Supplement to $L^2$ Estimates for the $\bar{\partial}$ Operator on a Stein Manifold

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Abstract

A revised version of the $L^2$ estimate of my previous note and an alternative proof of the approximation theorem on a Stein manifold are given.

1. Review of the $\bar{\partial}$ equation.

The setting is the same as my previous note [1], so we review briefly. Let $\Omega$ be a Stein manifold of complex dimension $n$. Let $\{\eta_\nu\}$ be a sequence of functions in $C^\infty_c(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and $\eta_\nu = 1$ on any compact subset of $\Omega$ when $\nu$ is large. Choose a Hermitian metric $ds^2 = h_{ik} dz^i \bar{dz}^k$ on $\Omega$ so that $|\partial \eta_\nu| \leq 1$ for $\nu = 1, 2, \ldots$. Denote by $dV$ the volume element defined by $ds^2$. Let $\varphi$ be a real valued continuous function on $\Omega$ and let $L^2_{(p,q)}(\Omega, \varphi)$ be the weighted $L^2$ space of $(p,q)$ forms such that $\|f\|_{\varphi}^2 = \int |f|^2 e^{-\varphi} dV < \infty$ where $\| \cdot \|_\varphi$ denotes the length with respect to $ds^2$. The $\bar{\partial}$ operator defines linear, closed, densely defined operators on these spaces.

In my previous note we give a $C^\infty$ function $\Psi$ on $\Omega$ which satisfies

(a) $\Psi$ is strictly plurisubharmonic
(b) $\Psi \geq 0$ on $\Omega$
(c) $\Omega_c = \{ z \in \Omega | \Psi(z) < c \} \subset \subset \Omega$ for every $c \in \mathbb{R}$
(d) $\|f\|_\varphi^2 \leq \|T^* f\|_\varphi^2 + \|S f\|_\varphi^2 \quad \text{for} \quad f \in D_{(p,q+1)}(\Omega)$.

And then we have the following existence theorem.

Theorem 1 [1]. Let $\varphi$ be any plurisubharmonic function on $\Omega$. For every $g \in L^2_{(p,q+1)}$
$(\Omega, \varphi)$ with $\bar{\partial} g = 0$, there exists a solution $u \in L^2_{p,q}(\Omega, \text{loc})$ of the equation $\bar{\partial} u = g$ such that

$$\int |u|^2 e^{-\varphi} dV \leq \int |g|^2 e^{-\varphi} dV.$$

### 2. Results

Denote by $A = A(\Omega)$ the space of all entire holomorphic functions on $\Omega$ with the Frechet topology of uniform convergence on all compact sets. The following is a revised version of Theorem 2 in [1].

**Theorem 2 (Revised).** Let $\varphi$ be any plurisubharmonic function on $\Omega$ and denote by $A_\varphi$ the set of entire holomorphic functions $u$ such that for some real number $N$,

$$\int |u|^2 e^{-\varphi - N\varphi} dV < \infty.$$

Then the closure $\text{cl} A_\varphi$ of $A_\varphi$ in $A$ contains all $u \in A$ such that $|u|^2 e^{-\varphi}$ is locally integrable, and $\text{cl} A_\varphi$ is equal to $A$ if and only if $e^{-\varphi}$ is locally integrable.

**Proof.** Given an entire function $U$ such that $|U|^2 e^{-\varphi}$ is locally integrable we shall approximate $U$ uniformly in a relatively compact set $\Omega_R = \{ z \in \Omega | \varphi(z) < R \}$ by functions in $A_\varphi$. To do so we choose a cut function $\chi \in C_c(\Omega)$ so that $\chi = 1$ on $\Omega_{R+1}$ and $\chi = 0$ on $\Omega \setminus \Omega_{R+2}$. Set $V = \chi U$. Then

$$V = U \text{ on } \Omega_R \text{ and } \bar{\partial} V = U \bar{\partial} \chi = 0 \text{ on } \Omega_{R+1} \cup (\Omega \setminus \Omega_{R+2}).$$

To make norms small we set weight functions $\varphi_t$ for $t > 0$ as

$$\varphi_t(z) = \varphi(z) + \max \{ 0, t(\varphi(z) - R - 1) \}.$$

Then $\varphi_t$ is plurisubharmonic and

$$\int |\bar{\partial} V|^2 e^{\varphi_t} dV = \int_{\Omega_{R+1}} |U|^2 |\bar{\partial} \chi|^2 e^{-\varphi - t(\varphi - R - 1)} dV$$

$$\leq \sup_{\Omega_{R+2}} |\bar{\partial} \chi|^2 \int_{\Omega_{R+1}} |U|^2 e^{-\varphi - t(\varphi - R - 1)} dV \rightarrow 0 \text{ as } t \rightarrow 0$$

since $|U|^2 e^{-\varphi} \in L^2_{\text{loc}}$. It follows from Theorem 1 that we can find a function $U_t$ with $\bar{\partial} u_t = \bar{\partial} V$ and

$$\int |u_t|^2 e^{-\varphi_t} dV \leq \int |\bar{\partial} V|^2 e^{-\varphi_t} dV \rightarrow 0 \text{ as } t \rightarrow \infty.$$
\[
\int_{\Omega_{R-1}} |u_i|^2 \, dV = \int_{\Omega_{R-1}} |u_i|^2 e^{-\varphi} e^{\phi^+} \, dV \\
\leq \sup_{\Omega_{R+1}} e^{\phi^+} \int_{\Omega_{R-1}} |u_i|^2 e^{-\varphi} \, dV \longrightarrow 0
\]
since \( \varphi = \varphi \) on \( \Omega_{R+1} \). Hence
\[
\sup_{\Omega_{R+1}} |u_i| \leq C \int_{\Omega_{R-1}} |u_i|^2 \, dV \longrightarrow 0, \quad \text{i.e.}
\]
u_i \longrightarrow 0 uniformly on \( \Omega_R \). We know that
\[
V = (V - u_i) + u_i \quad \text{and} \quad \overline{\partial} (V - u_i) = 0.
\]
And we have
\[
\int |V - u_i|^2 e^{-\varphi - N\varphi} \, dV \leq 2 \int |V|^2 e^{-\varphi - N\varphi} \, dV + 2 \int |u_i|^2 e^{-\varphi - N\varphi} \, dV.
\]
The 1-st term in the right hand side converges since \(|U|^2 e^{-\varphi} \in L^1_{\text{loc}} \). For the 2-nd term put \( N = 1 + t \). Then \( \varphi + \Psi = \varphi + \Psi \)
\[
\leq \varphi + (1 + t) \Psi \quad \text{on} \quad \Omega_{R+1} \quad \text{and} \quad \varphi + \Psi = \varphi + (1 + t) \Psi - t(R + 1)
\]
\[
\leq \varphi + (1 + t) \Psi \quad \text{on} \quad \Omega \setminus \Omega_{R+1} \quad \text{and so}
\]
\[
\int |u_i|^2 e^{-\varphi - (1 + t)\Psi} \, dV \leq \int |u_i|^2 e^{-\varphi - \Psi} \, dV < \infty.
\]
Hence \( V - u_i \in A_{\varphi} \). This proves the first assertion. For the second assertion we note that every function in \( A_{\varphi} \) must vanish at \( z \) if \( e^{-\varphi} \) is not integrable in any neighborhood of \( z \). Because if \( u \in A_{\varphi} \) and \( u(z) \neq \infty \) then there exists a neighborhood \( W \) of \( z \) such that \(|u| \geq \delta > 0 \) on \( W \) and a contradiction that
\[
\int |u|^2 e^{-\varphi - N\varphi} \, dV \quad \text{is} \quad \int |u|^2 e^{-\varphi} \, dV < \infty
\]
follows. From this it is easy to see that \( \text{cl} A_{\varphi} = A \) implies \( e^{-\varphi} \in L^1_{\text{loc}} \).

The same argument gives an alternative proof of the following approximation theorem on a Stein manifold.

\textbf{Theorem([4], 5.2.8).} Let \( \Omega \) be a complex manifold and \( \varphi \) a strictly plurisubharmonic \( C^\infty \) function on \( \Omega \) such that
\[
K_c = \{ z \in \Omega \mid \varphi(z) \leq c \} \subset \subset \Omega \quad \text{for every real number} \quad c.
\]
Every function which is holomorphic in a neighborhood of \( K_0 \) can then be approximated uniformly on \( K_0 \) by entire functions in \( \Omega \).
Proof. Let $U$ be a holomorphic function in $K_c(c>0)$. Choose a cut function $\chi \in C^\infty(\Omega)$ so that $\chi=1$ on $K_{c/2}$ and $\chi=0$ on $\Omega \setminus K_c$. Set $V=\chi U$ and

$$\varphi_t(z) = \varphi(z) + \max \{0, t(\Psi(z) - c/2)\}.$$ 

Then $\varphi_t$ is plurisubharmonic and

$$V = U \text{ on } K_0, \quad \bar{\partial}V = U\bar{\partial}\chi = 0 \text{ on } K_{c/2} \cup (\Omega \setminus K_c)$$

$$\int_{K_c \setminus K_{c/2}} |\bar{\partial}V|^2 e^{-\varphi_t} dV = \int_{K_c \setminus K_{c/2}} |\bar{\partial}V|^2 e^{-\varphi_t(\Psi - c/2)} dV \to 0 \text{ as } t \to \infty.$$ 

It then follows from Theorem 1 that we find a function $u_t$ such that $\bar{\partial}u_t = \bar{\partial}V$ and

$$\int_{K_{c/2}} |u_t|^2 e^{-\varphi_t} dV \leq \int_{K_c \setminus K_{c/2}} |\bar{\partial}V|^2 e^{-\varphi_t} dV \to 0.$$ 

In particular $\bar{\partial}u_t = 0$ on $K_{c/2}$ i.e. $u_t$ is holomorphic there and

$$\int_{K_c \setminus K_{c/2}} |u_t|^2 dV \to 0,$$

so $u_t \to 0$ uniformly on $K_0$. Since $V = (V - u_t) + u_t$ and $\bar{\partial}(V - u_t) = 0$, $U$ is uniformly approximated on $K_0$ by entire functions $V - u_t$. \qed

At this juncture we correct some errata in my previous note [1]. In Theorem 2 and 2', [1], the assumption that $\varphi \in C^2(\Omega)$ is dropped. In the proof of Therem 2, p.8, line 10, "We may assume that $\varphi \in C^2(\Omega)$" should be "must not". The case when $\varphi$ is not in $C^2$ is treated in this supplement.

References


