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NAOSITE: 長崎大学の学術研究成果リポジトリ
Hölder Estimates for the $\overline{\partial}$-Problem in some Convex Domains with Real Analytic Boundary

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Abstract

Let $\Omega$ be a convex domain which is a generalized type of the real ellipsoid. Then there is a solution for the $\overline{\partial}$-problem in $\Omega$ that satisfies the Hölder estimates.

1. Introduction. Let $D$ be a real ellipsoid, i.e.,

$$D = \{(x+iy) \in \mathbb{C}^N : \sum_{i=1}^{N} x_i^{2n_i} + \sum_{i=1}^{N} y_i^{2m_i} < 1\}$$

where $n_1, \ldots, n_N, m_1, \ldots, m_N$ are positive integers. Then Diederich-Fornaess-Wiegerinck [3] obtained $\frac{1}{q}$-Hölder estimates for solutions of $\overline{\partial}$-problem in $D$, where $q = \max_j \min\{2n_j, 2m_j\}$. On the other hand, Range [4] obtained $(\frac{1}{p} - \varepsilon)$-Hölder estimates, $\varepsilon > 0$, in the complex ellipsoid $E$, i.e.,

$$E = \{z : \sum_{j=1}^{N} |z_j|^{2n_j} < 1\}$$

where $p = \max_j 2n_j$. In the paper [3], it is shown that Range's solution satisfies $\frac{1}{p}$-Hölder estimates. Further, Bruna-Castillo [2] generalized Range's results to more general convex domains. In the present paper, we shall prove the existence of a solution that satisfies Hölder estimates in the domain $\Omega$ which is a somewhat generalized type of the real ellipsoid.

Finally we shall adopt the convention of denoting by $c$ any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries.

Let $s_i(x_i), t_i(y_i), i=1, \ldots, N$, be real analytic functions on $[0, a]$. We set
\(\phi_i(x_i) = s_i(x_i), \phi_i(y_i) = t_i(y_i)\)

Suppose that \(\phi_i, \phi_j, i=1, \ldots, N,\) satisfy the following conditions (i), (ii), (iii);

(i) \(\phi''(x_i) \geq 0, \phi''(y_i) \geq 0\)

(ii) \(\phi_i(0) = \phi_i(0) = 0, \phi_i(a) > 1, \phi_i(a) > 1\)

(iii) \(\phi''(x_i) + \phi''(y_i) > 0\) for \((x_i, y_i) \neq (0, 0)\).

We set

\[
\rho_i(z_i) = \phi_i(x_i) + \phi_i(y_i) \quad \text{for} \quad z_i = x_i + iy_i
\]

\[
\rho(z) = \sum_{j=1}^{N} \rho_j(z_j) \quad \text{for} \quad z = (z_1, \ldots, z_N),
\]

and

\[
\Omega = \{z : \rho(z) < 0\}.
\]

For \(\eta > 0\) sufficiently small, we define

\[
\Omega_{\eta} = \{z : \rho(z) < \eta\}.
\]

Then \(\Omega, \Omega_{\eta}\) are convex domains in \(\mathbb{C}^N\) with real analytic boundary. We define

\[
h_j(x_j, \xi_j) = \rho_j(x_j) - \rho_j(\xi_j) - \rho_j(\xi_j)(x_j - \xi_j)
\]

Then we have

**Lemma 1.** There exists \(\epsilon > 0\) such that

(1) \(h_j(x_j, \xi_j) > 0\) for \(|x_j| < \epsilon, |\xi_j| < \epsilon, x_j \neq \xi_j\).

**Proof.** In some neighborhood of 0, \(\phi_j(x_j)\) can be written in the following form.

\[
\phi_j(x_j) = b^{(0)} x_j^{2n} + b^{(1)} x_j^{2m+2} + \ldots (b^{(0)} > 0, n_j \geq 1)
\]

Therefore we have for some \(\epsilon > 0,

\[
\phi_j'(x_j) > 0, \phi_j''(x_j) > 0 \quad \text{for} \quad 0 < |x_j| < \epsilon.
\]

Thus we obtain the equality (1).

In view of lemma 2 of Adachi[1], we have the following.

**Lemma 2.** Let \(\xi_j = \xi_j + i\eta_j, z_j = x_j + iy_j\). Then there exist positive constant \(\epsilon\) and \(c\) such that

(2) \(\phi_j(x_j) - \phi_j(\xi_j) - \phi_j'(\xi_j)(x_j - \xi_j)
\]

\[
\geq c[\phi_j''(\xi_j)(x_j - \xi_j)^2 + (x_j - \xi_j)^{2m}]
\]

(3) \(\phi_j(y_j) - \phi_j(\eta_j) - \phi_j'(\eta_j)(y_j - \eta_j)
\]

\[
\geq c[\phi_j''(\eta_j)(y_j - \eta_j)^2 + (y_j - \eta_j)^{2m}]
\]

for \(|\xi_j| < \epsilon, |z_j| < \epsilon\).

We set

\[
q = \max_{j} \min\{2n_j, 2m_j\}.
\]

Let \(f = \sum f_v(z_v)dz_v\) be a \((0, 1)\)-form on \(\Omega\), and \(u\) be a function on \(\Omega\). We define

\[
\|f\|_{\omega(\Omega)} = \max_{v} \sup_{z \in \Omega} |f_v(z)|, \|u\|_\omega = \sup_{x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^2}.
\]
Then we shall prove the following.

**THEOREM.** Let $\Omega$ and $q$ be as above. Then there exists a constant $c$ such that for every bounded $\overline{\partial}$-closed $(0, 1)$ form $f$ on $\Omega$, there exists a $\frac{1}{q}$-Hölder continuous function $u$ on $\Omega$ such that

$$\overline{\partial} u = f \text{ and } \|u\|_{2,q} \leq c \|f\|_{L^q(\Omega)}.$$ 

3. **Proof of the theorem.** It is sufficient to prove the theorem for $f \in C_{(0, 1)}(\overline{\Omega})$. We assume $m_i \leq m_i$ for $i = 1, \ldots, N$. We set $\xi_j = \xi_i + i \eta_i$, $z_i = x_i + iy_i$,

$$p_i(\xi_i, z_i) = 2 \frac{\partial \rho_i}{\partial \xi_i}(\xi_i) + \gamma[(\phi_i'(\eta_i) - \phi_i'(\xi_i))(\xi_i - z_i) + (\xi_i - z_i)^{2m_i - 1}]$$

and

$$F_i(\xi_i, z_i) = p_i(\xi_i, z_i)(\xi_i - z_i).$$

Taking account of the equalities (2), (3), if we choose $\gamma > 0$ sufficiently small, we have (see Adachi [1]),

$$-p_i(\xi_i) + p_i(z_i) + \Re F_i(\xi_i, z_i) = c[(\phi_i'(\xi_i) + \phi_i'(\eta_i)) |z_i - \xi_i|^2 + |z_i - \xi_i|^{2m_i}]$$

for $|\xi_i| < \varepsilon$, $|z_i| < \varepsilon$, $j = 1, \ldots, N$.

We set

$$F(\xi, z) = \sum_{j=1}^{N} F_j(\xi_j, z_j) \text{ for } \xi = (\xi_1, \ldots, \xi_N), z = (z_1, \ldots, z_N).$$

Thus we obtain from the equalities (4),

$$-\rho(\xi) + \rho(z) + \Re F(\xi, z) \geq c \sum_{j=1}^{N} [(\phi_j'(\xi_j) + \phi_j'(\eta_j)) |z_j - \xi_j|^2 + |z_j - \xi_j|^{2m_j}]$$

for $|\xi| < \varepsilon$, $|z| < \varepsilon$. Since we cannot construct the support function $\Phi(\xi, z)$ which depends holomorphically on $z$, we apply the same method as the proof of Bruna-Castillo [2]. We set

$$G_i(\xi_i, z_i) = -2 \frac{\partial \rho_i}{\partial \xi_i}(\xi_i)(\xi_i - z_i) - \frac{\partial \rho_i}{\partial \xi_i}(\xi_i)(\xi_i - z_i)^2.$$ 

Then from the condition (iii), we have for some $\delta > 0$,

$$-\rho(\xi) + \rho(z) + \Re G_i(\xi_i, z_i) \geq c |z_i - \xi_i|^2$$

for $|\xi| \geq \varepsilon$, $|z - \xi| < \delta$.

Let $\phi(\xi)$ be a $C^\infty$ function in the complex plane with the properties that, $0 \leq \phi \leq 1$,

$$\phi(\xi) = \begin{cases} 1 & \text{for } |\xi| < \frac{\varepsilon}{2}, \\ 0 & \text{for } |\xi| \geq \frac{2\varepsilon}{3}. \end{cases}$$

We define

$$\tilde{F}_i(\xi_i, z_i) = \phi(\xi_i) F_i(\xi_i, z_i) + (1 - \phi(\xi_i)) G_i(\xi_i, z_i)$$

and

$$\tilde{P}_i(\xi_i, z_i) = \phi(\xi_i) P_i(\xi_i, z_i) + (1 - \phi(\xi_i)) \left(-2 \frac{\partial \rho_i}{\partial \xi_i}(\xi_i) - \frac{\partial \rho_i}{\partial \xi_i}(\xi_i)(\xi_i - z_i)\right)$$

Then we have
(7) \( \tilde{F}_j(\xi, z_i) = \tilde{F}_j(\xi, z_i)(\xi_i - z_i) \)

(8) \(-\rho_i(\xi_i) + \rho_i(z) + \text{Re} \tilde{F}_j(\xi, z) \geq c\sum_{j=1}^{N} (|\phi_j(\xi_j) + \phi_j''(\eta_j)| |z_j - \xi_j|^2 + |z_j - \xi_j|^{2m}) \)

for \( |\xi_i - z_i| < \delta \).

We define
\[
\tilde{F}(\xi, z) = \sum_{j=1}^{N} \tilde{F}_j(\xi, z_i) \text{ for } (\xi, z) \in \Omega \times \Omega.
\]

Then it holds from (8) that
\[
-\rho(\xi) + \rho(z) + \text{Re} \tilde{F}(\xi, z) \geq c\sum_{j=1}^{N} (|\phi_j(\xi_j) + \phi_j''(\eta_j)| |z_j - \xi_j|^2 + |z_j - \xi_j|^{2m}) \]

for \( |\xi - z| < \delta \).

To complete the Hölder estimates, we apply the elementary methods by Range[5] in order to construct the integral solution operator for the \( \overline{\partial} \)-problem. Choose \( \chi \in C^\infty(\mathbb{C}^N \times \mathbb{C}^N) \) such that, \( 0 \leq \chi \leq 1 \), \( \chi(\xi, z) = 1 \) for \( |\xi - z| \leq \frac{\delta}{2} \), and \( \chi(\xi, z) = 0 \) for \( |\xi - z| \geq \delta \). For \( j = 1, \ldots, N \), we define
\[
Q_j(\xi, z_i) = \chi \tilde{F}_j(\xi, z_i) + (1 - \chi)(\overline{\xi_j} - z_i)
\]
and
\[
\Phi(\xi, z) = \sum_{j=1}^{N} Q_j(\xi, z_i)(\xi_j - z_i).
\]
Then, from the equality (7), we have
\[
\Phi(\xi, z) = \chi \tilde{F}(\xi, z) + (1 - \chi) |\xi - z|^2.
\]

There exist positive numbers \( \eta \) and \( c \) such that
\[
|\Phi(\xi, z)| \geq c \text{ for } \xi \in \partial \Omega, \rho(z) < \eta, |\xi - z| \geq \frac{\delta}{2}.
\]

For \( t \in [0, 1] \) and \( \xi \in \partial \Omega \), we set
\[
w_j(\xi, z, t) = t Q_j(\xi, z_i) + (1 - t) \overline{\xi_j - z_i} \]
and
\[
W = \sum_{j=1}^{N} w_j d\xi_j.
\]
Then \( w_j(\xi, z, t) \) is well defined for
\[
z \in \Omega \cup \{ z : \rho(z) \leq \eta, |z - \xi| \geq \frac{\delta}{2} \}.
\]

For \( q = 0, 1, \) and \( f \in C^0(\overline{\Omega}) \), define
\[
K_q(W) = (2\pi i)^{-N} \binom{N-1}{q} W \wedge (\overline{\partial}_z W)^{N-q-1} \wedge (\overline{\partial}_z W)^q,
\]
\[
T_qf = \int_{\partial \times [0, 1]} f \wedge K_q(W) - \int_{\partial \times \{t \}} f \wedge K_q(W),
\]
\[
E_qf = \int_{\partial \times \{1 \}} f \wedge K_q(W).
\]
Then we have
(9) \( f = E_f + \overline{\partial} T_{\overline{\partial}} f \).

Moreover \( E_f \) has the following properties (see Range [5]).

(a) \( E_f \) is \( C^\infty \) on \( \overline{\Omega} \).

(b) \( \| E_f \|_{L^\infty(\partial \Omega)} \leq C \| f \|_{L^\infty(\partial \Omega)} \).

For \( (\zeta, z) \in \partial \Omega \times \partial \Omega \), we define
\[
\Gamma(\zeta, z) = \sum_{k=1}^{N} \frac{\partial}{\partial z_k} (\zeta_k - z_k).
\]

Then the convexity of \( \Omega \) implies
\[
\Gamma(\zeta, z) \neq 0 \text{ for } (\zeta, z) \in \partial \Omega \times \partial \Omega.
\]

Define
\[
S_0(\zeta, z) = \frac{\partial \rho}{\partial \zeta_0} (\zeta).
\]

\[
u_0(\zeta, z, \lambda) = \lambda \frac{S_0(\zeta, z)}{\Gamma(\zeta, z)} + (1 - \lambda) \frac{\overline{\zeta} - \overline{z}}{|\overline{\zeta} - \overline{z}|^2}
\]

for \( (\zeta, z, \lambda) \in \partial \Omega \times \partial \Omega \times [0, 1] \),
\[
U = \sum_{j=1}^{N} u_j d \zeta_j.
\]

Since \( S_j \) is holomorphic in \( z \), we have \( K_j(u) = 0 \). We define for \( g \in C^0(\partial \Omega, \overline{\Omega} \) ,
\[\begin{align*}
\| T_0 g \|_{C^\alpha(\Omega)} &\leq c(\alpha) \| E_f \|_{L^\infty(\partial \Omega)} \text{ for } \alpha < 1.
\end{align*}\]

Therefore \( Sf \) is the desired solution of the theorem if we can prove the following inequality.
\[\| T_0 f \|_{C^\alpha(\Omega)} \leq c \| f \|_{L^\infty(\partial \Omega)} \].

Since the denominator of \( K_0(W) \) does not vanish for \( \zeta \neq z \), it is sufficient to estimate the integral near the diagonal. If \( |\zeta - z| < \frac{\delta}{2} \), then \( Q_0(\zeta, z) = \overline{P}(\zeta, z) \). Therefore if we take
\[
L_j = \frac{\partial}{\partial z_j} \frac{\partial}{\partial \overline{z}_j} - \frac{\partial}{\partial \overline{z}_j} \frac{\partial}{\partial z_j} \quad (j = 1, \ldots, N - 1)
\]
as a base of the \((0, 1)\) tangential vector fields on \( \partial \Omega \cap B \), \( B \) being a small ball with center on \( \partial \Omega \), we have for \( i, j = 1, \ldots, N - 1 \),
\[
|L_j Q_0| \leq \delta_0 c(\phi_i(\zeta_i) + \phi_i(\eta_i) + (\phi_i(\zeta_i) + (\phi_i(\eta_i) |z_i - \zeta_i|)
\]
\[
|L_{\overline{z}} Q_0| \leq c(\zeta_i |z_i|^{n-1} + |\eta_i| |z_i|^{n-1} - 1).
\]

By the estimate in lemma 4 of Adachi [1], we can prove that \( T_0 f \) satisfies the inequality
(4), which completes the proof of the theorem.

References


