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Endogenous Technical Progress in a Model of CO₂ Emissions

Hans W. Gottinger

Abstract

In an energy model with endogenous technical progress it is shown that both a static equilibrium in which no growth occurs in the economy and a dynamic equilibrium in which the economy grows at a constant rate are possible. We obtain results that concern the impact of changes in the values of key parameters on the equilibrium growth rate. The results show that: (i) increases in the productivity of the energy sector increase the capital to knowledge ratio in equilibrium and the economy's rate of growth; (ii) an increase in either the social discount rate or the consumption elasticity of utility causes the economy to grow more slowly; (iii) an increase in the depreciation rate of capital lowers the capital to knowledge ration and the equilibrium growth rate.

Possible approaches to equilibrium are supported by numerical examples. These examples show that both the overall productivity of the fossil energy sector and the elasticity of production with respect to fossil energy use are important in determining the model's behaviour.

1. Introduction

In this paper we set out to model the economic aspects of the CO₂ problem under endogenous technical progress. Such models appear more
natural and provide increased flexibility and realism for policy-making purposes.

The model introduced here bears a strong resemblance to the simple model with exogenous technical progress in Gottinger (1991); however, here we specify the manner in which technical progress occurs. In this model the economy does not acquire technical progress for free but must invest in knowledge and physical capital. There is an additional economic rationale for including capital accumulation and knowledge in the analysis of the CO₂ problem, that is, a compensation argument for intertemporal or intergenerational equity (Spash and d'Arge, 1989). If a given fossil fuel consumption and possible ensuing greenhouse warming cannot be avoided in this generation or the next, then at least part of the capital and knowledge should be put to mitigate the effects, or to create technologies which future generations could use to protect themselves against any harmful effects. This is part of an insurance policy on CO₂ strategies (Manne and Richels, 1992; Schelling, 1991).

This model is more flexible because the level of investment is not fixed in advance but is determined within the optimization process. A limitation of this model is that only neutral technical progress is allowed, but its usefulness can be defended on the aggregate level, as intended (V. K. Smith, 1974, Ch. 2). The three control variables in the model are the level of fossil fuel use, F; the level of investment, I, and the distribution of investment, θ. Atmospheric CO₂, M; knowledge, S, and capital, K, are all state variables. F, M, S and K all determine production, Y.

As in previous models the objective is to maximize utility discounted at the social discount rate, r, over an infinite horizon.

We assume that the utility function has a constant consumption
Endogenous Technical Progress in a Model of CO$_2$ Emissions

elasticity of utility with elasticity $\gamma$ as shown below.

$$U(C) = (C^{1-\gamma}(1-\gamma) \frac{1}{\ln C}, \gamma \neq 1$$

$$= 1 \text{ nC}, \gamma = 1$$

(1)

The effects of CO$_2$ and energy use are determined by the function $h(F, M)$ and are separable from the effects of knowledge and capital. A Cobb–Douglas type term determines the impact of knowledge and capital on production. In this term the elasticities of production with respect to knowledge and capital are $\mu$ and $\nu$, respectively. Production is:

$$Y = h(F, M) S^\mu K^\nu$$

(2)

Because of the possibility of investment, this model is significantly different from those presented before. Consumption is $C = Y - I$, and investment is divided between knowledge and capital. $\theta I$ is invested in knowledge and $(1 - \theta) I$ is invested in capital. We assume that the control variables $F$ and $I$ are both greater than or equal to zero. $\theta$ is limited between zero and one which means that knowledge cannot be changed to capital, capital cannot be changed to knowledge and neither knowledge nor capital can be consumed. The atmospheric CO$_2$ level changes as in previous models,

$$dM/dt = F - \alpha M$$

Capital and knowledge differ in two major ways. First, capital depreciates at the rate $\rho$ while knowledge does not depreciate. Second, the change in capital is a linear function of capital investment, but the change in knowledge is a non-linear function of knowledge investment. These assumptions are specified by

$$dS/dt = (\theta I)^\sigma$$

(3)
where $\sigma$ can be seen as a \textquotedblleft society-dependent\textquotedblright transformation parameter of human capital investment into knowledge.

\[
\frac{dk}{dt} = (1 - \theta)I - \rho K
\]

(4)

For reasons of convenience, we refer to the derivative of the right-hand side of (3) with respect to $I$ divided by $\theta$ as the effective knowledge investment $ESI$. $ESI$ is the marginal change in $dS/dt$ per unit of investment in knowledge.

\[
ESI = \sigma (\theta I)^{\sigma-1}
\]

(5)

The paper is organised as follows. After having specified the model we present and briefly discuss the necessary conditions in Section 2. The analysis considers possible long run equilibrium solutions (Section 3) the model\'s simplification (Section 4), the classification and characterization of equilibrium solutions (Sections 5 and 6) and possible approach paths to equilibrium.

Simplification of the model allows numerical analysis (Section 7).

2. First-Order Necessary Conditions

The Hamiltonian, $H$, for this problem is

\[
H = U(C) - q(f - M) + \zeta (\theta I)^2 + \kappa ((1 - \theta)I - \rho K)
\]

(1)

Using optimal control we derive the necessary conditions in integral form. The first three equations determine the values of the adjoint variables or the shadow prices of fossil fuels, knowledge and capital. The shadow price of fossil fuels, $q$, is
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\[ q = - \int_{t}^{\infty} e^{-(x+\alpha)(r-t)} C^{-r}(Y/h(F, M))(\delta h/\delta M)d\tau \]  
(2)

As in earlier models the shadow price of energy at time t, \( q(t) \), represents the total future loss due to a one unit increase in atmospheric CO₂. The marginal disutility of increased CO₂ is the marginal utility of consumption times the marginal decrease in production with an increase in atmospheric CO₂. Equation (1) shows that the total future loss is the marginal disutility of increased CO₂ discounted at the social discount rate plus the absorption rate of CO₂ and summed over time.

\[ \zeta = \int_{t}^{\infty} e^{-\pi(\tau-t)} C^{-r} \mu(\eta/s)d\tau \]  
(3)

\[ \kappa = \int_{t}^{\infty} e^{-\pi(x+\rho)(\tau-t)} C^{-r} \nu(Y/K)d\tau \]  
(4)

\( \zeta \) and \( \kappa \) represent the long-run values of a unit increase in knowledge and capital, respectively. \( \zeta \) is the value of a unit increase in knowledge as its marginal utility discounted at the social discount rate and summed over time. In a similar way, we define \( \kappa \) as the value of a unit increase in capital, discounted at the social discount plus the depreciation rate. The marginal utility times the marginal productivity of energy must be less than or equal to the shadow price of CO₂, \( q \)

\[ C^{-r}(Y/h(F, M))(\delta h/\delta F) \leq q \]  
(5)

with a strict inequality holding if \( F=0 \). Similar relations hold between the adjoint variables \( \zeta, \kappa \) and the marginal utility of consumption. The reasoning goes like this. Because capital changes as a linear function of investment, the value of capital also measures the value of capital investment. However, the value of investment in knowledge is measured by the effec-
tive knowledge investment times the value of an increase in knowledge. Thus we can state the sum of the values of investment in knowledge and capital weighted by the distribution of investment in each must be less than or equal to the marginal utility of consumption.

\[ \theta (ESI) \xi + (1 - \theta) \kappa \leq C^{-1} \]  

(6)

If the 'less than' condition holds, I equals 0. Furthermore, if \( \sigma \) were less than one and I were equal to 0 the effective knowledge investment would be infinite and (6) would be violated. Therefore, if \( \sigma \) is less than one then I is always greater than 0,

\[
\begin{align*}
(ESI) &> \kappa, \quad \theta = 1 \\
(ESI) &= \kappa, \quad 0 \leq \theta \leq 1 \\
(ESI) &< \kappa, \quad \theta = 0
\end{align*}
\]  

(7)

(7) assures that we invest in both knowledge and capital only when it is equally effective in each. If the value of investment is unequal in knowledge and capital, we invest only in the more valuable factor. Adjoint variables must be continuous except at the boundary values of the associated state variables. This implies that C is continuous, and it follows that Y, I, M, K, S and F are continuous.

3. Long-run Equilibrium

We first establish the existence and characteristics of two possible long-run equilibria. In one equilibrium, consumption growth is zero, in the second equilibrium, the economy grows exponentially. In equilibrium the energy-CO\(_2\) sector of the economy behaves as in the simple static model
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with given exogenous technical process (Gottinger 1991).

Even if the economy grows, F and M are constant in the equilibrium. The necessary conditions can be considerably simplified. We first examine the conditions on the energy CO₂ sector. If F is greater than 0, then from 2 (2) and 2 (5):

\[ q^* = \frac{[r + \sigma + (\delta h/\delta M)]/\delta h/\delta F} \]  

As previously, rates are designated by *. If \( \theta < 1 \), from 2 (7), \( (ESI) \zeta \leq \kappa \). We can easily solve 2 (6) for \( \kappa \):

\[ \kappa = C^{-r} \text{ for } \theta < 1 \]  

Let g denote the rate of consumption growth. By differentiating (2) with respect to time combining (2) with 2 (4) and restating the equation in terms of rates of change

\[ -rg = r + \rho - \nu (Y/K) \text{ for } \theta < 1 \]  

If \( \theta > 0 \), then

\[ \zeta C^{-r}/ESI \text{ for } \theta > 0 \]  

and

\[ -rg(1 - \sigma) (I^* + \theta^*) = r - \mu (ESI) (Y/S) \text{ for } \theta < 0 \]  

The system of necessary conditions has been considerably simplified. Both \( \zeta \) and \( \kappa \) have now been solved out of the system. In our proposed equilibria, g equals zero or is constant and greater than zero; C, Y, and I all grow at g, and \( \theta \) is constant.

The equal growth rates of C,Y, and I allow equation 1 (3) to be
satisfied. Equations 1 (4), 2 (5) and (1) can be satisfied if $q$ increases at a constant rate of $g \left(1 - \gamma\right)$ and $F$ and $M$ are constant. As noted earlier, we assume that $h$ has the same properties as $f$ in the simple model. It follows that the pair of equations 1 (4) and (1), can be solved in the same manner as the similar equations for the simple model with exogenous exponential technical progress. $F$, $M$ and $h(F,M)$ are constant, and the following relation holds:

$$M = aF$$

(6)

Along a line in the $F$–$M$ plane where $h$ is constant

$$dM/dE = 1 / \left(r + \alpha - g(1 - \gamma)\right)$$

(7)

The familiar equations in (6) and (7) are very important, the intersection of the lines defined by these equations determines the equilibrium values of $F$ and $M$ in all cases.

If no investment is made in the capital, the ratio of knowledge to capital will grow as capital stock shrinks, consequently the marginal product of capital will grow; and when the marginal product of capital is large enough, it will always be optimal to sacrifice some consumption for investment in capital. In equilibrium, investment must be greater than $0$ and $\theta$ must be greater than $0$.

We can now define the static equilibrium. In the static equilibrium $g$ equals zero, and investment is made in capital only. $I$ equals $\rho K$. The ratio of $S^e$ to $K^e$ is treated as a single variable and (3), (6), (7) form a system of three equations in three unknowns which can be solved for the equilibrium.

We now examine an equilibrium in which $g > 0$. By our previous
argument, investment in capital must be greater than zero. By a similar argument, if the economy and capital stock are growing, investment must occur in knowledge or the marginal product of knowledge goes to infinity. A growth equilibrium is, therefore, also a balanced growth equilibrium in which knowledge and capital both receive investment and both (3) and (5) hold. Next we look back at 1 (5) and 1 (6). If $\theta$ constant, $S$ increases at a rate of $g\sigma$ and $K$ increases at a rate of $g$. Examining the production function 1 (2) we reach the conclusion that a balanced growth path equilibrium exists only if a critical condition is met.

Assume $\sigma \mu + \nu = 1$ \hfill (8)

This assumption might be more easily understood by comparing the effects of a constant rate of investment in economies where (8) does and does not hold. Suppose that current inputs and the rate of investment are constant in each of the following cases:

(i) If $\sigma \mu + \nu$ equals one, the economy grows uniformly at the investment rate. Most importantly, the marginal products of knowledge and capital and the fraction of production invested are constant.

(ii) If $\sigma \mu + \nu$ is greater than one, the marginal products of knowledge and capital continually rise and the fraction of production invested continually falls.

(iii) If $\sigma \mu + \nu$ is less than one, the opposite effects would occur.
A better consideration of the reasonableness of the assumption would require data on historical trends in output, current input productivity and use, and knowledge and capital investment. For example, if such data showed that historically production, research and capital investment all grew at a common rate but that the productivity of current inputs stagnated, the assumption that $\sigma \mu + \nu$ equals one would be supported. On the other hand, if production, research, capital investment, and the productivity of current inputs all grew at a common rate, this would suggest that $\sigma \mu + \nu$ is less than one. After this digression, using the relation in (8) we can state

$$Y/K = h(F, M)[S/K^\sigma]^\rho$$

(9)

and

$$(ESI) (Y/S) = h(F, M) \sigma (\mu g)^{(\sigma - 1)/\sigma} (K^\sigma /S)^{\mu/\sigma}$$

(10)

(3) and (5) can be rewritten as:

$$-\gamma g = r + \rho + \gamma h(F, M) (S/K^\sigma)^{\mu}$$

(11)

$$-\gamma g = r + (1 - \sigma) g - \mu (\sigma g)^{(\sigma - 1)/\sigma} h(F, M) (K/S)^{\nu/\sigma}$$

(12)

Equations (11) and (12) derived from (3) and (5) have straightforward interpretations.

In both we have one term giving the marginal value of investment. For capital, this equals the marginal product of capital; and for knowledge, this equals the marginal product of knowledge times the marginal effectiveness of investment. The investment in both cases must have an equal return to the drop over time in output value and production. Output value falls due to discounting and the change in marginal utility with growth in
consumption. Production from a unit of capital disappears due to depreciation. Equations (6), (7), (11), (12) are a system of four equations in four unknowns M, F, g, and the ratio of \( K^s \) to S. These equations define the growth equilibrium, in order to lend it to policy directed computational support we have to further specify and simplify the model. Such growth equilibrium paths essentially depend on the Cobb-Douglas technology (Dixit, 1977, 1990).

4. Simplification of Model For Computational Purposes

A simple model allows us to examine the two equilibria in more detail and determine which of the equilibria is appropriate.

Again we use the step model of CO\(_2\) impacts. We assume that a critical CO\(_2\) level exists, \( M_c \), below which atmospheric CO\(_2\) has no effect upon production and above which production drops to zero. Further we assume that there is a fossil fuel productivity factor (\( \varepsilon \)) constant at \( \delta \) and the fossil fuel sector is multiplied by a scaling factor \( \lambda \) (\( \lambda \) is only introduced for the computational analysis, it is of no importance to the economic analysis). In this model the equilibrium use of energy is \( \alpha M_c \).

To simplify notation, the variable \( \omega \) is equal to the energy sector output in equilibrium, defined as \( \lambda (\alpha M_c)^\lambda \). We also assume that, as with capital, knowledge increases approximately linearly with investment or research; therefore, \( \sigma \) equals one. This follows from the assumption made in 3 (8) that \( \nu \) equals one. These assumptions are expressed as

\[
Y = \lambda F^\delta S^\sigma K^{1-\sigma}, \quad M \leq M_c
\]
\[
0, \quad M > M_c
\]

(1)
With these simplifications 3 (11) and 3 (12) can be restated as

$$\frac{ds}{dt} = 0 \tag{2}$$

$$\lambda F^\nu (1 - \mu) [S/K]^\nu = r + \rho + \gamma g \tag{3}$$

$$\lambda F^\nu \mu [K/S]^{1-\nu} = r + \gamma g \tag{4}$$

5. Classification of Equilibria

5.1 Stationary Equilibria

In stationary equilibrium only capital receives investment, \( \theta = 0 \), the CO\(_2\) level equals \( M_c \), and investment equals \( K \). 3 (6) and 4 (3) form a two-equation system in two unknowns, \( F \) and \( K/S \). The equilibrium is similar to that in many growth models in that the ratio of the stocks determines the equilibrium. \( F \) equals \( aM_c \). The ratio of \( K/S \) is determined by the equation 4 (3). The specific level of knowledge, capital and consumption at the equilibrium is determined by the initial point.

A comparative statics analysis of equilibrium can be made. The equilibrium level of fossil fuel sector output, \( \omega \), increases with increases in \( \alpha, M_c, \) and \( \delta \). These parameters either make fossil fuels more productive or CO\(_2\) limits less severe. When \( g \) equals zero from 4 (3), four parameters, \( \omega, \mu, \rho \) and \( r \) affect the equilibrium value of \( K/S \). Because greater productivity reduces the importance of capital depreciation, increases in \( \omega \) increase the marginal product of capital at any given ratio of capital to knowledge. This causes an increase in the level of the capital relative to knowledge. Conversely, increases in the social discount rate, \( r \), increase the value of current consumption versus investment in capital and increase the shadow cost of
capital. Increases in depreciation, \( \rho \), cause capital to disappear more rapidly and likewise increase the shadow cost of capital. Increases in \( r \) or \( \rho \) result in a lower relative use of capital. The impacts of changes in \( \mu \) are uncertain.

5.2 Balanced Growth Equilibrium

In the balanced growth equilibrium \( F \) again equals \( \alpha M \). Both 4(3) and 4(4) hold and these two equations can be solved for the ratio of \( K \) to \( S \) and for the rate of economic growth, \( g \). Production \( Y \) equals \( \omega S^\nu K^{1-\nu} \). Using 1(3) and 1(4), \( I \) and \( \omega \) can be found.

By taking the total differential of 4(3) and 4(4) we can examine changes in the equilibrium values of \( g \) and the ratio of \( K \) to \( S \) with respect to changes in parameter values. Again the impact of \( \mu \) on either value is uncertain. An increase in the productivity of the energy sector as measured by \( \omega \) causes both the rate of economic growth, \( g \), and the ratio of capital to knowledge to increase. Again, a more productive economy makes depreciation less important. An increase in consumption elasticity of utility, \( \gamma \), causes the economy to grow more slowly but has no effect on the capital to knowledge ratio. An increase in the social discount rate has the same effects as an increase in \( \gamma \). An increase in the capital depreciation rate lowers the equilibrium growth rate and the capital to knowledge ratio.

5.3 Optimality of Equilibrium

We assume that optimal behaviour leads to an equilibrium. However, the proofs of stability and optimality, used before, cannot be applied to this problem, because the Hamiltonian is not differentiable with respect to atmospheric \( \text{CO}_2 \) at the equilibrium. We know that for optimality the fossil fuel use and the atmospheric \( \text{CO}_2 \) level must converge to \( \alpha M_c \) and \( M_c \) respec-
If a path does not raise CO₂ to the critical level, it leaves unused capacity in the economy. For any such path we can construct a dominant path which does raise CO₂ to the critical level. If the CO₂ level remains at the critical level, the problem is reduced to a capital two-sector growth model where it will be optimal for knowledge and capital to converge to an equilibrium. If it is not optimal for CO₂ to remain at the critical level then oscillations down from and back to the critical level must be optimal. Such oscillations seem highly unlikely to be optimal, because of both discounting and the curvature of the utility function. Oscillations cause periods of reduced consumption, discounting tends to move higher levels of consumption to earlier time periods. The greater the curvature of the utility function the greater the loss in marginal value as consumption moves from low points to high points. In general, the curvature of the utility function tends to smooth out the consumption stream and eliminate cycles in consumption.

Within our means we were unable to prove that cycles are not optimal though tools along this line could be put to test (Goodwin, 1990). Instead, in what follows, I will use a simple graphical representation to argue that at least one of the equilibria classified represents long-run optimal behaviour if an optimum exists.

If the economy is very productive, an optimum defined by regular maximisation does not exist. If the utility growth rate, \( g(1 - r) \), is greater than the social discount rate, \( r \), welfare will be infinite. In this case the investment required to maintain the growth rate will be greater than the production.

Figures 1 and 2 illustrate how the appropriate equilibrium is determin
ed. The figures show levels of knowledge, OA, and of capital, OB, both as lines in the knowledge–capital plane. Along the knowledge line the marginal product of knowledge (MPS) equals the social discount rate, r, and along the capital line the marginal product of capital (MPK) equals the sum of the social discount rate, r, plus the depreciation rate, ρ

\[ MPK = \lambda F^\nu(S/K), \quad MPS = \lambda F^\nu(K/S)^{1-\nu} \]

\[ MPK - (r + \rho) = 0 \]
\[ MPS - r < 0 \]
\[ MPK - (r + \rho) > 0 \]

Figure 1: Stationary Equilibrium

In Fig. 1 the capital line lies above the knowledge line. If MPK is greater than or equal to \( r + \rho \), MPS must be less than r. In this situation \( g \) along the equilibrium path is negative. Since it is never optimal for knowledge to decrease this is clearly not an optimal equilibrium.
In Fig. 2 the knowledge line, OA, lies above the capital line, OB, and a line exists along which both the marginal product of knowledge minus the social discount rate and the marginal product of capital minus the sum of the social discount rate plus the depreciation rate are equal and greater than zero. In this case $g$ is positive and the balanced growth equilibrium is optimal.

6. Characterization of Equilibrium

We now analytically characterize the rates of change of energy use, knowledge, and capital as the equilibria are reached. The solution of these
equations requires the continuity of consumption and investment.

Fig. 3 illustrates possible transition paths to a balanced growth equilibrium, labelled 1 to 4. When fossil fuel use equals $\alpha M_c$ OA in the figure is the line along which balanced investment takes place. When the level of output from the fossil fuel sector is greater than the equilibrium output, OB is the line along which balanced investment will proceed.

As indicated by the figure the line OB lies below and moves up to coincide with OA as fossil fuel use moves toward equilibrium. Above line OA the ratio of knowledge to capital is too high and investment takes place in capital only, path 1, below OB the opposite results in path 2.

The possibility exists of combined paths in which we first invest only in capital and then invest only in knowledge, path 3. Along path four the
ratio of knowledge and capital, by reaching OA, stays on OA as OA moves. The possibility of being on such a path will be further examined later on.

Paths to a stationary equilibrium would be similar. However, when converging to a stationary equilibrium a region exists in which it is not optimal to invest in either knowledge or capital, the region between OA and OB in Fig. 1. In this region the capital stock is allowed to shrink as the equilibrium is approached.

At the moment equilibrium is reached, the continuity of production and consumption determine investment and the continuity of the left hand sides of 4 (3) and 4 (4) determine g. Knowing g and I allows us to solve for the rates of change of the variables just as the equilibrium is reached and allows the analysis of this small portion of the transition path. Because of the similarity of the approach to the growth and stationary equilibrium, we will only discuss the approach to the growth equilibrium.

From 3 (1) we derive a condition on fossil fuel use which holds when CO₂ is not at its critical level:

$$C^{-\tau} = q^*_0$$

By differentiating both sides of (1) and dividing out common elements, we obtain:

$$-\gamma g + \eta* - F* = r + \alpha$$

(2)

4 (1) provides an expression for Y in terms of F, S and K, and it can be differentiated with respect to time and substituted into (2) obtaining

$$\mu S* + (1 - \mu)K* - (1 - \delta)F* = \alpha + \gamma g$$

(3)

Because investment is continuous, the total investment in knowledge and
capital, the sum of the right–hand sides of 1 (4) and 4 (2) must be equal just before and after the equilibrium is reached. We can then derive the following equation which holds as equilibrium is reached.

$$SS^* + KK^* = g(S+K)$$  \hspace{1cm} (4)

Along regular transition paths to optimal growth, investment is either in knowledge or capital alone when equilibrium is reached. Knowledge and capital reach their equilibrium ratio after or, less likely, just as fossil fuel use reaches its equilibrium level. In all of these cases, (3) and (4) are sufficient to determine the rates of change of all variables. It is much more difficult to determine the possibility of balanced investment equilibrium. If equilibrium can be reached with investment in both knowledge and capital, we can equate 3 (11) and 3 (12), differentiate with respect to time, and develop

$$(\mu \rho + r + \gamma g)S^* - (\mu \rho + r + \gamma g)K^* + \delta \rho F^* = 0$$  \hspace{1cm} (5)

(3) – (5) form a system of three linear equations in three unknowns. One method of solving this system is to invert the matrix of coefficients of the three variables $S^*$, $K^*$ and $F^*$ and multiply the vector of the right–hand sides of these equations. A first step in finding the inverse is to find the determinant of the matrix, that is

$$D = \delta \rho S - (S+K) (\mu \rho + (1 - \delta)(r+\gamma g))$$  \hspace{1cm} (6)

We then can calculate three equations to determine $S^*$, $K^*$ and $F^*$ as the equilibrium is approached.

$$S^* = g - [\delta \rho K (\alpha + r - (1 - \gamma)g)]/[\delta \rho S - (S+K) (\mu \rho + (1 - \delta)(r+\gamma g))]$$  \hspace{1cm} (7)
\[ K^* = g + \left[ \delta \rho S (\alpha + r - (1 - \gamma)g) / \delta \rho S - (S+K) (\mu \rho + (1 - \delta) (r+\gamma g)) \right] \]  

(8)

\[ F^* = \left[ (K+S) (\mu \rho + r+\gamma g) (\alpha + r - (1 - \gamma)g) \right] / \left[ \delta \rho S (S+K) (\mu \rho + (1 - \delta) (r+\gamma g)) \right] \]  

(9)

It should be clear that if \( D \) is positive, the equilibrium cannot be reached along a balanced investment path. If \( D \) is positive, then fossil fuel use is increasing. To reach equilibrium, fossil fuel use must be greater than \( \alpha M \) and increasing at equilibrium, \( \text{CO}_2 \) must either exceed the critical level or the fossil fuel path must be discontinuous. Neither of these can be optimal. \( D \) must therefore not be positive. A high social discount rate encourages high early use of fossil fuels and makes the possibility of a positive \( D \) less likely. High values of the consumption elasticity of utility encourage high early fossil fuel use to level out consumption in a growing economy and affect \( D \) in the same manner as the social discount rate. The role of the fossil fuel productivity factor, \( \delta \) and depreciation, \( \rho \), are less clear. (5) shows that if \( \delta \) and \( \rho \) are high the ratio of knowledge to capital must change rapidly along the balanced growth path. This suggests that, when fossil fuel use is dropping and \( \delta \) and \( \rho \) are large, the investment required to maintain a balanced approach path is too high to be optimal. If \( D \) is zero, an infinity of solutions or no solutions may exist. Further, very small variations in parameters, while allowing us to find unique solutions to the problem, are likely to cause completely varying answers.

7. Numerical Examples

Further study of this model requires the use of numerical examples. The
model is not detailed or realistic enough to allow an estimation of parameters, therefore we examine the behaviour of the model over a wide range of parameter values. Because our purpose is to discover anomalies in the model's behaviour and to examine the shapes of possible optimal paths, the ability to scale the model is not of immediate concern. One major feature of equilibrium discussion has been the balanced growth equilibrium. In this context we see that the individual parameters of the energy sector, $\lambda$, $\alpha$, $M_e$ and $\delta$ do not affect the equilibrium. It is only necessary to specify the total output of the energy sector in equilibrium, $w$. The other parameters we need to specify are the consumption elasticity of utility, $\gamma$, the discount rate, $r$; the elasticity of production with respect to knowledge, $\mu$, and the depreciation rate of capital, $\rho$. In order to reduce the number of cases examined and because the value of the capital depreciation rate is less uncertain than the others, the depreciation rate of capital was set at 0.1 in all cases. The range of values of the parameters are listed:

- $\rho = 0.1$
- $\mu = 0.1 - 0.4$, $r = 0.01 - 0.1$, $\gamma = 0.5 - 0.95$
- $w = 0.1 - 0.4$

All combinations of the extreme values of the parameters provide sixteen cases which were examined for possible balanced equilibria. In four cases the optimum is not defined because $(1 - \gamma) g$ is greater than $r$. These cases are characterized by high equilibrium energy sector output, $w$, which results in a high growth rate. When this is coupled with either a low discount rate or low relative risk aversion, a finite optimum does not exist. In six more cases the balanced growth rate is negative, this condition is associated with low fossil fuel sector output. This result suggests that, when
fossil fuel use is severely limited due to CO$_2$ and substitutes are not available, the economy may stagnate even if opportunities for investment exist. The cases in which the balanced growth path represents a possible optimal equilibrium are listed below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.10</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>$r$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.95</td>
<td>0.50</td>
<td>0.95</td>
<td>0.95</td>
<td>0.50</td>
<td>0.95</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.40</td>
<td>0.10</td>
<td>0.10</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>Growth ($%$)</td>
<td>10.60</td>
<td>1.10</td>
<td>0.54</td>
<td>14.85</td>
<td>10.10</td>
<td>5.34</td>
</tr>
<tr>
<td>K/S</td>
<td>1.55</td>
<td>0.20</td>
<td>0.20</td>
<td>0.90</td>
<td>0.90</td>
<td>0.90</td>
</tr>
</tbody>
</table>

The examples show a wide variation in both the equilibrium growth rate and the equilibrium capital to knowledge ratio. Cases 2 and 3 again illustrate that low energy sector output causes low equilibrium growth rates. Cases 5 and 6 show how sensitive the model is to the curvature of the utility function. When the consumption elasticity of utility, $\gamma$, nearly doubles, the optimal growth rate is cut nearly by half. If progress and growth continue in spite of CO$_2$ the present generation is the poorest generation. The optimality of lowering fossil fuel use and/or consumption in the present to promote future growth is tempered by our desire to raise consumption for the poorest consumers, ourselves. Therefore, a high consumption elasticity leads to higher present consumption and lower growth.
Other relationships in Table 1 are not as transparent. As shown in 4 (3) and 4 (4) the marginal product of knowledge minus the marginal product of capital must equal the rate of depreciation $\rho$, in equilibrium. Increases in $\mu$ raise the marginal product of knowledge and lower the marginal product of capital, therefore, to maintain a difference in the marginal products equal to $\rho$, the K/S ratio must fall when $\mu$ rises. This is reflected in cases 1 and 4. The level of energy sector output is low, the K/S ratio becomes very low to maintain the required difference in marginal products, as in cases 2 and 3. Capital depreciation is in a sense a fixed cost of maintaining capital. When the energy sector is large this fixed cost is not too important. However, when energy sector output and growth is low, the fixed cost is important and the level of capital fails to raise the marginal product of capital. This suggests a rule of thumb that if the economy slows down due to CO$_2$ or other influences, a shift away from inputs with substantial fixed costs would possibly occur. Furthermore, we can conclude: when the fossil fuel sector output is shrinking the K/S ratio must be changing to maintain the difference in the marginal products. It follows that the rate of change in capital and knowledge will be very different in these cases.

If the fossil fuel output shrinks rapidly to low levels, the structure of the economy must change very rapidly. Non-depreciating assets such as knowledge must increase rapidly compared to regular capital assets. A fundamental shift to a service economy? The economic interpretation of the conditions on balanced investment approaches to the balanced growth equilibrium is limited because of the complex expression for the variables’ rates of change. We use numerical examples to gain some insights into possible balanced approach paths. In all examples production is normalized at one. To determine the rate of change in fossil fuel use the absorption rate of
carbon dioxide and the fossil fuel productivity factor must be specified. In all these examples $\alpha$ is 0.001 and $\epsilon$ is 0.8.

The rates of change as the equilibrium is neared are listed in Table 2.

<table>
<thead>
<tr>
<th>Table 2: Rates of Change as Equilibrium is Approached, Balanced Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha=0.001$</td>
</tr>
<tr>
<td>$\epsilon=0.8$</td>
</tr>
<tr>
<td>Case</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>Parameters:</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>$\nu$</td>
</tr>
<tr>
<td>$\omega$</td>
</tr>
<tr>
<td>Results:</td>
</tr>
<tr>
<td>$g(%)$</td>
</tr>
<tr>
<td>$S^*$</td>
</tr>
<tr>
<td>$K^*$</td>
</tr>
<tr>
<td>$F^*$</td>
</tr>
</tbody>
</table>

The most significant feature of this table is that in the three cases with low discount rates-cases two, three, and four-fossil fuel use is increasing or decreasing very slowly. The effect of the social discount rate of fossil fuel was discussed earlier. Growth in consumption along a balanced equilibrium is too slow to cause fossil fuel use, higher initial investment, and higher consumption growth. The higher fossil fuel use would cause CO$_2$ to reach equilibrium before knowledge and capital.

As noted earlier, when both receive investment, the marginal pro-
ducts of knowledge and capital must be different by the depreciation rate, $\delta$

We also noted earlier that low energy sector output reduced the differences in these marginal products. This same relationship is evident here in cases one, five and six. When the fossil fuel sector output is shrinking the ratio of capital to knowledge must be changing to maintain the difference in the marginal products; it follows, that the rate of change in capital and knowledge will be very different in these cases. If the fossil fuel output shrinks rapidly to low levels, the structure of the economy must change very rapidly. Non-depreciating assets such as knowledge must increase rapidly compared to regular capital assets.

References


