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Hölder Estimates for the $\overline{\partial}$-Problem in some Convex Domains with Real Analytic Boundary

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Abstract

Let $\Omega$ be a convex domain which is a generalized type of the real ellipsoid. Then there is a solution for the $\overline{\partial}$-problem in $\Omega$ that satisfies the Hölder estimates.

1. Introduction. Let $D$ be a real ellipsoid, i.e.,

$$D = \{ x + iy \in \mathbb{C}^N : \sum_{i=1}^{N} x_i^{2n_i} + \sum_{j=1}^{N} y_j^{2m_j} < 1 \}$$

where $n_1, \ldots, n_N, m_1, \ldots, m_N$ are positive integers. Then Diederich-Fornaess-Wiegerinck [3] obtained $\frac{1}{q}$-Hölder estimates for solutions of $\overline{\partial}$-problem in $D$, where $q = \max_j \min\{2n_j, 2m_j\}$. On the other hand, Range [4] obtained $(\frac{1}{p} - \varepsilon)$-Hölder estimates, $\varepsilon > 0$, in the complex ellipsoid $E$, i.e.,

$$E = \{ z : \sum_{j=1}^{N} |z_j|^{2n_j} < 1 \}$$

where $p = \max_j 2n_j$. In the paper [3], it is shown that Range's solution satisfies $\frac{1}{p}$-Hölder estimates. Further, Bruna-Castillo [2] generalized Range's results to more general convex domains. In the present paper, we shall prove the existence of a solution that satisfies Hölder estimates in the domain $\Omega$ which is a somewhat generalized type of the real ellipsoid.

Finally we shall adopt the convention of denoting by $c$ any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries.

Let $s_i(x_i), t_i(y_i), i = 1, \ldots, N$, be real analytic functions on $[0, a]$. We set
\[ \phi_i(x) = s_i(x^i), \phi_j(y) = t_j(y^j) \]

Suppose that \( \phi_i, \phi_j, i = 1, \ldots, N, \) satisfy the following conditions (i), (ii), (iii):

(i) \( \phi_i''(x_i) \geq 0, \phi_j''(y_j) \geq 0 \)

(ii) \( \phi_i(0) = \phi_j(0) = 0, \phi_i(a) > 1, \phi_j(a) > 1 \)

(iii) \( \phi_i''(x_i) + \phi_j''(y_j) > 0 \) for \( (x_i, y_j) \neq (0, 0). \)

We set

\[ \rho_i(z_i) = \phi_i(x_i) + \phi_j(y_j) \text{ for } z_i = x_i + iy_j, \]

\[ \rho(z) = \sum_{i=1}^{N} \rho_i(z_i) \text{ for } z = (z_1, \ldots, z_N), \]

and

\[ \Omega = \{ z : \rho(z) < 0 \}. \]

For \( \eta > 0 \) sufficiently small, we define

\[ \Omega_\eta = \{ z : \rho(z) < \eta \}. \]

Then \( \Omega, \Omega_\eta \) are convex domains in \( C^n \) with real analytic boundary. We define

\[ h_i(x_i, \xi_i) = \rho_i(x_i) - \rho_i(\xi_i) - \rho_i(\xi_i)(x_i - \xi_i) \]

Then we have

**Lemma 1.** There exists \( \varepsilon > 0 \) such that

(1) \( h_i(x_i, \xi_i) > 0 \) for \( |x_i| < \varepsilon, |\xi_i| < \varepsilon, x_i \neq \xi_i \).

**Proof.** In some neighborhood of 0, \( \phi_i(x_i) \) can be written in the following form.

\[ \phi_i(x_i) = b_1^{(1)} x_i^{2n_1} + b_2^{(1)} x_i^{2n_2} + \cdots + (b_1^{(1)} > 0, n_1 \geq 1). \]

Therefore we have for some \( \varepsilon > 0, \)

\[ \phi_i'(x_i) > 0, \phi_i''(x_i) > 0 \text{ for } 0 < |x_i| < \varepsilon. \]

Thus we obtain the equality (1).

In view of lemma 2 of Adachi[1], we have the following.

**Lemma 2.** Let \( \xi_i = \xi_j + i\eta_j, z_j = x_j + iy_j \). Then there exist positive constant \( \varepsilon \) and \( c \) such that

(2) \( \phi_i(x_i) - \phi_i(\xi_i) - \phi_i(\xi_j)(x_i - \xi_j) \geq c \phi_i''(\xi_j)(x_i - \xi_j)^2 + (x_i - \xi_j)^{2n_1} \]

(3) \( \phi_i(y_j) - \phi_i(\eta_j) - \phi_i(\eta_j)(y_j - \eta_j) \geq c \phi_i''(\eta_j)(y_j - \eta_j)^2 + (y_j - \eta_j)^{2n_2} \)

for \( |\xi_i| < \varepsilon, |z_j| < \varepsilon. \)

We set

\[ q = \max \min \{2n_i, 2m_i\}. \]

Let \( f = \sum_{i,j} f_{ij}(z)d\bar{z}_j \) be a \( (0, 1) \)-form on \( \Omega \), and \( u \) be a function on \( \Omega \). We define

\[ \|f\|_{L^\infty(\Omega)} = \max_{v} \sup_{z \in \Omega} |f_v(z)|, \|u\|_\infty = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^q}. \]
Then we shall prove the following.

**Theorem.** Let $\Omega$ and $q$ be as above. Then there exists a constant $c$ such that for every bounded $\bar{\partial}$-closed $(0, 1)$ form $f$ on $\Omega$, there exists a $\frac{1}{q}$-Hölder continuous function $u$ on $\Omega$ such that

$$\bar{\partial}u = f$$

and

$$\|u\|_{\frac{1}{q}} \leq c \|f\|_{L^1(\Omega)}.$$

3. Proof of the theorem. It is sufficient to prove the theorem for $f \in C_{(0, 1)}(\bar{\Omega})$. We assume $m_i \leq n_i$ for $i = 1, \ldots, N$. We set for $\zeta_i = \xi_i + i\eta_i$, $z_i = x_i + iy_i$,

$$p_i(\zeta_i, z_i) = 2 \frac{\partial \rho_i}{\partial \xi_i}(\zeta_i) + \gamma[(\psi_i''(\eta_i) - \phi_i''(\xi_i))(\zeta_i - z_i) + (\zeta_i - z_i)^{2m_i - 1}]$$

and

$$F_i(\zeta_i, z_i) = p_i(\zeta_i, z_i)(\zeta_i - z_i).$$

Taking account of the equalities (2), (3), if we choose $\gamma > 0$ sufficiently small, we have (see Adachi [1]),

$$-\rho_i(\zeta_i) + \rho_i(z_i) + \Re F_i(\zeta_i, z_i) \geq c[(\phi_i''(\xi_i) + \phi_i''(\eta_i)) |z_i - \xi_i|^2 + |z_i - \zeta_i|^{2m_i}]$$

for $|\zeta_i| < \varepsilon$, $|z_i| < \varepsilon$, $i = 1, \ldots, N$.

We set

$$F(\zeta, z) = \sum_{i=1}^{N} F_i(\zeta_i, z_i)$$

for $\zeta = (\zeta_1, \ldots, \zeta_N)$, $z = (z_1, \ldots, z_N)$.

Thus we obtain from the equalities (4),

$$-\rho(\zeta) + \rho(z) + \Re F(\zeta, z) \geq c \sum_{i=1}^{N} [(\phi_i''(\xi_i) + \phi_i''(\eta_i)) |z_i - \xi_i|^2 + |z_i - \zeta_i|^{2m_i}]$$

for $|\zeta| < \varepsilon$, $|z| < \varepsilon$.

Since we cannot construct the support function $\Phi(\zeta, z)$ which depends holomorphically on $z$, we apply the same method as the proof of Bruna-Castillo [2]. We set

$$G_i(\zeta_i, z_i) = -2 \frac{\partial \rho_i}{\partial \xi_i}(\zeta_i)(\zeta_i - z_i) - \frac{\partial^2 \rho_i}{\partial \xi_i^2}(\zeta_i)(\zeta_i - z_i)^2.$$

Then from the condition (iii), we have for some $\delta > 0$,

$$-\rho(\zeta) + \rho(z) + \Re G_i(\zeta_i, z_i) \geq c |z_i - \zeta_i|^2$$

for $|\zeta| \geq \frac{\varepsilon}{2}$, $|z - \zeta| < \delta$.

Let $\phi(\zeta)$ be a $C^\infty$ function in the complex plane with the properties that, $0 \leq \phi \leq 1$, $\phi(\zeta) = 1$ for $|\zeta| < \frac{\varepsilon}{2}$, $\phi(\zeta) = 0$ for $|\zeta| \geq \frac{2\varepsilon}{3}$. We define

$$\tilde{F}_i(\zeta_i, z_i) = \phi(\zeta_i)F_i(\zeta_i, z_i) + (1 - \phi(\zeta_i))G_i(\zeta_i, z_i)$$

and

$$\tilde{P}_i(\zeta_i, z_i) = \phi(\zeta_i)P_i(\zeta_i, z_i) + (1 - \phi(\zeta_i)){\left(-2 \frac{\partial \rho_i}{\partial \xi_i}(\zeta_i) - \frac{\partial^2 \rho_i}{\partial \xi_i^2}(\zeta_i)(\zeta_i - z_i)\right)}.$$
(7) \( \hat{F}_j(\xi, z_i) = F_j(\xi, z_i)(\xi_i - z_i) \),
(8) \(- \rho_j(\xi) + \rho(z) + \text{Re} \hat{F}(\xi, z) \geq c \sum_{j=1}^{N} [ (\phi_j^*(\xi_j) + \phi_j^*(\eta_j))|z_j - \xi_j|^2 + |z_j - \xi_j|^{2m} ] \)
for \(|\xi - z_i| < \delta\).

We define
\[
\Phi(\xi, z) := \sum_{j=1}^{N} \hat{F}_j(\xi, z_i) \text{ for } (\xi, z) \in \Omega \times \Omega.
\]
Then it holds from (8) that
\[- \rho(\xi) + \rho(z) + \text{Re} \hat{F}(\xi, z) \geq c \sum_{j=1}^{N} [ (\phi_j^*(\xi_j) + \phi_j^*(\eta_j))|z_j - \xi_j|^2 + |z_j - \xi_j|^{2m} ] \]
for \(|\xi - z| < \delta\).

To complete the Hölder estimates, we apply the elementary methods by Range[5] in order to construct the integral solution operator for the \( \bar{\partial} \)-problem. Choose \( \chi \in C^{\infty}(CN \times CN) \) such that, \( 0 \leq \chi \leq 1 \), \( \chi(\xi, z) = 1 \) for \(|\xi - z| < \frac{\delta}{2}\), and \( \chi(\xi, z) = 0 \) for \(|\xi - z| \geq \delta\). For \( j = 1, \ldots, N \), we define
\[
Q_j(\xi, z) = \chi \hat{F}_j(\xi, z_i) + (1 - \chi)(\xi_j - z_i),
\]
and
\[
\Phi(\xi, z) := \sum_{j=1}^{N} Q_j(\xi, z_i)(\xi_j - z_i).
\]
Then, from the equality (7), we have
\[
\Phi(\xi, z) = \chi \hat{F}(\xi, z) + (1 - \chi)|\xi - z|^2.
\]
There exist positive numbers \( \eta \) and \( c \) such that
\[
|\Phi(\xi, z)| \geq c \text{ for } \xi \in \partial \Omega, \rho(z) < \eta, |\xi - z| \geq \frac{\delta}{2}.
\]
For \( t \in [0, 1] \) and \( \xi \in \partial \Omega \), we set
\[
w_j(\xi, z, t) = tQ_j(\xi, z) + (1 - t)(\xi_j - z_i)
\]
and
\[
W = \sum_{j=1}^{N} w_j d\xi_j.
\]
Then \( w_j(\xi, z, t) \) is well defined for
\[
z \in \Omega \cup \{ z : \rho(z) \leq \eta, |z - \xi| = \frac{\delta}{2} \}.
\]
For \( q = 0, 1 \), and \( f \in C(0, 1)(\Omega) \), define
\[
K_q(W) = (2\pi)^{-N-1}{\binom{N-1}{q}}W \wedge (\bar{\partial}z)^{N-q-1} \wedge (\bar{\partial}_z W)^q,
\]
\[
T_{sf} = \int_{\partial x_1} f(\xi) W \wedge \partial x_1 f \wedge K(W),
\]
\[
E_{sf} = \int_{\partial x_1} f(\xi) W \wedge K(W).
\]
Then we have
(9) \( f = Ef + \bar{\partial} T_{\partial f}. \)
Moreover \( Ef \) has the following properties (see Range[5]).
(a) \( Ef \) is \( C^\infty \) on \( \tilde{\Omega}_g \)
(b) \( \|Ef\|_{L^\infty(\partial\Omega)} \leq c \|f\|_{L^\infty(\partial\Omega)} \).
For \( (\zeta, z) \in \partial\Omega_g \times \tilde{\Omega}_g \), we define
\[
\Gamma(\zeta, z) = \sum_{k=1}^{N} \frac{\partial P}{\partial \xi_k}(\zeta)(\zeta_k - z_k).
\]
Then the convexity of \( \tilde{\Omega}_g \) implies
\[
\Gamma(\zeta, z) \neq 0 \text{ for } (\zeta, z) \in \partial\Omega_g \times \tilde{\Omega}_g.
\]
Define
\[
S_j(\zeta, z) = \frac{\partial P}{\partial \xi_j}(\zeta),
\]
\[
u_j(\zeta, z, \lambda) = \lambda S_j(\zeta, z) + (1 - \lambda) \frac{\zeta_j - \bar{z}_j}{|\zeta - \bar{z}|^2}
\text{ for } (\zeta, z, \lambda) \in \partial\Omega_g \times \tilde{\Omega}_g \times [0, 1],
\]
\[
U = \sum_{j=1}^{N} \nu_j d\xi_j.
\]
Since \( S_j \) is holomorphic in \( z \), we have \( K_1(u) = 0 \). We define for \( g \in \mathcal{C}^{0,1}(\tilde{\Omega}_g) \),
\[
(10) \quad T_{\partial} g = \int_{\partial\Omega \times [0, 1]} g \wedge K_0(U) - \int_{\partial\Omega \times [0, 1]} g \wedge K_0(U)
\]
Then we have \( \bar{\partial}(T_{\partial} g) = g \) provided \( g \in \mathcal{C}^{0,1}(\tilde{\Omega}_g) \), \( \bar{\partial} g = 0 \). We define the operator
\[
(11) \quad S = T_{\partial} E + T_{\partial}.
\]
Then, for \( f \in \mathcal{C}^{0,1}(\tilde{\Omega}) \) with \( \bar{\partial} f = 0 \), we have from the equality (9), (11),
\[
(12) \quad S f = f.
\]
Since the first integral in (10) is \( C^\infty \) in \( \tilde{\Omega}_g \) and the kernel of the second integral is the
Bochner–Martinelli kernel, we have
\[
(13) \quad \|T_{\partial}(Ef)\|_{\mathcal{C}^{\alpha}(\tilde{\Omega}_g)} \leq c(\alpha) \|Ef\|_{L^\infty(\partial\Omega)} \text{ for } \alpha < 1.
\]
Therefore \( Sf \) is the desired solution of the theorem if we can prove the following
inequality.
\[
(14) \quad \|T_{\partial} f\|_{\mathcal{C}^{\alpha}(\tilde{\Omega}_g)} \leq c \|f\|_{L^\infty(\partial\Omega)}.
\]
Since the denominator of \( K_0(W) \) does not vanish for \( \zeta \neq z \), it is sufficient to estimate
the integral near the diagonal. If \( |\zeta - z| < \frac{\delta}{2} \), then \( Q_1(\zeta, z) = \tilde{P}_1(\zeta, z) \). Therefore if we take
\[
L_j = \frac{\partial P}{\partial \xi_j} \frac{\partial}{\partial \xi_k} - \frac{\partial P}{\partial \xi_j} \frac{\partial}{\partial \xi_j} - \frac{\partial P}{\partial \xi_j} \frac{\partial}{\partial \xi_N} (j = 1, \ldots, N - 1)
\]
as a base for the \((0, 1)\) tangential vector fields on \( \partial\Omega \cap B \), \( B \) being a small ball with
center on \( \partial\Omega \), we have for \( i, j = 1, \ldots, N - 1 \),
\[
|L_j Q_1| \leq \delta_0 c \left( \phi_i'(\xi_i) + \phi_i''(\eta_i) + |\phi_i'''(\eta_i)| + |\phi_i'''(\xi_i)| \right) |z_i - \xi_i|,
\]
\[
|L_j Q_N| \leq c \left( |\xi_i|^{m_i - 1} + |\eta_i|^{m_i - 1} \right).
\]
By the estimate in lemma 4 of Adachi[1], we can prove that \( T_{\partial} f \) satisfies the inequality
which completes the proof of the theorem.

References


