ON 2-CLASS FIELD TOWERS OF IMAGINARY QUADRATIC NUMBER FIELDS

YUTAKA SUEYOSHI

Abstract

By the results of Golod-Shafarevich and Vinberg-Gaschütz, the 2-class field tower of an imaginary quadratic number field $K$ is infinite if the 2-rank of the ideal class group of $K$ is greater than or equal to 5. In our earlier paper, we examined the case where the 2-class rank of $K$ is equal to 4, and proved that the 2-class field tower of $K$ is infinite if $K$ has only one negative prime discriminant, except for one type of Rédei matrix of $K$. In this paper, we investigate the case where all the prime discriminants of $K$ are negative, by classifying the Rédei matrices of $K$.

1. Introduction

Let $K$ be an imaginary quadratic number field with discriminant $d$, and $C_K$ denote the ideal class group of $K$. We mean by the 2-class field tower of $K$ the sequence of fields

$$K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots,$$

where $K_{i+1}$ is the Hilbert 2-class field (i.e. the maximal unramified abelian 2-extension) of $K_i$ for $i \geq 0$. If $K_{i+1} \neq K_i$ for all $i$, then we say that the 2-class field tower of $K$ is infinite.

By the results of Golod-Shafarevich [3] and Vinberg-Gaschütz [12, 16], the 2-class field tower of $K$ is infinite if 2-rank $C_K := \dim_{F_2} C_K / C_K^2 \geq 5$, where $F_2$ is the finite field with two elements and we consider the elementary abelian 2-group $C_K / C_K^2$ as a vector space over $F_2$. When 2-rank $C_K = 3$ and 4-rank $C_K := \dim_{F_2} C_K^2 / C_K^4 = 0$, there are some examples of infinite families of $K$ with infinite (resp. finite) 2-class field towers [4, 7]. However, when

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2-rank $C_K = 4$, no example of $K$ with finite 2-class field tower has ever been known.

Concerning the estimate of the upper bound of the inferior limit of the root discriminants of imaginary number fields, Martinet [8] has conjectured that the 2-class field tower of $K$ with 2-rank $C_K = 4$ is always infinite. In this direction, Koch [6] and Hajir [4, 5] proved that the 2-class field tower of $K$ is infinite if 4-rank $C_K \geq 3$. Further, Benjamin [1, 2] proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and 4-rank $C_K = 2$, except for some types of Rédei matrices of $K$.

In our earlier paper [15], we proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and exactly one negative prime discriminant divides $d$, except for one type of Rédei matrix of $K$. In this paper, we investigate the case where 2-rank $C_K = 4$ and all the prime discriminants of $d$ are negative, by classifying the Rédei matrices of $K$.

2. Martinet’s inequalities and Rédei matrices

Let $K$ be an imaginary quadratic number field with discriminant $d$. To prove that the 2-class field tower of $K$ is infinite, we use Martinet’s inequalities and their corollaries [8, 15], as in [1, 2, 4, 5, 15].

**Martinet’s inequality (general case)** [8]. Let $E/F$ be a quadratic extension of number fields. Denote by $r_1$ (resp. $r_2$) the number of real (resp. imaginary) places of $F$. Also denote by $t$ (resp. $\rho$) the number of finite (resp. infinite) places of $F$ which ramify in $E$. If

$$t \geq r_1 + r_2 - \rho + 3 + 2\sqrt{2(r_1 + r_2) - \rho + 1},$$

then the 2-class field tower of $E$ is infinite.

The method to prove that the 2-class field tower of $K$ is infinite is as follows. Let $F$ be a subfield of the genus field of $K$, then $F/Q$ is a 2-extension and the Galois group of $F/Q$ is an elementary abelian 2-group. Put $E = FK$. If the extension $E/F$ satisfies Martinet’s inequality, then the 2-class field tower of $E$ is infinite. Since $E/K$ is an unramified 2-extension, the 2-class field tower of $K$ would also be infinite.
Martinet’s inequalities (special cases). (i) Let \( F \) be a totally real number field of degree \( n \), and \( E \) a totally imaginary quadratic extension of \( F \). Let \( t \) be the number of prime ideals of \( F \) which ramify in \( E \). If
\[
t \geq 3 + 2\sqrt{n + 1},
\]
then the 2-class field tower of \( E \) is infinite.

(ii) Let \( F \) be a totally imaginary number field of degree \( n \), and \( E \) a quadratic extension of \( F \). Let \( t \) be the number of prime ideals of \( F \) which ramify in \( E \). If
\[
t \geq \frac{n}{2} + 3 + 2\sqrt{n + 1},
\]
then the 2-class field tower of \( E \) is infinite.

Proof. (i) Since \( r_1 = \rho = n \) and \( r_2 = 0 \), the assertion follows from the general case of Martinet’s inequality.

(ii) Since \( r_1 = \rho = 0 \) and \( r_2 = \frac{n}{2} \), the assertion follows from the general case of Martinet’s inequality.

Proposition. Let \( F \) be a subfield of the genus field of \( K \).

(i) Suppose that \( F/Q \) is a real quadratic extension. If three rational primes split in \( F \) and another rational prime is unramified in \( F \) and these four rational primes ramify in \( K \), then the 2-class field tower of \( K \) is infinite.

(ii) Suppose that \( F/Q \) is a totally real biquadratic extension. If two rational primes split completely in \( F \) and ramify in \( K \), or if a rational prime splits completely in \( F \) and two other rational primes are unramified in \( F \) and these three rational primes ramify in \( K \), then the 2-class field tower of \( K \) is infinite.

(iii) Suppose that \( F/Q \) is an imaginary quadratic extension. If four rational primes split in \( F \) and ramify in \( K \), then the 2-class field tower of \( K \) is infinite.

(iv) Suppose that \( F/Q \) is a totally imaginary biquadratic extension. If two rational primes split completely in \( F \) and another rational prime is unramified in \( F \) and these three rational primes ramify in \( K \), then the 2-class field tower of \( K \) is infinite.
Proof. Put $E = FK$. Since $F/Q$ is a 2-extension contained in the genus field of $K$ in all cases, it is sufficient to prove that the 2-class field tower of $E$ is infinite. Note also that if $F/Q$ is a biquadratic extension and a rational prime $p$ is unramified in $F$, then $p$ splits into at least two primes in $F$.

(i) Since $n = 2$ and $t = 7$, $t \geq 7 \geq 3 + 2\sqrt{2} + 1 = 6.464 \cdots$ in this case, the 2-class field tower of $E$ is infinite by the case (i) of Martinet’s inequalities.

(ii) Since $n = 4$, $t = 8 \geq 3 + 2\sqrt{4} + 1 = 7.472 \cdots$ in these cases, the 2-class field tower of $E$ is infinite by the case (ii) of Martinet’s inequalities.

(iii) Since $n = 2$ and $t = 8 \geq 3 + 2\sqrt{2} + 1 = 7.464 \cdots$ in this case, the 2-class field tower of $E$ is infinite by the case (ii) of Martinet’s inequalities.

(iv) Since $n = 4$ and $t = 10 \geq 3 + 2\sqrt{4} + 1 = 9.472 \cdots$ in this case, the 2-class field tower of $E$ is infinite by the case (ii) of Martinet’s inequalities.

Next, we recall some properties of Rédei matrices of quadratic number fields $[9, 10, 11, 13, 14]$. A rational integer is called a discriminant if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a prime discriminant. Prime discriminants are denoted by $p^* = (-1)^{\frac{n-1}{2}}p$ (if $p$ is an odd prime), or $p^* = -4, 8$ or $-8$ (if $p$ is equal to 2). Let $d = p_1^{m_1}p_2^{m_2} \cdots p_t^{m_t}$ be the unique factorization of $d$ into a product of prime discriminants. By genus theory, the genus field of $K$ is $Q(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_t})$ and we have 2-rank $C_K = t - 1$.

Using Kronecker symbols $\left(\frac{D}{p}\right)$, where $D$ is a discriminant and $p$ is a prime number satisfying $p \nmid D$, we define the Rédei matrix $R_K = (a_{ij}) \in M_{t \times t}(F_2)$ of $K$ by

$$(-1)^{a_{ij}} = \begin{cases} \left(\frac{p_i^{m_i}}{p_j}ight) & (i \neq j), \\ \left(\frac{d/p_i^{m_i}}{p_i}ight) & (i = j). \end{cases}$$

By the definition of the Kronecker symbol, we have $a_{ij} = 0$ $(i \neq j)$ if and only if the rational prime $p_j$ splits in $Q(\sqrt{p_i})$. Note that the sum of all row vectors of $R_K$ is equal to the zero vector $\mathbf{o}$ in $F_2^t$ so that rank $R_K \leq t - 1$ and the solution space $X$ of the linear equations $xR_K = \mathbf{o}$ $(x \in F_2^t)$ contains
the vector \( \mathbf{1} = (1, 1, \ldots, 1) \). By the results of Rédei and Rédei-Reichardt [9, 11], we have

\[ 4 \text{-rank} C_K = t - 1 - \text{rank} R_K. \]

In the case where \( p_i^* \neq -4 \) and \( p_j^* \neq -4 \), we have \( a_{ij} = a_{ji} \) \((i \neq j)\) if and only if \( p_i^* > 0 \) or \( p_j^* > 0 \), by the quadratic reciprocity law.

In [15], we proved that the 2-class field tower of \( K \) is infinite if the 2-rank of \( C_K \) is equal to 4 and exactly one negative prime discriminant divides \( d \), except the case where

\[
R_K = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

with \( p_i^* \neq -4 \) negative, by changing the order of \( p_i \)'s \((2 \leq i \leq 5)\). In the exceptional case, we have 4-rank \( C_K = 0 \).

It is also proved in [1, 2, 15] that the 2-class field tower of \( K \) is infinite if 2-rank \( C_K = 4 \) and 4-rank \( C_K = 2 \), except the case where \( R_K \) is of the type

\[
\begin{pmatrix}
* & 1 & 1 & 0 & 0 \\
* & 1 & 1 & 1 & 1 \\
* & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & * & * \\
1 & 1 & 1 & * & *
\end{pmatrix}
\]

with \( p_1^* = -4 \), \( p_2^* < 0 \), \( p_3^* < 0 \), \( p_4^* > 0 \) and \( p_5^* > 0 \), by changing the order of \( p_i \)'s \((2 \leq i \leq 5)\). Therefore, if \( a_{44} = 0 \), then \( a_{45} = a_{54} = a_{55} = 0 \) and \( a_{11} = 1 \). On the other hand, if \( a_{44} = 1 \), then \( a_{45} = a_{54} = a_{55} = 1 \) and \( a_{21} = 1 \). Hence, there are four types of exceptional Rédei matrices.

3. The case with five negative prime discriminants

Let \( K \) be an imaginary quadratic number field with discriminant \( d \). In this section, we examine whether the 2-class field tower of \( K \) is infinite, in the case where 2-rank \( C_K = 4 \) and all the prime discriminants of \( d \) are negative.
In this case, by Rédei’s theorem [10], we have \( \text{rank } R_K \geq 2 \) and therefore 4-rank \( C_K \leq 2 \).

First, we classify the Rédei matrices of \( K \). Let \( d = p_1^\epsilon_1 p_2^\epsilon_2 p_3^\epsilon_3 p_4^\epsilon_4 p_5^\epsilon_5 \) be the unique factorization of \( d \) into a product of prime discriminants. If \(-4\) is contained in the prime discriminants of \( d \), then we may assume that \( p_1^\epsilon_1 = -4 \) without loss of generality.

If \( p_1^\epsilon_1 \neq -4 \), then the Rédei matrix \( R_K = (a_{ij}) \) is an adjacency matrix of a tournament, that is, it is a \( \{0, 1\} \)-matrix satisfying \( a_{ij} + a_{ji} = 1 \) for all \( i, j \) (\( 1 \leq i, j \leq 5, i \neq j \)). Therefore, there are 12 types of Rédei matrices, by changing the order of \( p_i \)'s, if necessary.

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(a) (b) (c)

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(d) (e) (f)

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(g) (h) (i)

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(j) (k) (l)

If \( p_1^\epsilon_1 = -4 \), then the matrix \((a_{ij})_{2 \leq i, j \leq 5, i \neq j}\) is an adjacency matrix of a tournament. Therefore, there are following types of Rédei matrices, by changing the order of \( p_i \)'s (\( 2 \leq i \leq 5 \)), if necessary.
Here, the asterisks "*" mean 0 or 1. In the cases (m) and (p), there are 16 types of $R_K$, respectively. In the cases (n) and (o), there are 8 types of $R_K$, by changing the order of $p_i$’s (3 $\leq$ i $\leq$ 5) or $p_i$’s (2 $\leq$ i $\leq$ 4), respectively.

In the above cases, the 4-ranks of $C_K$ are as in the following table.

<table>
<thead>
<tr>
<th>case</th>
<th>(a)</th>
<th>(b) $\sim$ (g), (j)</th>
<th>(h), (i), (k), (l)</th>
<th>(m)</th>
<th>(n) $\sim$ (p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1, 2</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

In the cases (m) $\sim$ (p), we can determine the 4-ranks of $C_K$ as follows:

(m) The 4-rank of $C_K$ is equal to 2 if and only if $a_{31} = a_{41}$ and $a_{51} = 0$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 2$ and 12 types of $R_K$ such that 4-rank $C_K = 1$.

(n) The 4-rank of $C_K$ is equal to 1 if and only if $a_{11} = a_{21}$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 1$ and 4 types of $R_K$ such that 4-rank $C_K = 0$, by changing the order of $p_i$’s (3 $\leq$ i $\leq$ 5).

(o) The 4-rank of $C_K$ is equal to 1 if and only if $a_{51} = 0$. Therefore, there are 4 types of $R_K$ such that 4-rank $C_K = 1$ and 4 types of $R_K$ such that 4-rank $C_K = 0$, by changing the order of $p_i$’s (2 $\leq$ i $\leq$ 4).

(p) The 4-rank of $C_K$ is equal to 1 if and only if $a_{31} = a_{41}$. Therefore, there are 8 types of $R_K$ such that 4-rank $C_K = 1$ and 8 types of $R_K$ such that 4-rank $C_K = 0$. 
For some of the above Rédéi matrices, we can find subfields \( F = Q(\sqrt{p_i}) \), \( Q(\sqrt{p_i^2}, \sqrt{p_j}) \) or \( Q(\sqrt{p_i^2}, \sqrt{p_j^2} \sqrt{p_k}) \) \((i, j, k \in \{1, 2, 3, 4, 5\})\) of the genus field of \( K \), for which we can apply Proposition (iii), (iv) or (ii). The results are in the following table. In the row of “primes”, we describe the primes which split completely in \( F \) and ramify in \( K \), and in the parentheses we describe primes which are unramified in \( F \) and ramify in \( K \). We see that the 2-class field tower of \( K \) is infinite in the cases (a) \( \sim \) (j) and (m). In the cases (n) \( \sim \) (p), the 2-class field tower of \( K \) is infinite under some additional conditions described below.

<table>
<thead>
<tr>
<th>case</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( Q(\sqrt{p_i^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j}) )</td>
<td>( Q(\sqrt{p_i^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
</tr>
<tr>
<td>primes</td>
<td>( p_i, p_2, p_3, p_4 )</td>
<td>( p_1, p_2, (p_3) )</td>
<td>( p_1, p_2, p_3, p_4 )</td>
<td>( p_1, p_2, p_3, p_4 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(e)</th>
<th>(f)</th>
<th>(g)</th>
<th>(h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2} \sqrt{p_k}) )</td>
</tr>
<tr>
<td>primes</td>
<td>( p_1, p_3, (p_2) )</td>
<td>( p_1, p_2, p_3, p_4 )</td>
<td>( p_2, p_3, (p_1) )</td>
<td>( p_1, p_2, (p_3) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(i)</th>
<th>(j)</th>
<th>(k)</th>
<th>(l)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
</tr>
<tr>
<td>primes</td>
<td>( p_1, p_5 )</td>
<td>( p_1, p_4, (p_3) )</td>
<td>( p_1, p_2, (p_3) )</td>
<td>( p_1, p_2, (p_3) )</td>
<td>( p_2, p_3, (p_1) )</td>
</tr>
<tr>
<td>add. cond.</td>
<td>( a_{41} = a_{51} = 0 )</td>
<td>( a_{31} = a_{41} = a_{51} = 1 )</td>
<td>( a_{51} = 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>case</th>
<th>(n)</th>
<th>(o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
<td>( Q(\sqrt{p_i^2}, \sqrt{p_j^2}) )</td>
</tr>
<tr>
<td>primes</td>
<td>( p_1, p_2, (p_3) )</td>
<td>( p_1, p_2, (p_3) )</td>
</tr>
<tr>
<td>add. cond.</td>
<td>( a_{21} = a_{31} = a_{41}, a_{51} = 1 )</td>
<td>( a_{31} = a_{41} = 0 )</td>
</tr>
<tr>
<td>add. cond.</td>
<td>( a_{31} = a_{41} = 1 )</td>
<td>( a_{41} = a_{51} = 0 )</td>
</tr>
</tbody>
</table>

We summarize the result in the following table. The numbers in the table are the numbers of Rédéi matrices \( R_K \) with five negative prime discriminants, and the numbers in the parentheses are the numbers of \( R_K \) for which we can prove that the 2-class field tower of \( K \) is infinite.

<table>
<thead>
<tr>
<th>4-rank ( C_K )</th>
<th>( (a) \sim (l) )</th>
<th>( (m) )</th>
<th>( (n) )</th>
<th>( (o) )</th>
<th>( (p) )</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 (1)</td>
<td>4 (4)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>5 (5)</td>
</tr>
<tr>
<td>1</td>
<td>7 (7)</td>
<td>12 (12)</td>
<td>4 (2)</td>
<td>4 (4)</td>
<td>8 (4)</td>
<td>35 (29)</td>
</tr>
<tr>
<td>0</td>
<td>4 (2)</td>
<td>0 (0)</td>
<td>4 (4)</td>
<td>4 (2)</td>
<td>8 (2)</td>
<td>20 (10)</td>
</tr>
<tr>
<td>total</td>
<td>12 (10)</td>
<td>16 (16)</td>
<td>8 (6)</td>
<td>8 (6)</td>
<td>16 (6)</td>
<td>60 (44)</td>
</tr>
</tbody>
</table>
Therefore we obtain the following

**Theorem.** Let $K$ be an imaginary quadratic number field with discriminant $d$. Suppose that $2$-rank $C_K = 4$ and all the prime discriminants of $d$ are negative. Let $d = p_1^*p_2^*p_3^*p_4^*p_5^*$ be the unique factorization of $d$ into a product of prime discriminants. If $-4$ is contained in the prime discriminants of $d$, then we assume that $p_1^* = -4$. There are 12 types of Rédei matrices in the case $p_1^* \neq -4$ by changing the order of $p_i$’s, and 48 types of Rédei matrices in the case $p_1^* = -4$ by changing the order of $p_i$’s ($2 \leq i \leq 5$). Among them, the $2$-class field tower of $K$ is infinite for 44 types of $R_K$.

**References**


[13] Y. Sueyoshi, On a comparison of the 4-ranks of the narrow ideal class groups of $\mathbb{Q}(\sqrt{m})$ and $\mathbb{Q}(\sqrt{-m})$, Kyushu J. Math. 51 (1997), 261-272.


Department of Computer and Information Sciences
Faculty of Engineering
Nagasaki University
1-14 Bunkyo-machi
Nagasaki 852-8521, Japan
E-mail: sueyoshi@cis.nagasaki-u.ac.jp