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H^1 Estimates for Extensions of Holomorphic Functions in Certain Pseudoconvex Domains

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Abstract

Let Ω be some weakly pseudoconvex domain in C^n with C^2-boundary, and V be a one dimensional subvariety in general position in Ω. Then for any function f ∈ H^1(V), there exists F ∈ H^1(Ω) such that F |_V = f.

Introduction. Let D be the ellipsoid (real or complex) in C^n, and M be a subvariety in a neighborhood $\bar{D}$ of $\bar{D}$ which intersects $\partial D$ transversally. We set $M = \bar{M} \cap D$. In the previous papers ([1], [2]), the author proved that if $f \in H^p(M)$, 1 ≤ p < ∞, then there exists $F \in H^p(D)$ satisfying $F |_M = f$, where $H^p(G)$ is the Hardy class on a domain G. In the present paper, we study the above problem for some pseudoconvex domain Ω which does not possess the real analytic boundary.

Let $\Phi \in C^\omega[0, 1]$ be a real function satisfying

(A. 1) $\Phi(0) = 0$, $\Phi(1) = 2$

(A. 2) $\Phi'(t) > 0$ (0 < t ≤ 1)

(A. 3) $\Phi''(t) + \Phi'(t) > 0$ (0 < t < 1)

(A. 4) there exists $r > 0$ such that $\Phi''(t) > 0$ (0 < t < r)

(A. 5) $\int_0^1 \log \Phi(t) t^{-1/2} dt < -\infty$.

For $0 < a < \frac{1}{2}$, write $W_a(t) = 2e^t \cdot \exp(-a t^2)$.

Then $\Psi \in C^\omega[0, 1]$ satisfies the above conditions. We set

$\rho(z) = \sum_{j=1}^{N-1} |z_j|^2 + \Psi(\sum_{j=1}^{N-1} |z_j|^2) - 1$, $\Omega = \{z : \rho(z) < 0\}$.

Let $\bar{V}$ be a one dimensional subvariety in a neighborhood $\bar{\Omega}$ of $\bar{\Omega}$ which intersects $\partial \Omega$ transversally. Suppose that $\bar{V}$ is written in the following form

$\bar{V} = \{z \in \Omega : h_i(z) = 0, 1 \leq i \leq N-1\}$

where $h_1, \ldots, h_{N-1}$ are holomorphic functions in $\bar{\Omega}$ satisfying
THEOREM. Let $f \in H^1(V)$. Then there exists a function $H \in H^1(\Omega)$ such that $H(z) = f(z)$ for $z \in V$.

In what follows we shall adopt the convention of denoting by $c$ any positive constant which does not depend on the relevant parameters in the estimate.

1. Support functions. Now we begin with the following lemma which was proved by Verdera [5].

**Lemma 1.** There exists a constant $\eta = \eta(\Psi) > 0$ such that for $L = \frac{1}{16}$, the following inequality holds

$$\Psi(|z + v|^2) - \Psi(|z|^2) - 2Re\left(\frac{\partial \Psi}{\partial \zeta}(|z|^2)v\right) \geq \Psi(L|v|^2),$$

for $\zeta, v \in C$, $|\zeta| < \eta$, $|v| < \eta$.

Let $U$ be an open neighborhood of $\bar{\Omega}$ in which $\rho$ is defined. We set

$$F_0(\zeta, z) = \sum_{j=1}^{N} \frac{\partial \rho}{\partial \zeta_j}(\zeta_j - z_j) \quad \text{for } (\zeta, z) \in U \times C.$$

Then we have

**Lemma 2.** There exist positive constants $\eta, c$ and $M$, depending only on $\Psi$, such that

$$2Re F_0(\zeta, z) \geq \rho(\zeta) - \rho(z) + c \Psi(M|\zeta - z|^2)$$

for $(\zeta, z) \in U \times C, |\zeta_N| < \eta, |\zeta - z| < \eta$.

**Proof.** From lemma 1, we have

$$\Psi(|z_N|^2) - \Psi(|\zeta_N|^2) \geq 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N)\right) + \Psi(L|z_N - \zeta_N|^2)$$

Therefore we obtain

$$\rho(z) = \rho(\zeta) - 2Re F_0(\zeta, z) + (\rho(z) - \rho(\zeta) + 2Re F_0(\zeta, z))$$

$$\geq \rho(\zeta) - 2Re F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j|^2 - \sum_{j=1}^{N-1} |\zeta_j|^2$$

$$+ 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(z_N - \zeta_N)\right) + 2Re\sum_{j=1}^{N-1} \zeta_j(z_j - z)$$

$$+ 2Re\left(\frac{\partial \Psi}{\partial \zeta_N}(|\zeta_N|^2)(\zeta_N - z_N)\right) + \Psi(L|z_N - \zeta_N|^2)$$

$$= \rho(\zeta) - 2Re F_0(\zeta, z) + \sum_{j=1}^{N-1} |z_j - \zeta_j|^2 + \Psi(L|z_N - \zeta_N|^2).$$
From the conditions (A. 1), (A. 2) and (A. 4), we have for small \(|x|, |y|,\)
\[
\psi\left(\frac{x^2 + y^2}{2}\right) \leq \frac{1}{2} \psi(x^2) + \frac{1}{2} \psi(y^2) \leq c(x^2 + \psi(y^2)).
\]

Therefore we have for some \(M > 0,\)
\[
\sum_{j=1}^{N} |z_j - \zeta_j|^2 + \psi(L |z_N - \zeta_N|^2) \geq c \psi\left(\frac{1}{2} \sum_{j=1}^{N} |z_j - \zeta_j|^2 + L |z_N - \zeta_N|^2\right) \geq c \psi(M |z - \zeta|^2).
\]
Thus we obtained the required inequality.

We denote by \(B(z, r)\) the open ball in \(\mathbb{C}^N\) with center \(z\) and radius \(r\). For \(\epsilon, \delta > 0,\) we set
\[
U_\epsilon = \{z \in U : \rho(z) < \delta\}, \quad V_\delta = \{z \in U : |\rho(z)| < \delta\},
\]
\[
U_{\epsilon, \delta} = \{(\xi, z) \in V_\delta \times U_\epsilon : |z - z_N| < \epsilon\},
\]
\[
Z = \{z : z_N = 0\}.
\]

By applying the technique of Henkin and Cirka [3], we have the following.

**Lemma 3.** There exists \(\epsilon, \delta, c > 0\) depending on \(\psi,\) and continuously differentiable functions
\[
\Phi : V_\delta \times U_\epsilon \to C, \quad F : U_{\epsilon, \delta} \to C \quad \text{and} \quad G : U_{\epsilon, \delta} \to C
\]
which are holomorphic in \(z \in U_\delta\) for each fixed \(\zeta \in V_\delta,\) such that

(a) \(\Phi = FG\) in \(U_{\epsilon, \delta}\)

(b) \(F(\zeta, \zeta) = 0, |G| > c\) in \(U_{\epsilon, \delta}, |\Phi| > c\) in \((V_\delta \times U_\delta)/U_{\epsilon, \delta}\)

(c) For some \(M > 0,\) the following inequality holds
\[
2\text{Re} F(\zeta, z) \geq \rho(\zeta) - \rho(z) + c\psi(M |\zeta - z|^2), (\zeta, z) \in U_{\epsilon, \delta}
\]

(d) \(d_{\zeta}F(\zeta, z)|_{z=\zeta} = \partial\rho(z)\)

(e) the function \(\Phi\) can be written in the form
\[
\Phi(\zeta, z) = \sum_{j=1}^{N} (\zeta_j - z_j) P_j(\zeta, z)
\]

where \(P_j(1 \leq j \leq N)\) are continuously differentiable in \(V_\delta \times U_\epsilon,\) holomorphic in \(z \in U_\delta\) for each fixed \(\zeta \in V_\delta.\)

**Proof.** For \(\alpha > 0,\) define
\[
W_{\alpha} = \left\{\zeta : \sum_{j=1}^{N} |\zeta_j|^2 - 1 < \alpha, |\zeta_N| < \alpha\right\}.
\]

We choose \(\alpha\) so small that \(W_{2\alpha} \subset U\) and \(2\alpha < \eta.\) Let \(\zeta^* \in \partial D/Z.\) From the condition (A. 3), \(\zeta^*\) is a strongly pseudoconvex boundary point of \(D.\) Then there exists a biholomorphic mapping \(\phi\) of some neighborhood \(S_{\zeta^*}\) of \(\zeta^*\) onto some neighborhood \(W_{\zeta^*}\) of \(0\) such that \(\rho_c(W) = \rho(\phi^{-1}(W))\) is strictly convex in \(W_{\zeta^*}.\) From the Taylor expansion, there exist constants \(\bar{\epsilon} > 0\) and \(\bar{\delta} > 0\) such that
\[2 \text{Re} \sum_{i=1}^{N} \frac{\partial \rho_i}{\partial w_i} (w')(w'_i - w_i) \geq \rho_i (w') - \rho_i (w) + \gamma^* |w'_i - w_i|^2\]

if \(|w'_i| < \varepsilon, |w'_j - w_j| < \varepsilon\).

Since the mapping \(\phi\) is a diffeomorphism, there exist \(\zeta, z\) such that

\[2 \text{Re} \sum_{i=1}^{N} \frac{\partial \rho_i}{\partial w_i} (\phi (\zeta))(\phi (\zeta) - \phi (z)) \geq \rho (\zeta) - \rho (z) + \gamma^* |\zeta - z|^2\]

if \(|\zeta - \zeta^*| < \varepsilon^*\) and \(|\zeta - z| < \varepsilon^*\).

We set

\[F_i (\zeta, z) = \sum_{i=1}^{N} \frac{\partial \rho_i}{\partial z_i} (\zeta)(\phi_i (\zeta) - \phi_i (z)).\]

Then we obtain

\[\frac{\partial F_{i \zeta}}{\partial \zeta_k} \bigg|_{\zeta = \zeta} = \frac{\partial \rho_i}{\partial \zeta_k} (\zeta)\]

and

\[2 \text{Re} F_i (\zeta, z) \geq \rho (\zeta) - \rho (z) + \gamma^* |\zeta - z|^2\]

for \(|\zeta - \zeta^*| < \varepsilon^*, |\zeta - z| < \varepsilon^*\).

We select points \(\zeta_1^*, \ldots, \zeta_p^*\) on \(\partial \Omega|Z\) in such a way that the balls \(B(\zeta_i^*, \varepsilon_i^*), i = 1, 2, \ldots, p\), cover \(\partial \Omega|W\). Let the infinitely differentiable functions \(\lambda_i, i = 1, \ldots, p\), form a partition of unity in a neighborhood \(E\) of \(\partial \Omega|W\), where \(\text{supp} \lambda_i \subset B(\zeta_i^*, \varepsilon_i^*)\). We set

\[F_i (\zeta, z) = \sum_{i=1}^{p} \lambda_i (\zeta) F_i (\zeta, z)\]

for \(\zeta \in E, |\zeta - z| < 2\varepsilon^*\). We choose \(\delta > 0\) so small that \(V_\delta|W \subset E\). Let \(\chi \in C^\infty (C^n)\) be a function, with support contained in \(W_2\), which is identically 1 on \(W\).

We set

\[F (\zeta, z) = \chi (\zeta) F_0 (\zeta, z) + (1 - \chi (\zeta)) F_i (\zeta, z)\]

for \((\zeta, z) \in U_2\).

Then we have for \(\zeta \in W, |\zeta - z| < 2\varepsilon, 2 \text{Re} F (\zeta, z) = 2 \text{Re} F_0 (\zeta, z) \geq \rho (\zeta) - \rho (z) + c \text{Vol} |\zeta - z|^2|\).

For \(\zeta \in V_\delta|W, |\zeta - z| < 2\varepsilon\), we have

\[2 \text{Re} F (\zeta, z) \geq \chi (\zeta)(\rho (\zeta) - \rho (z) + c \text{Vol} |\zeta - z|^2|) + (1 - \chi (\zeta))(\rho (\zeta) - \rho (z) + c |\zeta - z|^2|).

Thus we obtain (c). Further, using the equality (1),

\[d_{\partial} F (\zeta, z) |_{\zeta = \zeta} = \partial \rho.\]

If we choose \(\delta > 0\) sufficiently small, we have from (c), \(\text{Re} F (\zeta, z) > 0\) for \(\zeta \in V_\delta, z \in U_\delta, \varepsilon < |\zeta - z| < 2\varepsilon\). Let \(\mu \in C^\infty (C^n)\) be such that \(\text{supp} \mu \subset B(0, 2\varepsilon), \mu (z) = 1\) for \(|z| < \varepsilon\). We define for \((\zeta, z) \in V_\delta \times U_\delta,\)

\[F (\zeta, z) = \begin{cases} \partial_{\partial} (\log F) (\chi (\zeta - z)) (x \in \text{supp grad}_x (\chi (\zeta - z)) & (z \in \text{supp grad}_x (\chi (\zeta - z))) \\ 0 & (z \notin \text{supp grad}_x (\chi (\zeta - z)) \end{cases}\]

By \(L^2_b (U_\delta)\), we denote the Hilbert space of closed differential forms of type \((0, 1)\) with coefficients belonging to \(L^2 (U_\delta)\). Since \(F (\zeta, z) \in C^\infty (V_\delta, L^2_b (U_\delta))\), and \(U_\delta\) is
pseudoconvex, from the Oka–Hörmander theorem there exists the function $C(\zeta, z) \in C^1(V_0, L^2(U_0))$ with the property $\partial_x C(\zeta, z) = \Gamma(\zeta, z)$. We set

$$\Phi(\zeta, z) = \begin{cases} F(\zeta, z) \exp(-C(\zeta, z)) & (\zeta, z) \in U_{\epsilon, 0} \\ \exp(\log F \cdot \chi(\zeta - z) - C(\zeta, z)) & (\zeta, z) \in V_0 \times U_0 \mid U_{\epsilon, 0} \end{cases}$$

Then we have $\partial_x \Phi(\zeta, z) = 0$. Thus we proved that

$$\Phi(\zeta, z) \in C^1(V_0, O(U_0)).$$

We set

$$A(\zeta, z, w) = \Phi(\zeta, z) - \Phi(\zeta, w).$$

Then $A(\zeta, z, w)$ satisfies

$$A(\zeta, z, w) \in C^1(V_0, O(U_0 \times U_0))$$

and $A(\zeta, z, z) = 0$.

By Hefer's theorem, taking account of the pseudoconvexity of $\Omega$, there exist functions $Q(\zeta, z, w) \in C^1(V_0, O(U_0 \times U_0))$ such that

$$A(\zeta, z, w) = \sum_{i=1}^{N} Q_i(\zeta, z, w)(w_i - z_i).$$

Therefore we have

$$\Phi(\zeta, z) = A(\zeta, z, \zeta) = \sum_{i=1}^{N} Q_i(\zeta, z, \zeta)(w_i - \zeta_i).$$

If we set $P_i(\zeta, z) = Q_i(\zeta, z, \zeta)$, we obtain (e).

2. Proof of the theorem. For $\epsilon > 0$, we set

$$\Omega_\epsilon = \{z \in \Omega : \rho(z) < \epsilon\}.$$

Let $f^*$ be the boundary value of $f \in H^1(V)$. Then $f^* \in L^1(\partial V)$. Then by Hatziafratis [3], we have

**Lemma 4.** For $f \in H^1(V)$, $z \in V$, we have the formula

$$f(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z),$$

where $K(\zeta, z)$ is written in the following form

$$K(\zeta, z) = \sum_{i=1}^{N} K_i(\zeta, z) d_{\zeta_i} = \sum_{i=1}^{N} \frac{a_i(\zeta, z) d_{\zeta_i}}{\Phi(\zeta, z)},$$

$a_i(\zeta, z)$ being smooth on $V_0 \times U_0$, holomorphic in $z \in U_0$.

For $z \in \Omega$, we set

$$H(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z).$$

Then $H$ is holomorphic in $\Omega$ and satisfies $H \mid_\nu = f$. Let $z_0 \in \partial V$. We set

$$\tilde{B} = B\left(z_0, \frac{\delta}{2}\right), \quad B = B\left(z_0, \frac{\delta}{4}\right)$$

and $H(z) = \int_{\partial V \cap B} f^*(\zeta) K(\zeta, z)$

It is sufficient to show that $H \in H^1(\Omega)$. Let $dS_\epsilon$ be the surface element on $\partial \Omega_\epsilon$. By Fubini's theorem, we have

$$\int_{\partial \Omega \cap B} |H(z)| dS_\epsilon(z) \leq \sum_{i=1}^{N} \int_{\partial V \cap B} \left(\int_{\partial \Omega \cap B} |K_i(\zeta, z)| dS_\epsilon(z)\right) d\sigma_i(\zeta).$$
Now we estimate
\[ I(\xi) = \int_{\partial \Omega} \frac{dS_\xi(z)}{|\Phi(\xi, z)|}. \]

There are \( \delta_0, c > 0 \) such that for each \( z \) sufficiently close to \( \partial \Omega \), one can find a smooth (of class \( C^1 \)) change of coordinates \( T(\xi) = (T_1(\xi), \ldots, T_N(\xi)) \) satisfying

1. \( T_1(\xi) = \rho(\xi) - \rho(z) + \text{Im} \ F(\xi, z) \)
2. \( T_j(\xi) = \xi - z_j \quad (j = 2, \ldots, N) \)

with \( c^{-1} |\xi - z| \leq |T(\xi)| = c |\xi - z| \quad (\xi \in B(z, \delta_0)). \)

We set
\[ t_1 = \rho(\xi) - \rho(z), \quad t_2 = \text{Im} \ F(\xi, z), \quad t = (T_1, \ldots, T_N). \]

Then we have
\[ I(\xi) \leq c \int_{t_1 \in \mathbb{R}} \cdots \int_{t_N \in \mathbb{R}} \frac{dt_1 \cdots dt_N}{([\xi + \mathcal{Y}(M|t|^3)]^2 + t_1^2)^{1/2}}. \]

Now we introduce spherical polar coordinates
\[ t_2 = r \cos a, \quad r = (t_2^2 + t_3^2 + t_4^2)^{1/2} \]

Then we obtain
\[ I(\xi) \leq c \int_0^\infty dr \int_0^\pi \frac{r^2 \sin a}{(\mathcal{Y}(Mr^3))^2 + r^2 \cos^2 a} \frac{ds}{(\mathcal{Y}(Mr^3))^2 + s} \leq c \int_0^\infty \log \frac{1}{\mathcal{Y}(r^3)} \log \frac{1}{r} dt < \infty. \]

Thus we have
\[ \sup_{e} \int_{\partial \Omega} |\bar{H}(z)|dS_\xi(z) < \infty, \]

which completes the proof of the theorem.

References