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An Approximation Theorem on Some Convex Domains

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Abstract

Let Ω be a convex domain with real analytic boundary which is a generalized type of the complex ellipsoid. Then the approximation theorem in the $H^p$-sense holds in Ω.

Introduction. Let $G$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then Stout [3] proved that the approximation theorem in the $H^p$-sense, $1 \leq p < \infty$, holds in $G$. Beatrous[1] studied the approximation theorem in a weighted Bergman space.

In the present paper, we shall prove that the results of Stout are also true for some convex domain $\Omega$ with real analytic boundary. That is, the following theorem holds.

**Theorem.** If $f \in H^p(\Omega)$, $1 \leq p < \infty$, then there exists a sequence $\{f_n\}$ in $O(\overline{\Omega})$ that converges in the $H^p$-sense to $f$.

Finally we shall adopt the convention of denoting by $c$ any positive constant which does not depend on the relevant parameters in the estimate.

1. Preliminaries. Let $s_i$, $1 \leq i \leq n$, be real analytic functions in an interval $[0, a_i]$ such that
   (i) $s_i(0) \geq 0$, $s_i(t) + 2ts_i'(t) > 0$ for $0 < t < a_i$
   (ii) $s_i(0) = 0$, $s_i(a_i) > 1$.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ of the type

$$\Omega = \{z: \rho(z) < 0\}$$

where

$$\rho(z) = \sum_{i=1}^{n}s_i(|z_i|^2) - 1 \text{ for } z = (z_1, \ldots, z_n).$$

For example,
is one of the above domains, where \( m_i \)s are positive even integers. Bruna and Castillo [2] proved the following fundamental inequality.

\[ (1) \quad p(z) - p(z + 2 \Re F(z, z)) \geq c(L_{m}(\Omega)(\xi - z)^2 + |\xi - z|^m)(\xi, z \in \Omega) \]

where \( m \) is a positive integer,

\[ F(z, z) = \sum_{i,j=1}^{n} \frac{\partial^2 \rho(\xi)}{\partial z_i \partial \bar{z}_j} (\xi - z_i)(\bar{z}_j - z). \]

We set

\[ H(\xi, z) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^2 \rho(\xi)}{\partial z_i \partial \bar{z}_j} |F(\xi, z)|^n. \]

Let \( f^* \) be the boundary value of \( f \in H^p(\Omega), 1 \leq p < \infty \). Then \( f^* \in L^p(\partial \Omega) \). Now we have

\[ (2) \quad f(z) = \int_{\partial \Omega} f^*(\xi) \mathcal{H}(\xi, z) (z \in \Omega). \]

We define

\[ a_i(\xi) = \frac{\partial^2 \rho(\xi)}{\partial z_i \partial \bar{z}_j}. \]

They by the fundamental inequality (1), we obtain

\[ a_i(\xi) \leq |\rho(\xi)| + |\xi - z|^2 + |\Im F(\xi, z)| + |\xi - z|^m. \]

Let \( \gamma \) be a \( C^\infty \) function in \( \Omega \), and \( g \in L^p(\partial \Omega), 1 \leq p < \infty \).

We set

\[ Tg(z) = \int_{\partial \Omega} g(\xi)(\gamma(\xi) - \gamma(z)) \mathcal{H}(\xi, z) \]

Then we have the following.

**Proposition 1.** If \( g \in L^p(\partial \Omega), 1 \leq p < \infty \), then

\[ \sup_{r \leq r_{\delta}} \int_{r_{\delta}^2} |Tg(z)| \delta \sigma(z) < \infty, \]

where \( \delta \sigma \) is the surface measure on \( \{\rho = r\} \).

**Proof.** First we prove that \( Tg \) is bounded, provided \( g \) is bounded. We set

\[ \rho(z) = t_1, \quad \Im F(\xi, z) = t_2, \]

\[ t_{2j-1} + it_{2j} = \xi_j - z, \quad j = 2, \ldots, n, \]

\[ t' = (t_3, \ldots, t_{2n}), \quad dt' = dt_3 \ldots dt_{2n}. \]

Then it holds that

\[ |\xi - z| \sim |t_1| + |t_2| + |t'|. \]

We denote by \( b(\xi, z) \) each coefficient of \( H(\xi, z) \). Then we have

\[ |Tg(z)| \leq c \int_{\partial \Omega} |b(\xi, z)| |\xi - z| \delta \sigma(\xi) \]
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\[ \leq c \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{(t_1 + |t_2| + |t'|) \alpha_1(\xi) \ldots \alpha_{n-1}(\xi) dt_2 dt'}{|F(\xi, z)|^m} \]

\[ \leq c \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{(t_1 + |t_2| + |t'|) dt_2 dt'}{(t_1 + |t_2| + |t'|^m) \prod_{j=2}^{n} (t_1 + |t_2| + t_{2j-1} + t_{2j} + |t'|^m)} \]

We set \( w_j = t_{2j-1} + it_{2j} \). We choose \( \delta (0 < \delta < 1) \) so small that \( nm \delta < 1 \). We set

\[ P(t) = (t_1 + |t_2| + |t'|^m) \prod_{j=2}^{n} (t_1 + |t_2| + w_j |t|^2 + |t'|^m). \]

Then we have

\[ \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{|t'| dt_2 dt'}{P(t)} \leq \]

\[ \leq c \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{|t'| dt_2 dt'}{\prod_{j=2}^{n} (t_1 + |t_2| + w_j |t|^2 + |t'|^m) \prod_{j=2}^{n} |t_1|^2 (1 - \delta)^2 |t_2|^2} \]

\[ \leq c \prod_{j=2}^{n} \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{dt_2 dt'}{|t_1|^2 (1 - \delta)^2 < \infty.} \]

\[ \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} P(t) \frac{dt_2 dt'}{\prod_{j=2}^{n} (t_1 + |t_2| + w_j |t|^2 + |t'|^m)} \leq c \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{dt_2 dt'}{|w_j|^2 (1 - \delta)^2 |t_2|^2} \]

\[ \leq c \prod_{j=2}^{n} \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{dt_2 dt'}{|w_j|^2 (1 - \delta)^2 < \infty.} \]

Therefore \( Tg(z) \) is bounded. Next we prove proposition 1 when \( p = 1 \). By the Fubini's theorem, we have

\[ \int_{\rho = r} |Tg(z)| \, d\sigma(z) \leq c \int_{\rho = r} |g(\xi)| \left( \int_{\rho = r} |b(\xi, z)| \, |\xi - z| \, d\sigma(z) \right) \, d\sigma(\xi). \]

On the other hand we have

\[ \int_{\rho = r} |b(\xi, z)| \, |\xi - z| \, d\sigma(z) \]

\[ \leq c \int_{|t_2| \leq \delta_0, \ |t'| \leq \delta_0} \frac{(r + |t_2| + |t'|) dt_2 dt'}{(r + |t_2| + |t'|^m) \prod_{j=2}^{n} (r + |t_2| + w_j |t|^2 + |t'|^m)} \]

By the estimate above, we have

\[ \sup \int_{\rho = r} |b(\xi, z)| \, |\xi - z| \, d\sigma(z) < \infty. \]

Thus we obtain

\[ \sup \int_{\rho = r} |Tg(z)| \, d\sigma(z) \leq c \int_{\rho = r} |g(\xi)| \, d\sigma(\xi). \]

For \( r < 0 \) near \( 0 \), let \( \varOmega_r = \{ \rho(z) < r \} \), and let \( T^{(r)} : L^1(\partial \Omega) \to C(\partial \Omega_r) \) be the linear operator defined by \( T^{(r)} g = Tg |_{\partial \Omega_r} \). From the above proof, there is a constant \( c \), independent of \( r \) such that

\[ \| T^{(r)} g \|_{L^1(\partial \Omega_r)} \leq c \| g \|_{L^1(\partial \Omega)}, \]

\[ \| T^{(r)} g \|_{L^1(\partial \Omega_r)} \leq c \| g \|_{L^1(\partial \Omega)}. \]
The Riesz–Thorin theorem implies that if \( g \in L^p(\partial \Omega), \ 1 < p < \infty \), then
\[
\|T^\gamma g\|_{L^q(\partial \Omega)} \leq c \|g\|_{L^p(\partial \Omega)}.
\]
Therefore proposition 1 is proved.

**Proposition 2.** If \( f \in H^p(\Omega), \ 1 \leq p < \infty \), and if \( \gamma \) is a \( C^\infty \) function on \( C^\infty \), then function defined by
\[
f_\gamma(z) = \int_{\partial \Omega} f^*(\xi) \gamma(\xi) H(\xi, z)
\]
belongs to \( H^p(\Omega) \).

**Proof.** From the formula (2), we have
\[
\bar{f}(z) = \int_{\partial \Omega} \bar{f}^*(\xi) (\gamma(\xi) - \gamma(z)) H(\xi, z) + \gamma(z) f(z).
\]
We write in the form \( \bar{f}(z) = f_1(z) + f_2(z) \), say. Then in view of proposition 1,
\[
\left( \int_{\partial \Omega} |\bar{f}(z)|^p |d\sigma(z)| \right)^\frac{1}{p} \leq \left( \int_{\partial \Omega} |f_1|^p |d\sigma(z)| \right)^\frac{1}{p} + \left( \int_{\partial \Omega} |f_2|^p |d\sigma(z)| \right)^\frac{1}{p} \leq c.
\]
Therefore \( \bar{f} \in H^p(\Omega) \), which completes the proof.

2. **Proof of the theorem.** The proof of the theorem can be obtained by following proofs of Stout [3]. But we sketch the proof briefly. Let \( U = \{U_1, \ldots, U_q\} \) be an open cover of \( \partial \Omega \) such that if \( P_j \in U_j \) and \( \nu_j \) is unit outward normal to \( \partial \Omega \) at \( P_j \), then \( z - \epsilon \nu_j \) approach \( z \) nontangentially through \( \Omega \) as \( \epsilon \to 0^+ \). Let \( \{\gamma_1, \ldots, \gamma_q\} \) be a smooth partition of unity on \( \partial \Omega \) that is subordinate to \( U \), and let
\[
f_j(z) = \int_{\partial \Omega} f^*(\xi) \gamma_j(\xi) H(\xi, z).
\]
Then, by proposition 2, we have \( f_j \in H^p(\Omega) \). Moreover, \( f_j \) is holomorphic on a neighborhood of the compact set \( \partial \Omega \setminus U_j \) and satisfies \( f = f_1 + \cdots + f_q \). Define
\[
f_j^{(\epsilon)}(z) = f_j(z - \epsilon \nu_j).
\]
Then it holds that \( f_j^{(\epsilon)} \in O(\Omega) \) and
\[
\lim_{\epsilon \to 0} \int_{\partial \Omega} |f_j - f_j^{(\epsilon)}| |d\sigma| = 0.
\]
This completes the proof of the theorem.

**References**

