ON THE INFINITUDE OF 2-CLASS FIELD TOWERS OF SOME IMAGINARY QUADRATIC NUMBER FIELDS

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Abstract

By the results of Golod-Shafarevich and Vinberg-Gaschütz, the 2-class field tower of an imaginary quadratic number field $K$ is infinite if the 2-rank of the ideal class group of $K$ is greater than or equal to 5. In our earlier papers, we examined the case where the 2-class rank of $K$ is equal to 4, by classifying the Rédei matrices of $K$. We proved that the 2-class field tower of $K$ is infinite if $K$ has only one negative prime discriminant, except for one type of Rédei matrix of $K$, and that the 2-class field tower of $K$ is infinite for many cases if all the prime discriminants of $K$ are negative. In this paper, we investigate the remaining case where $K$ has exactly three negative prime discriminants.

1. Introduction

Let $K$ be an imaginary quadratic number field with discriminant $d$, and $C_K$ denote the ideal class group of $K$. We mean by the 2-class field tower of $K$ the sequence of fields

$$K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_i \subseteq \cdots,$$

where $K_{i+1}$ is the Hilbert 2-class field (i.e., the maximal unramified abelian 2-extension) of $K_i$ for $i \geq 0$. If $K_{i+1} \neq K_i$ for all $i$, then we say that the 2-class field tower of $K$ is infinite.

By the results of Golod-Shafarevich [3] and Vinberg-Gaschütz [11, 16], the 2-class field tower of $K$ is infinite if 2-rank $C_K := \dim_{F_2} C_K/C_K^{2} \geq 5$, where $F_2$ is the finite field with two elements and we consider the elementary abelian 2-group $C_K/C_K^{2}$ as a vector space over $F_2$. Concerning the estimate

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of the upper bound of the inferior limit of the root discriminants of imaginary number fields, Martinet [7] has conjectured that the 2-class field tower of $K$ with 2-rank $C_K = 4$ is always infinite. In this direction, Koch [6] and Hajir [4, 5] proved that the 2-class field tower of $K$ is infinite if 4-rank $C_K := \dim \mathbf{F}_2 C_K^2 / C_K^4 \geq 3$. Further, Benjamin [1, 2] proved that the 2-class field tower of $K$ is infinite if 2-rank $C_K = 4$ and 4-rank $C_K = 2$, except for some types of Rédei matrices of $K$.

In our earlier papers [14, 15], we examined the case where 2-rank $C_K = 4$, by classifying the Rédei matrices of $K$. We proved that the 2-class field tower of $K$ is infinite if exactly one negative prime discriminant divides $d$, except for one type of Rédei matrix of $K$, and that the 2-class field tower of $K$ is infinite for many cases if all the prime discriminants of $K$ are negative. In this paper, we investigate the remaining case where exactly three negative prime discriminants divide $d$.

2. Martinet’s Inequalities and Rédei Matrices

Let $K$ be an imaginary quadratic number field with discriminant $d$. To prove that the 2-class field tower of $K$ is infinite, we use the following proposition which is a corollary of Martinet’s inequalities [7].

**Proposition** [14, 15]. Let $F$ be a subfield of the genus field of $K$.

(i) Suppose that $F/\mathbf{Q}$ is a real quadratic extension. If three rational primes split in $F$ and another rational prime is unramified in $F$ and these four rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.

(ii) Suppose that $F/\mathbf{Q}$ is a totally real biquadratic extension. If two rational primes split completely in $F$ and ramify in $K$, or if a rational prime splits completely in $F$ and two other rational primes are unramified in $F$ and these three rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.

(iii) Suppose that $F/\mathbf{Q}$ is an imaginary quadratic extension. If four rational primes split in $F$ and ramify in $K$, then the 2-class field tower of $K$ is infinite.

(iv) Suppose that $F/\mathbf{Q}$ is a totally imaginary biquadratic extension. If two rational primes split completely in $F$ and another rational prime is unramified
in $F$ and these three rational primes ramify in $K$, then the 2-class field tower of $K$ is infinite.

Next, we recall some properties of Rédei matrices of quadratic number fields [8, 9, 10, 12, 13]. A rational integer is called a discriminant if it is the discriminant of a quadratic number field or equal to 1. A discriminant which is divisible by only one prime is called a prime discriminant. Prime discriminants are denoted by $p^* = (-1)^{\frac{p-1}{2}}p$ (if $p$ is an odd prime), or $p^* = -4$, 8 or $-8$ (if $p$ is equal to $2$). Let $d = p_1^* p_2^* \cdots p_t^*$ be the unique factorization of $d$ into a product of prime discriminants. By genus theory, the genus field of $K$ is $\mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*}, \cdots, \sqrt{p_t^*})$ and we have 2-rank $C_K = t - 1$.

Using Kronecker symbols $\left( \frac{D}{p} \right)$, where $D$ is a discriminant and $p$ is a prime number satisfying $p \not| D$, we define the Rédei matrix $R_K = (a_{ij}) \in M_{t \times t}(\mathbb{F}_2)$ of $K$ by

$$(-1)^{a_{ij}} = \begin{cases} 
\left( \frac{p_i^*}{p_j} \right) & (i \neq j), \\
\left( \frac{d/p_i^*}{p_i} \right) & (i = j).
\end{cases}$$

By the definition of the Kronecker symbol, we have $a_{ij} = 0$ ($i \neq j$) if and only if the rational prime $p_j$ splits in $\mathbb{Q}(\sqrt{p_i^*})$. Note that the sum of all row vectors of $R_K$ is equal to the zero vector $\mathbf{0}$ in $\mathbb{F}_2^t$ so that rank $R_K \leq t - 1$ and the solution space $X$ of the linear equations $x R_K = \mathbf{0}$ ($x \in \mathbb{F}_2^t$) contains the vector $\mathbf{1} = (1, 1, \cdots, 1)$. By the results of Rédei and Rédei-Reichardt [8, 10], we have

$$4\text{-rank } C_K = t - 1 - \text{rank } R_K.$$

In the case where $p_i^* \neq -4$ and $p_j^* \neq -4$, we have $a_{ij} = a_{ji}$ ($i \neq j$) if and only if $p_i^* > 0$ or $p_j^* > 0$, by the quadratic reciprocity law.

3. The Case with Three Negative Prime Discriminants

Let $K$ be an imaginary quadratic number field with discriminant $d$. In this section, we examine whether the 2-class field tower of $K$ is infinite, in the
case where 2-rank $C_K = 4$ and exactly three negative prime discriminants divide $d$. In this case, by Rédei’s theorem [9], we have rank $R_K \geq 1$ and therefore 4-rank $C_K \leq 3$.

First, we classify the Rédei matrices of $K$. Let $d = p_1^* p_2^* p_3^* p_4^* p_5^*$ be the unique factorization of $d$ into a product of prime discriminants, with $p_1^*$, $p_2^*$ and $p_3^*$ negative. If $-4$ is contained in the prime discriminants of $d$, then we may assume that $p_1^* = -4$ without loss of generality. There are three types of Rédei matrices, according as $p_1^* \neq -4$ (cases (A) and (B)) or $p_1^* = -4$ (case (C)):

$$
\begin{align*}
\text{(A)} & : \begin{pmatrix}
* & 1 & 1 & * & * \\
0 & * & 1 & * & * \\
0 & 0 & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{pmatrix} \\
\text{(B)} & : \begin{pmatrix}
* & 1 & 0 & * & * \\
0 & * & 1 & * & * \\
1 & 0 & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{pmatrix} \\
\text{(C)} & : \begin{pmatrix}
* & 1 & 1 & 0 & 0 \\
0 & * & 1 & * & * \\
0 & 0 & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{pmatrix}
\end{align*}
$$

Here, the asterisks “*” mean 0 or 1.

Since $p_1^* > 0$ and $p_3^* > 0$, we have $a_{i4} = a_{4i}$ and $a_{i5} = a_{5i}$ for $1 \leq i \leq 3$ (resp. $2 \leq i \leq 5$) in the cases (A) and (B) (resp. in the case (C)).

In the case (A), the matrix $(a_{ij})_{1 \leq i \leq 3, 4 \leq j \leq 5}$ is one of the following, by changing the order of $p_4$ and $p_5$, if necessary.

$$
\begin{align*}
\text{(1)} & : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix} \\
\text{(2)} & : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(3)} & : \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(4)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(5)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix} \\
\text{(6)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(7)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(8)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(9)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(10)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(11)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(12)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(13)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix} \\
\text{(14)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(15)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(16)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(17)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
\text{(18)} & : \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}
$$
In the case (B), the matrix \((a_{ij})_{1\leq i\leq 3, 4\leq j\leq 5}\) is one of the following, by changing the orders of \(p_i\)'s (1 \leq i \leq 3) and \(p_j\)'s (4 \leq j \leq 5), if necessary.

\[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix}
\]

\((19)\) \((20)\) \((21)\) \((22)\) \((23)\) \((24)\)

\[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 1 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
\end{pmatrix}
\]

\((25)\) \((26)\) \((27)\) \((28)\) \((29)\) \((30)\)

\[
\begin{pmatrix}
0 & 1 \\
1 & 1 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 1 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

\((31)\) \((32)\) \((33)\) \((34)\) \((35)\) \((36)\)

In the case (C), the matrix \((a_{ij})_{2\leq i\leq 3, 4\leq j\leq 5}\) is one of the following, by changing the order of \(p_4\) and \(p_5\), if necessary.

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix}
\]

\((37)\) \((38)\) \((39)\) \((40)\) \((41)\)

\[
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
1 & 0 \\
\end{pmatrix}
\]

\((42)\) \((43)\) \((44)\) \((45)\) \((46)\)

\[
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

\((47)\) \((48)\) \((49)\) \((50)\)

In the case (C), the matrix \((a_{ij})_{2\leq i\leq 3, 4\leq j\leq 5}\) is one of the following, by changing the order of \(p_4\) and \(p_5\), if necessary.

\[
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{pmatrix}
\]

\((51)\) \((52)\) \((53)\) \((54)\) \((55)\)
\[
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\]

For some of the above Rédei matrices, we can find subfields \( F = \mathbb{Q}(\sqrt{p_i^1}) \), \( \mathbb{Q}(\sqrt{p_i^1, \sqrt{p_i^2}}) \) or \( \mathbb{Q}(\sqrt{p_i^2, p_j^3, \sqrt{p_i^1 p_k^3}}) \) \( i, j, k \in \{1, 2, 3, 4, 5\} \) of the genus field of \( K \), for which we can apply Proposition (i), (ii) or (iv).

The case (A): In the cases (1) \( \sim \) (13) and (18) \( \sim \) (26), we put \( F = \mathbb{Q}(\sqrt{p_i^1}, \sqrt{p_i^2}) \), then at least one of \( p_i \)'s \( (1 \leq i \leq 3) \) splits completely in \( F \) and ramifies in \( K \), and the other two primes are unramified and split into at least two primes in \( F \) and ramify in \( K \). Therefore, the 2-class field tower of \( K \) is infinite by Proposition (ii). In the cases (14) and (36), we put \( F = \mathbb{Q}(\sqrt{p_4 p_5}, \sqrt{p_1 p_2}) \), then \( p_4 \) and \( p_5 \) split completely in \( F \) and ramify in \( K \). Therefore, the 2-class field tower of \( K \) is infinite by Proposition (ii). In these cases, the 4-ranks of \( C_K \) are as in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>(1)</th>
<th>(2), (3), (4)</th>
<th>(5), (6), (7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
<th>(13)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank ( C_K )</td>
<td>2, 3</td>
<td>1, 2</td>
<td>1</td>
<td>1, 2</td>
<td>0, 1</td>
<td>1, 2</td>
<td>1</td>
<td>0, 1, 1</td>
<td></td>
</tr>
</tbody>
</table>

In the cases where there are two possibilities, the 4-rank of \( C_K \) takes the larger value if and only if \( a_{45} = 0 \).

In the cases (15), (17), (27) and (29), we can apply Proposition (iv). The results are in the following table. In the row of “primes”, we describe the primes which split completely in \( F \) and ramify in \( K \), and in the parentheses we describe primes which are unramified in \( F \) and ramify in \( K \). In these cases, the 2-class field tower of \( K \) is infinite.

<table>
<thead>
<tr>
<th>Case</th>
<th>(15)</th>
<th>(17)</th>
<th>(27)</th>
<th>(29)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank ( C_K )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( F )</td>
<td>( \mathbb{Q}(\sqrt{p_3^1, \sqrt{p_5^1}}) )</td>
<td>( \mathbb{Q}(\sqrt{p_5^2, \sqrt{p_3^1}}) )</td>
<td>( \mathbb{Q}(\sqrt{p_3^1, \sqrt{p_4}}) )</td>
<td>( \mathbb{Q}(\sqrt{p_5^2, \sqrt{p_3^1}}) )</td>
</tr>
<tr>
<td>Primes</td>
<td>( p_1, p_2, (p_4) )</td>
<td>( p_1, p_5, (p_4) )</td>
<td>( p_1, p_2, (p_5) )</td>
<td>( p_1, p_4, (p_5) )</td>
</tr>
</tbody>
</table>

In the cases (16), (28), (30) \( \sim \) (33) and (35), the 2-class field tower of \( K \) is infinite under the additional condition \( a_{45} = 0 \). The results are in the following table.
In the case (28), the 4-rank of $C_K$ is equal to 1 if and only if $a_{45} = 0$, and in the case (31), the 4-rank of $C_K$ is equal to 1 if and only if $a_{45} = 1$.

In the remaining case (34), we have 4-rank $C_K = 0$ and we cannot prove that the 2-class field tower of $K$ is infinite.

The case (B): In the cases (37) ~ (41) and (44) ~ (46), we put $F = Q(\sqrt{p_1}, \sqrt{p_5})$, then at least one of $p_i$’s ($1 \leq i \leq 3$) splits completely in $F$ and ramifies in $K$, and the other two primes are unramified and split into at least two primes in $F$ and ramify in $K$. Therefore, the 2-class field tower of $K$ is infinite by Proposition (ii). In the cases (42) and (50), we put $Q(\sqrt{p_1p_2}, \sqrt{p_1p_5})$, then $p_4$ and $p_5$ split completely in $F$ and ramify in $K$. Therefore, the 2-class field tower of $K$ is infinite by Proposition (ii). In these cases, the 4-ranks of $C_K$ are as in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>(37)</th>
<th>(38)</th>
<th>(39)</th>
<th>(40)</th>
<th>(41)</th>
<th>(42)</th>
<th>(44)</th>
<th>(45)</th>
<th>(46)</th>
<th>(50)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>1, 2</td>
<td>0, 1</td>
<td>0, 1</td>
<td>0, 1</td>
<td>0, 1</td>
<td>1, 2</td>
<td>0, 1</td>
<td>0, 1</td>
<td>1, 2</td>
<td>0</td>
</tr>
</tbody>
</table>

In the cases (37), (38), (40) and (42), the 4-rank of $C_K$ takes the larger value if and only if $a_{45} = 0$, and in the cases (41), (44), (45) and (46), the 4-rank of $C_K$ takes the larger value if and only if $a_{45} = 1$.

In the cases (43), (47) and (48), the 2-class field tower of $K$ is infinite under the additional condition $a_{45} = 0$. The results are in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>(43)</th>
<th>(47)</th>
<th>(48)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>0, 1</td>
<td>0, 1</td>
<td>0, 1</td>
</tr>
<tr>
<td>$F$</td>
<td>$Q(\sqrt{p_5})$</td>
<td>$Q(\sqrt{p_4})$</td>
<td>$Q(\sqrt{p_2}, \sqrt{p_5})$</td>
</tr>
<tr>
<td>Primes</td>
<td>$p_1, p_2, p_4, (p_5)$</td>
<td>$p_1, p_2, p_5, (p_3)$</td>
<td>$p_1, p_5, (p_3)$</td>
</tr>
<tr>
<td>Add. cond.</td>
<td>$a_{45} = 0$</td>
<td>$a_{45} = 0$</td>
<td>$a_{45} = 0$</td>
</tr>
</tbody>
</table>
In these cases, the 4-rank of $C_K$ is equal to 1 if and only if $a_{45} = 1$.

In the remaining case (49), we have 4-rank $C_K = 0$ or 1 and we cannot prove that the 2-class field tower of $K$ is infinite. In this case, the 4-rank of $C_K$ is equal to 1 if and only if $a_{45} = 1$.

The case (C): In the cases (51) \(\sim\) (55), we put $F = \mathbb{Q}(\sqrt{p_4}, \sqrt{p_5})$, then at least one of $p_i$’s ($2 \leq i \leq 3$) splits completely in $F$ and ramifies in $K$, and the other prime and $p_1 = 2$ are unramified and split into at least two primes in $F$ and ramify in $K$. Therefore, the 2-class field tower of $K$ is infinite by Proposition (ii). In these cases, the 4-ranks of $C_K$ are as in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>(51)</th>
<th>(52)</th>
<th>(53)</th>
<th>(54)</th>
<th>(55)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>1, 2, 3</td>
<td>0, 1, 2</td>
<td>0, 1, 2</td>
<td>0, 1</td>
<td></td>
</tr>
</tbody>
</table>

In the case (51), the 4-rank of $C_K$ takes the largest value if and only if $a_{31} = a_{41} = a_{51} = a_{45} = 0$, and takes the smallest value if and only if $a_{45} = 1$ and $a_{11} \neq a_{21}$, or $a_{45} = 1$, $a_{11} = a_{21}$ and $a_{31} = 1$. In the case (52), the 4-rank of $C_K$ takes the largest value if and only if $a_{45} = 0$ and $a_{31} = a_{51} = 0$, and takes the smallest value if and only if $a_{45} = a_{31} = 1$. In the case (53), the 4-rank of $C_K$ takes the largest value if and only if $a_{45} = 0$, $a_{31} = 0$, and $a_{11} = a_{21}$, and takes the smallest value if and only if $a_{45} = 1$ and $a_{11} \neq a_{21}$. In the case (54), the 4-rank of $C_K$ takes the larger value if and only if $a_{31} = 0$. In the case (55), the 4-rank of $C_K$ takes the larger value if and only if $a_{11} = a_{21}$.

In the remaining cases, the 2-class field tower of $K$ is infinite under the additional conditions described in the following table.

<table>
<thead>
<tr>
<th>Case</th>
<th>(56)</th>
<th>(57)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-rank $C_K$</td>
<td>0, 1, 2</td>
<td>0, 1, 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F$</th>
<th>$Q(\sqrt{p_4}, \sqrt{p_5})$</th>
<th>$Q(\sqrt{p_2}, \sqrt{p_5})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primes</td>
<td>$p_1, p_2, p_3, (p_4)$</td>
<td>$p_1, p_3, (p_4)$</td>
</tr>
<tr>
<td>Add. cond.</td>
<td>$a_{51} = 0$</td>
<td>$a_{51} = 1, a_{54} = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$F$</th>
<th>$Q(\sqrt{p_2}, \sqrt{p_5})$</th>
<th>$Q(\sqrt{p_4}, \sqrt{p_5})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primes</td>
<td>$p_2, p_4, (p_1)$</td>
<td>$p_1, (p_2, p_3)$</td>
</tr>
<tr>
<td>Add. cond.</td>
<td>$a_{54} = 0$</td>
<td>$a_{54} = 1, a_{41} = a_{51} = 0$</td>
</tr>
</tbody>
</table>
In the case (56), the 4-rank of $C_K$ takes the largest value if and only if $a_{51} = a_{54} = 0$ and $a_{21} = a_{31}$, and takes the smallest value if and only if $a_{54} = 1$ and $a_{21} = a_{51}$. In the case (57), the 4-rank of $C_K$ takes the larger value if and only if $a_{54} = 0$ and $a_{51} = a_{31}$, or $a_{54} = 1$ and $a_{21} = a_{51}$. In the case (58), the 4-rank of $C_K$ takes the larger value if and only if $a_{54} = 0$ and $a_{31} = a_{41}$, or $a_{54} = 1$ and $a_{21} = a_{51}$. In the case (59), the 4-rank of $C_K$ takes the larger value if and only if $a_{54} = 0$ and $a_{51} = a_{41} = a_{54} = 0$, or $a_{54} = 1$ and $a_{21} = a_{51}$.

In the case (60), the 4-rank of $C_K$ takes the larger value if and only if $a_{54} = 0$ and $a_{51} = a_{41} = a_{54} = 0$, or $a_{54} = 1$ and $a_{21} = a_{51}$.

We summarize the result in the following table, where we divide the cases (1) ~ (50) into six classes as follows:

Case (A1): (1) ~ (15), (17) ~ (27), (29) and (36)
Case (A2): (16), (28), (30) ~ (33) and (35)
Case (A₃): (34)
Case (B₁): (37) ~ (42), (44) ~ (46) and (50)
Case (B₂): (43), (47) and (48)
Case (B₃): (49)

The numbers in the table are the numbers of types of Rédei matrices $R_K$ with exactly three negative prime discriminants, and the numbers in the parentheses are the numbers of $R_K$ for which we can prove that the 2-class field tower of $K$ is infinite.

<table>
<thead>
<tr>
<th>4-rank $C_K$</th>
<th>(A₁)</th>
<th>(A₂)</th>
<th>(A₃)</th>
<th>(B₁)</th>
<th>(B₂)</th>
<th>(B₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1 (1)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>2</td>
<td>11 (11)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>2 (2)</td>
<td>0 (0)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>1</td>
<td>38 (38)</td>
<td>6 (3)</td>
<td>0 (0)</td>
<td>8 (8)</td>
<td>3 (0)</td>
<td>1 (0)</td>
</tr>
<tr>
<td>0</td>
<td>6 (6)</td>
<td>8 (4)</td>
<td>2 (0)</td>
<td>10 (10)</td>
<td>3 (3)</td>
<td>1 (0)</td>
</tr>
<tr>
<td>Total</td>
<td>56 (56)</td>
<td>14 (7)</td>
<td>2 (0)</td>
<td>20 (20)</td>
<td>6 (3)</td>
<td>2 (0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4-rank $C_K$</th>
<th>(51)</th>
<th>(52)</th>
<th>(53)</th>
<th>(54)</th>
<th>(55)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2 (2)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>2</td>
<td>14 (14)</td>
<td>4 (4)</td>
<td>4 (4)</td>
<td>0 (0)</td>
<td>0 (0)</td>
</tr>
<tr>
<td>1</td>
<td>8 (8)</td>
<td>20 (20)</td>
<td>20 (20)</td>
<td>12 (12)</td>
<td>12 (12)</td>
</tr>
<tr>
<td>0</td>
<td>0 (0)</td>
<td>8 (8)</td>
<td>8 (8)</td>
<td>12 (12)</td>
<td>12 (12)</td>
</tr>
<tr>
<td>Total</td>
<td>24 (24)</td>
<td>32 (32)</td>
<td>32 (32)</td>
<td>24 (24)</td>
<td>24 (24)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4-rank $C_K$</th>
<th>(56)</th>
<th>(57)</th>
<th>(58)</th>
<th>(59)</th>
<th>(60)</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>3 (3)</td>
</tr>
<tr>
<td>2</td>
<td>4 (4)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>0 (0)</td>
<td>8 (4)</td>
<td>47 (43)</td>
</tr>
<tr>
<td>1</td>
<td>20 (18)</td>
<td>16 (11)</td>
<td>16 (7)</td>
<td>16 (8)</td>
<td>16 (6)</td>
<td>212 (171)</td>
</tr>
<tr>
<td>0</td>
<td>8 (4)</td>
<td>16 (11)</td>
<td>16 (7)</td>
<td>16 (8)</td>
<td>0 (0)</td>
<td>126 (93)</td>
</tr>
<tr>
<td>Total</td>
<td>32 (26)</td>
<td>32 (22)</td>
<td>32 (14)</td>
<td>32 (16)</td>
<td>24 (10)</td>
<td>388 (310)</td>
</tr>
</tbody>
</table>

Therefore we obtain the following:

**Theorem.** Let $K$ be an imaginary quadratic number field with discriminant $d$. Suppose that 2-rank $C_K = 4$ and exactly three negative prime discriminants divide $d$. Let $d = p_1^s_1p_2^s_2p_3^s_3p_4^s_4p_5^s_5$ be the unique factorization of $d$ into a product of prime discriminants, with $p_1^s_1$, $p_2^s_2$ and $p_3^s_3$ negative. If $-4$ is contained in the prime discriminants of $d$, then we assume that $p_1^s_1 = -4$. There are 100 types of Rédei matrices in the case $p_1^s_1 ≠ -4$ by changing the order of $p_i$’s, and 288 types of Rédei matrices in the case $p_1^s_1 = -4$ by changing the order of $p_i$’s ($2 ≤ i ≤ 5$). Among them, the 2-class field tower of $K$ is infinite for 310 types of $R_K$. 

10
References


[12] Y. Sueyoshi, On a comparison of the 4-ranks of the narrow ideal class groups of $\mathbb{Q}(\sqrt{m})$ and $\mathbb{Q}(\sqrt{-m})$, Kyushu J. Math. 51(2) (1997), 261-272.


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