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A Note on $L^2$ Estimates for the $\overline{\partial}$ Operator on a Stein Manifold

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Abstract

Two results concerning the $\overline{\partial}$ operator on a Stein manifold are obtained. We construct a strictly plurisubharmonic function. First one is the existence of solutions of the $\overline{\partial}$ equation in the weighted $L^2$ space of the weight diminished by the function. Second one is the denseness of a set of related holomorphic functions in the space of entire functions. These are variants of the similar results in $\mathbb{C}^n$ ([3], Theorem 4.4.1, 4.4.2, and 4.4.4) to a Stein manifold.

1. Preliminaries — The $\overline{\partial}$ equation on a Stein manifold.

Let $\Omega$ be a Stein manifold of complex dimension $n$. We fix such $\Omega$ throughout this note. We recall here the fundamental setting of the $\overline{\partial}$ equation on $\Omega$ as [3], 5.1 and 5.2. Let $\{\eta_\nu\}$ be a sequence of functions in $C^\infty_c(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and $\eta_\nu = 1$ on any compact subset of $\Omega$ when $\nu$ is large. Then we can choose a Hermitian metric $ds^2 = h_{ij} dz^i d\bar{z}^j$ on $\Omega$ so that $|\overline{\partial} \eta_\nu| \leq 1$ for $\nu = 1, 2, \ldots$. Keep the Hermitian metric $ds^2$ and the sequence $\eta_\nu$ fixed. Denote by $dv$ the volume element defined by $ds^2$.

Let $\varphi$ be a real valued function in $C^2(\Omega)$ and let $L^2_{(p,q)}(\Omega, \varphi)$ be the space of all (equivalence classes of) forms of type $(p,q)$, such that the coefficients are measurable in any local coordinate system, and

$$\|f\|^2 = \int |f|^2 e^{-\varphi} \, dV \leq \infty.$$
The operator \( \overline{\partial} \) defines linear, closed, densely defined operators
\[
L^2(\mathcal{O}, \varphi) \xrightarrow{T} L^2_{(p, q+1)}(\mathcal{O}, \varphi) \xrightarrow{S} L^2_{(p, q+2)}(\mathcal{O}, \varphi).
\]
The solvability of the \( \overline{\partial} \) equation: \( RT = N_s \) follows from an estimate
\[
\|f\|_2^2 \leq C(\|T^*f\|_p^2 + \|Sf\|_p^2), f \in D_{(p, q+1)}(\Omega),
\]
where \( D_{(p, q+1)}(\Omega) \) is the space of \((p, q+1)\) forms with the coefficients in \( D(\Omega) = C_c(\Omega) \). \( D_{(p, q+1)}(\Omega) \) is dense in the domain \( D_T \cap D_s \) for the graph norm \( \|f\|_p + \|T^*f\|_p + \|Sf\|_p \) (see 4.1 and Lemma 5.2.1 in [3]).

We construct a \( C^\infty \) strictly plurisubharmonic function \( \psi \) on \( \Omega \) which satisfies the above estimate and is convenient to our purpose. A Stein \( \Omega \) has a function \( \psi \in C^\infty(\Omega) \) such that
\( \psi \)
- is strictly plurisubharmonic
- \( \psi \geq 0 \) on \( \Omega \)
- \( \Omega_c = \{ z \in \Omega ; \psi(z) < c \} \subset \subset \Omega \) for every \( c \in \mathbb{R} \).

This is [3], Theorem 5.1.6 for example and the existence of such functions with properties (a) and (c) is equivalent to the fact that a complex manifold \( \Omega \) is Stein by [3], Theorem 5.2.10. Adopting \( \psi \) as a weight and applying (5.2.12) in [3], we have
\[
\int(\lambda - C)|f|^2 e^{-\psi} dV \leq 4(\|T^*f\|_p^2 + \|Sf\|_p^2), f \in D_{(p, q+1)}.
\]
Here \( C \) is a positive continuous function on \( \Omega \) independent of \( f \) and \( \psi \), and \( \lambda \) is the lowest eigenvalue of the Hermitian symmetric form
\[
H_\psi(t, t) = \sum \psi_{ij} t^i \bar{t}^j, t \in C^n
\]
where \( \partial \overline{\partial} \psi = \sum \psi_{ij} w^i \wedge \bar{w}^j \) with respect to a local orthonormal basis \( w^1, \cdots, w^n \) for \((1, 0)\) forms. \( \lambda \) is a positive continuous function by the strict plurisubharmonicity of \( \psi \). We construct \( \Psi \) as the form \( \Psi = \chi_0(\psi) \) where \( \chi_0 \) is a \( C^\infty \) convex increasing function of real one variable. The lower bound \( \lambda \) for \( H_\psi \) is replaced by \( \chi_0'(\psi) \lambda \) for \( H_\varphi \). We can choose \( \chi_0 \) so rapidly increasing that it satisfies (1) \( \chi_0'(\psi) \lambda - C \geq 4 \) (2) \( \chi_0 \geq 0 \) and (3) \( \lim_{t \to +\infty} \chi_0(t) = +\infty \). It then follows by (5.2.12) and Lemma 5.2.1 in [3] that
\[
\int|f|^2 e^{-\Psi} dV \leq \|T^*f\|_p^2 + \|Sf\|_p^2, f \in D_{(p, q+1)}(\Omega),
\]
and that the function \( \Psi \) satisfies the same properties (a), (b) and (c) as \( \psi \). This weight function \( \Psi \) is a substitute of \( \log(1 + |z|^2) \) in the case of \( C^n \) (see [3], Theorem 4.4.2, 4.4.4).

2. Results — Two estimates for the \( \partial \overline{\partial} \) operator.

**Theorem 1.** Let \( \Omega \) be a Stein manifold and let \( \Psi \) be the strictly plurisubharmonic function constructed as above. Let \( \varphi \) be any plurisubharmonic function on \( \Omega \). Then for every \( g \in L^2_{(p, q+1)}(\Omega, \varphi) \) satisfying \( \partial \overline{\partial} g = 0 \) one can find a solution \( u \in L^2_{(p, q)}(\Omega, \varphi) \).
loc) of the equation $\bar{\partial} u = g$ such that
\[
\int |u|^2 e^{-\varphi - \psi} \, dV \leq \int |g|^2 e^{-\varphi} \, dV.
\]

**Proof.** Since $\Psi \geq 0$,
\[
\int |g|^2 e^{-\varphi - \psi} \, dV \leq \int |g|^2 e^{-\psi} \, dV \leq \infty; \text{ i.e. } g \in L^2_{\nu, q+1}(\Omega, \varphi + \Psi).
\]

First assume $\varphi \in C^2$. The lower bound for the Hermitian form $H_{\varphi + \Psi}$ of $\varphi + \Psi$ is greater than that of $\Psi$. It follows that
\[
\|f\|_{\nu, q}^2 \leq \|T^*f\|_{\nu, q}^2 + \|Sf\|_{\nu, q}^2, \quad f \in D_{\nu, q+1}.
\]
This implies that the $\bar{\partial}$ equation is solvable in $L^2(\Omega, \varphi + \Psi)$. That is, the equation $\bar{\partial} u = g$ has a solution $u \in L^2_{\nu, q}(\Omega, \varphi + \Psi)$ for every $g \in L^2_{\nu, q+1}(\Omega, \varphi + \Psi)$ satisfying $\bar{\partial} g = 0$, and $u$ can be chosen so that $\|u\|_{\nu, q} \leq \|g\|_{\nu, q+1}$ by an argument involving the Hahn-Banach theorem. Next assume that $\varphi$ is not in $C^2$. Since $\varphi$ is plurisubharmonic, there exists a decreasing sequence $\{q_n\}_{n=1}$ of $C^\omega$ plurisubharmonic functions on $\Omega$ such that $\lim_{n \to \infty} q_n = \varphi$ almost everywhere on $\Omega$. For every $\nu$ we can find $u_\nu$ so that $\bar{\partial} u_\nu = g$ and
\[
\int |u_\nu|^2 e^{-\varphi - \psi} \, dV \leq \int |g|^2 e^{-\varphi} \, dV \leq \int |g|^2 e^{-\varphi} \, dV.
\]
Since $\varphi_\nu$ decreases, this shows that the $L^2$ norm of $u_\nu$ over any compact set is bounded. We can therefore choose a subsequence $\{u_\mu\}$ which converges weakly on every compact set to a limit function $u \in L^2_{\nu, q}(\Omega, \loc)$ by a diagonal argument. Hence $\bar{\partial} u = g$. For fixed $\nu$ and $R > 0$, $u_\mu \to u$ weakly in $L^2_{\nu, q}(\Omega_\nu, \varphi_\nu + \Psi)$ where $\Omega_\nu = \{z \in \Omega : \Psi(z) < R\}$ is relatively compact. And for $\mu > \nu$, $-\varphi_\nu \leq -\varphi_\mu$ so the norms are bounded by
\[
\int_{\Omega_\nu} |u_\mu|^2 e^{-\varphi_\nu - \psi} \, dV \leq \int_{\Omega_\nu} |u_\mu|^2 e^{-\varphi_\nu} \, dV \leq \int |g|^2 e^{-\psi} \, dV.
\]
Hence the weak limit has the same bound
\[
\int_{\Omega_\nu} |u|^2 e^{-\varphi_\nu - \psi} \, dV \leq \int |g|^2 e^{-\psi} \, dV.
\]
Letting $\nu \to \infty$ and $R \to \infty$, we get the desired estimate.

**Theorem 2.** Let $\Omega$ be a Stein manifold and let $\Psi$ be the strictly plurisubharmonic function constructed as above. Let $\varphi$ be any plurisubharmonic function on $\Omega$. Then the set of entire functions $u \in A(\Omega)$ such that
\[
\int |u|^2 e^{-\varphi - N\nu} \, dV < \infty
\]
for some nonnegative integer $N$, contains functions not identically zero. In fact, these functions are dense in the space $A = A(\Omega)$ of all entire holomorphic functions with the Frechet topology of uniform convergence on compact sets.
PROOF. The topology in \( A \) of uniform convergence on all compact sets is equivalent to the Frechet topology of \( L^2 \) convergence on all compact sets. For every continuous linear functional \( L \) on \( A \) one can therefore by F. Riesz' representation theorem find a function \( \nu \in L^2 \) with compact support such that

\[
L(u) = \int u \overline{\nu} \, dV, \quad u \in A.
\]

Put \( B = \{ u \in A : \int |u|^2 e^{-\nu} \, dV < \infty \text{ for some integer } N \} \). We have to show that \( L = 0 \) on \( A \) if \( L = 0 \) on \( B \). Our claim is then a consequence of the Hahn-Banach theorem. Take any functional such that \( L = 0 \) on \( B \). Then \( \int u \overline{\nu} \, dV = 0 \) for \( u \in B \). Denote by \( R \) a positive number such that \( \text{supp } \nu \subset \Omega_R \).

We may assume that \( \varphi \in C^\infty \) since every plurisubharmonic function has the approximation from above of \( C^\infty \) plurisubharmonic functions. Set \( \varphi_N(z) = \varphi(z) + \Psi(z) + N\chi(\Psi(z)) \), where \( \chi \) is a convex increasing \( C^\infty \) function vanishing precisely on \( (-\infty, R] \) and \( \chi(t) = t + c \) near \( +\infty \). Then \( \varphi_N \) is strictly plurisubharmonic and the lower bound for the Hermitian form of \( \varphi_N \) is greater than that of \( \Psi \). It follows that

\[
\|f\|_{L^\infty} \leq \|T^*f\|_{L^\infty} + \|Sf\|_{L^\infty}, \quad f \in D_{T^*} \cap D_S.
\]

We shall consider the \( \overline{\partial} \) operator on the function space

\[
L^2(\Omega, \varphi_N)^T \rightarrow L^2_{(0,1)}(\Omega, \varphi_N) \rightarrow L^2_{(0,2)}(\Omega, \varphi_N).
\]

We know from the above estimate that \( R_T = N_5 \) and that \( R_T \) and \( R_{T^*} \) are closed (see [2], Theorem 1.1.1., 1.1.2. or [3] 4.1). Notice that elements in the null space \( N_T \) are analytic functions satisfying the estimate of \( B : N_T \subset B \). So if \( u \in N_T \) then

\[
0 = \int u \overline{v} \, dV = \int u \overline{\nu e^{\psi_N}} e^{-\psi_N} \, dV = (u, \nu e^{\psi_N})_{\varphi_N},
\]

i.e. \( \nu e^{\psi_N} \) belongs to \( N_T^\perp = R_{T^*} \). We can choose \( f \in L^2_{(0,1)}(\Omega, \varphi_N) \) so that \( T^*f = \nu e^{\psi_N} \) and \( f \) is orthogonal to the null space \( N_{T^*} \). Since \( (N_{T^*})^\perp = R_T = N_5, Sf = 0 \). In view of the above estimate,

\[
\int |f|^2 e^{-\psi_N} \, dV \leq \|T^*f\|^2 = \int |\nu|^2 e^{\psi_N} \, dV = \int_{\Omega_R}.
\]

Since \( \chi(\psi) = 0 \) on \( \Omega_R \) the right-hand side is a constant \( C \) independent of \( N \) by the definition of \( \varphi_N \). Write \( g^N = f e^{-\psi_N} \). Then the above inequality becomes

\[
\int |g^N|^2 e^{\psi_N} \, dV \leq C.
\]

For fixed \( N \), if \( M \geq N \) then \( \int |g^M|^2 e^{\psi_N} \, dV \leq \int |g^M|^2 e^{\psi_N} \, dV \leq C \). So the sequence \( \{g^N\} \) has a subsequence \( \{g^M\} \) which has a weak limit \( g \) in \( L^2_{(0,1)}(\Omega, \varphi_N) \) such that \( \int |g|^2 e^{\psi_N} \, dV \leq C \). Since \( g^M \rightarrow g \) almost everywhere on \( \Omega \) the inequality above and the Lebesgue-Fatou lemma imply that
\[ \int |g|^2 \, e^{\varphi_N} \, dV \leq C, \text{ for every } N. \]

Since \( \varphi_N \) increases with \( N \) and tends to \( \infty \) on \( \Omega \setminus \bar{\Omega}_k \), we deduce that \( g = 0 \) on \( \Omega \setminus \bar{\Omega}_k \) (almost everywhere). The equality \( T^* f = v \, e^{\varphi_N} \) becomes

\[ \int \langle \bar{\partial} u, f \rangle \, e^{-\varphi_N} \, dV = (\bar{\partial} u, f)_{\varphi_N} = (u, T^* f)_{\varphi_N} = (u, v e^{\varphi_N})_{\varphi_N} = \int u \bar{v} \, dV, \]

Hence

\[ \int \langle \bar{\partial} u, g^N \rangle \, dV = \int u \bar{v} \, dV, \quad u \in D(\Omega). \]

Letting \( N \to \infty \) we deduce that

\[ \int \langle \bar{\partial} u, g \rangle \, dV = \int u \bar{v} \, dV, \quad u \in D(\Omega). \]

Take \( \xi \in C^\infty(\Omega) \) such that \( \xi = 1 \) on a neighborhood of \( \bar{\Omega}_k \). If \( u \in A \) then the equality above implies that

\[ L(u) = \int_{\partial \Omega} u \bar{v} \, dV = \int (\xi u) \bar{v} \, dV = \int \langle \bar{\partial} (\xi u), g \rangle \, dV \]

\[ = \int_{\partial \Omega} \langle \bar{\partial} u, g \rangle \, dV = 0, \]

which shows our assertion.

We give here another variant of Theorem 2 which can be proved by the same argument.

\textbf{Theorem 2'.} Let \( \Omega \) be a Stein manifold and let \( \Psi \) be a strictly plurisubharmonic function constructed as above. Let \( \varphi \) be a plurisubharmonic function such that \( \{ z \in \Omega ; \varphi(z) < c \} \subset \subset \Omega \) for any \( c \in R \). Then the set of entire functions \( u \) such that

\[ \int |u|^2 \, e^{-\Psi - \varphi_N} \, dV < \infty \]

for some nonnegative integer \( N \), is dense in the space \( A \) of entire holomorphic functions on \( \Omega \).

\section*{References}


