \( p \) splits into at least two primes in \( F \).

(i) Since \( n = 2 \) and \( t \geq 7 \geq 3 + 2\sqrt{2+1} = 6.464 \cdots \) in this case, the 2-class field tower of \( E \) is infinite by the case (i) of Martinet’s inequalities.

(ii) Since \( n = 4 \) and \( t \geq 8 \geq 3 + 2\sqrt{4+1} = 7.472 \cdots \) in these cases, the 2-class field tower of \( E \) is infinite by the case (i) of Martinet’s inequalities.

(iii) Since \( n = 2 \) and \( t \geq 8 \geq \frac{3}{2} + 3 + 2\sqrt{2+1} = 7.464 \cdots \) in this case, the 2-class field tower of \( E \) is infinite by the case (ii) of Martinet’s inequalities.

(iv) Since \( n = 4 \) and \( t \geq 10 \geq \frac{4}{2} + 3 + 2\sqrt{4+1} = 9.472 \cdots \) in this case, the 2-class field tower of \( E \) is infinite by the case (ii) of Martinet’s inequalities.

Next, we recall some properties of Rédei matrices of quadratic number fields [9, 10, 11, 13, 14]. A rational integer is called a discriminant if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a prime discriminant. Prime discriminants are denoted by \( p^* = (-1)^{\frac{p-1}{2}}p \) (if \( p \) is an odd prime), or \( p^* = -4, 8 \) or \( -8 \) (if \( p \) is equal to 2). Let \( d = p_1^*p_2^*\cdots p_t^* \) be the unique factorization of \( d \) into a product of prime discriminants. By genus theory, the genus field of \( K \) is \( Q(\sqrt{p_1^*}, \sqrt{p_2^*}, \cdots, \sqrt{p_t^*}) \) and we have 2-rank \( C_K = t - 1 \).

Using Kronecker symbols \( \left( \frac{D}{p} \right) \), where \( D \) is a discriminant and \( p \) is a prime number satisfying \( p \nmid D \), we define the Rédei matrix \( R_K = (a_{ij}) \in M_{t \times t}(F_2) \) of \( K \) by

\[
(-1)^{a_{ij}} = \begin{cases} 
\left( \frac{p_i^*}{p_j} \right) & (i \neq j), \\
\left( \frac{d/p_i^*}{p_i} \right) & (i = j).
\end{cases}
\]

By the definition of the Kronecker symbol, we have \( a_{ij} = 0 \) \((i \neq j)\) if and only if the rational prime \( p_j \) splits in \( Q(\sqrt{p_i^*}) \). Note that the sum of all row vectors of \( R_K \) is equal to the zero vector \( o \) in \( F_2^t \) so that rank \( R_K \leq t - 1 \) and the solution space \( X \) of the linear equations \( xR_K = o \) \((x \in F_2^t)\) contains
the vector \( \mathbf{1} = (1, 1, \cdots, 1) \). By the results of Rédei and Rédei-Reichardt [9, 11], we have

\[
4 \text{-rank } C_K = t - 1 - \text{rank } R_K.
\]

In the case where \( p_1^* \neq -4 \) and \( p_j^* \neq -4 \), we have \( a_{ij} = a_{ji} \) \((i \neq j)\) if and only if \( p_i^* > 0 \) or \( p_j^* > 0 \), by the quadratic reciprocity law.

In [15], we proved that the 2-class field tower of \( K \) is infinite if the 2-rank of \( C_K \) is equal to 4 and exactly one negative prime discriminant divides \( d \), except the case where

\[
R_K = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
\end{pmatrix}
\]

with \( p_1^* \neq -4 \) negative, by changing the order of \( p_i \)'s \((2 \leq i \leq 5)\). In the exceptional case, we have 4-rank \( C_K = 0 \).

It is also proved in [1, 2, 15] that the 2-class field tower of \( K \) is infinite if 2-rank \( C_K = 4 \) and 4-rank \( C_K = 2 \), except the case where \( R_K \) is of the type

\[
\begin{pmatrix}
* & 1 & 1 & 0 & 0 \\
* & 1 & 1 & 1 & 1 \\
* & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & * \\
\end{pmatrix}
\]

with \( p_1^* = -4 \), \( p_2^* < 0 \), \( p_3^* < 0 \), \( p_4^* > 0 \) and \( p_5^* > 0 \), by changing the order of \( p_i \)'s \((2 \leq i \leq 5)\). Therefore, if \( a_{44} = 0 \), then \( a_{45} = a_{54} = a_{55} = 0 \) and \( a_{41} = 1 \). On the other hand, if \( a_{44} = 1 \), then \( a_{45} = a_{54} = a_{55} = 1 \) and \( a_{21} = 1 \). Hence, there are four types of exceptional Rédei matrices.

3. The case with five negative prime discriminants

Let \( K \) be an imaginary quadratic number field with discriminant \( d \). In this section, we examine whether the 2-class field tower of \( K \) is infinite, in the case where 2-rank \( C_K = 4 \) and all the prime discriminants of \( d \) are negative.
In this case, by Rédei’s theorem [10], we have rank \( R_K \geq 2 \) and therefore 4-rank \( C_K \leq 2 \).

First, we classify the Rédei matrices of \( K \). Let \( d = p_1^* p_2^* p_3^* p_4^* p_5^* \) be the unique factorization of \( d \) into a product of prime discriminants. If \(-4\) is contained in the prime discriminants of \( d \), then we may assume that \( p_4^* = -4 \) without loss of generality.

If \( p_4^* \neq -4 \), then the Rédei matrix \( R_K = (a_{ij}) \) is an adjacency matrix of a tournament, that is, it is a \( \{0, 1\} \)-matrix satisfying \( a_{ij} + a_{ji} = 1 \) for all \( i, j \) (\( 1 \leq i, j \leq 5, \ i \neq j \)). Therefore, there are 12 types of Rédei matrices, by changing the order of \( p_i \)'s, if necessary.

\[
\begin{align*}
&\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} & \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix} & \begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix} & \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1
\end{pmatrix} \\
&\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix} & \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} & \begin{pmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}
\]

If \( p_4^* = -4 \), then the matrix \((a_{ij})_{2 \leq i \leq 5, \ 2 \leq j \leq 5}\) is an adjacency matrix of a tournament. Therefore, there are following types of Rédei matrices, by changing the order of \( p_i \)'s (\( 2 \leq i \leq 5 \)), if necessary.
Here, the asterisks "*" mean 0 or 1. In the cases (m) and (p), there are 16 types of $R_K$, respectively. In the cases (n) and (o), there are 8 types of $R_K$, by changing the order of $p_i$'s $(3 \leq i \leq 5)$ or $p_i$'s $(2 \leq i \leq 4)$, respectively.

In the above cases, the 4-ranks of $C_K$ are as in the following table.

<table>
<thead>
<tr>
<th>case</th>
<th>(a)</th>
<th>(b) ~ (g), (j)</th>
<th>(h), (i), (k), (l)</th>
<th>(m)</th>
<th>(n) ~ (p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1, 2</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

In the cases (m) ~ (p), we can determine the 4-ranks of $C_K$ as follows:

(m) The 4-rank of $C_K$ is equal to 2 if and only if $a_{31} = a_{41}$ and $a_{51} = 0$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 2$ and 12 types of $R_K$ such that 4-rank $C_K = 1$.

(n) The 4-rank of $C_K$ is equal to 1 if and only if $a_{11} = a_{21}$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 1$ and 4 types of $R_K$ such that 4-rank $C_K = 0$, by changing the order of $p_i$'s $(3 \leq i \leq 5)$.

(o) The 4-rank of $C_K$ is equal to 1 if and only if $a_{51} = 0$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 1$ and 4 types of $R_K$ such that 4-rank $C_K = 0$, by changing the order of $p_i$'s $(2 \leq i \leq 4)$.

(p) The 4-rank of $C_K$ is equal to 1 if and only if $a_{31} = a_{41}$. Therefore, there are 8 types of $R_K$ such that 4-rank $C_K = 1$ and 8 types of $R_K$ such that 4-rank $C_K = 0$. 

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For some of the above Rédei matrices, we can find subfields $F = \mathbb{Q}(\sqrt{p_i^j})$, $\mathbb{Q}(\sqrt{p_i^j}, \sqrt{p_j^k})$ or $\mathbb{Q}(\sqrt{p_i^j}, \sqrt{p_j^k}, \sqrt{p_k^l})$ ($i, j, k \in \{1, 2, 3, 4, 5\}$) of the genus field of $K$, for which we can apply Proposition (iii), (iv) or (ii). The results are in the following table. In the row of “primes”, we describe the primes which split completely in $F$ and ramify in $K$, and in the parentheses we describe primes which are unramified in $F$ and ramify in $K$. We see that the 2-class field tower of $K$ is infinite in the cases (a) $\sim$ (j) and (m). In the cases (n) $\sim$ (p), the 2-class field tower of $K$ is infinite under some additional conditions described below.

<table>
<thead>
<tr>
<th>case</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$\mathbb{Q}(\sqrt{p_1^2})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_3^4})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3})$</td>
<td>$\mathbb{Q}(\sqrt{p_3^4})$</td>
</tr>
<tr>
<td>primes</td>
<td>$p_1, p_2, p_3, p_4$</td>
<td>$p_1, p_2, (p_3)$</td>
<td>$p_1, p_2, p_3, p_4$</td>
<td>$p_1, p_2, p_3, p_4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(e)</th>
<th>(f)</th>
<th>(g)</th>
<th>(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$\mathbb{Q}(\sqrt{p_1^2}, \sqrt{p_2^3})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_3^4})$</td>
<td>$\mathbb{Q}(\sqrt{p_3^4})$</td>
</tr>
<tr>
<td>primes</td>
<td>$p_1, p_2, (p_3)$</td>
<td>$p_1, p_2, p_3, p_4$</td>
<td>$p_1, p_2, p_3, (p_1)$</td>
<td>$p_1, p_2, (p_3)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(i)</th>
<th>(j)</th>
<th>(k)</th>
<th>(l)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_2^4})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_3^4})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3})$</td>
<td>$\mathbb{Q}(\sqrt{p_3^4})$</td>
<td></td>
</tr>
<tr>
<td>primes</td>
<td>$p_1, p_2$</td>
<td>$p_1, p_4, (p_3)$</td>
<td>$p_1, p_2, p_3, p_4$</td>
<td>$p_1, p_2, (p_3)$</td>
<td></td>
</tr>
<tr>
<td>add. cond.</td>
<td>$a_{41} = a_{51} = 0$</td>
<td>$a_{31} = a_{41} = a_{51} = 1$</td>
<td>$a_{51} = 1$</td>
<td>$a_{41} = a_{51} = 0$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(n)</th>
<th>(o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_2^4})$</td>
<td>$\mathbb{Q}(\sqrt{p_2^3}, \sqrt{p_3^4})$</td>
</tr>
<tr>
<td>primes</td>
<td>$p_1, p_5$</td>
<td>$p_1, p_2, (p_3)$</td>
</tr>
<tr>
<td>add. cond.</td>
<td>$a_{21} = a_{31} = a_{41}, a_{51} = 1$</td>
<td>$a_{31} = a_{41} = 0, a_{31} = 1, a_{41} = a_{51} = 0$</td>
</tr>
</tbody>
</table>

We summarize the result in the following table. The numbers in the table are the numbers of types of Rédei matrices $R_K$ with five negative prime discriminants, and the numbers in the parentheses are the numbers of $R_K$ for which we can prove that the 2-class field tower of $K$ is infinite.

<table>
<thead>
<tr>
<th>4-rank $C_K$</th>
<th>(a) $\sim$ (l)</th>
<th>(m)</th>
<th>(n)</th>
<th>(o)</th>
<th>(p)</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 (1)</td>
<td>4 (4)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>5 (5)</td>
</tr>
<tr>
<td>1</td>
<td>7 (7)</td>
<td>12 (12)</td>
<td>4 (2)</td>
<td>4 (4)</td>
<td>8 (4)</td>
<td>35 (29)</td>
</tr>
<tr>
<td>0</td>
<td>4 (2)</td>
<td>0 (0)</td>
<td>4 (4)</td>
<td>4 (2)</td>
<td>8 (2)</td>
<td>20 (10)</td>
</tr>
<tr>
<td>total</td>
<td>12 (10)</td>
<td>16 (16)</td>
<td>8 (6)</td>
<td>8 (6)</td>
<td>16 (6)</td>
<td>60 (44)</td>
</tr>
</tbody>
</table>

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Therefore we obtain the following

**Theorem.** Let $K$ be an imaginary quadratic number field with discriminant $d$. Suppose that 2-rank $C_K = 4$ and all the prime discriminants of $d$ are negative. Let $d = p_1^* p_2^* p_3^* p_4^* p_5^*$ be the unique factorization of $d$ into a product of prime discriminants. If $-4$ is contained in the prime discriminants of $d$, then we assume that $p_1^* = -4$. There are 12 types of Rédei matrices in the case $p_1^* \neq -4$ by changing the order of $p_i$’s, and 48 types of Rédei matrices in the case $p_1^* = -4$ by changing the order of $p_i$’s $(2 \leq i \leq 5)$. Among them, the 2-class field tower of $K$ is infinite for 44 types of $R_K$.

**References**


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