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<th>Extension of Holomorphic Functions from Subvarieties to Convex Domains</th>
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Abstract

In this paper we shall prove that any holomorphic L^p function on \( V \), \( (1 \leq p < \infty) \), can be extended to a holomorphic L^p function on \( D \) when \( D \) is the real ellipsoid and \( M \) is a submanifold in general position in \( D \). We also study the \( H^m \) case.

1. Introduction. Let \( D \) be the domain such that

\[
D = \left\{ x + iy \in \mathbb{C}^N : \sum_{j=1}^N (x_j^n + y_j^m) < 1 \right\}
\]

where \( n_j, m_j \) are positive integers. We set

\[
\rho(z) = \sum_{j=1}^N (x_j^n + y_j^m) - 1 \quad \text{for} \quad z = x + iy.
\]

Let \( \mathring{V} \) be a subvariety in a neighborhood \( \mathring{D} \) of \( D \) which intersects \( \partial D \) transversally. Suppose that \( \mathring{V} \) is written in the form

\[
\mathring{V} = \{ z \in \mathring{D} : h_1(z) = \ldots = h_m(z) = 0 \} \quad (m < n)
\]

where \( h_1(z), \ldots, h_m(z) \) are holomorphic in \( \mathring{D} \) which satisfy \( \partial h_1 \wedge \ldots \wedge \partial h_m \wedge \partial \rho 
eq 0 \) on \( \mathring{V} \cap \partial D \). Let \( V = \mathring{V} \cap D \). Under the above assumption concerning \( V \), we shall show that

**THEOREM 1.** Suppose that \( f \) is a bounded holomorphic function in \( V \) and \( 0 < \varepsilon < 1 \). Then there exist a holomorphic function \( F \) in \( D \) such that \( F \mid V = f \), \( \rho(z)^a F(z) \in \Delta_a(D) \) for any \( 0 < a < \varepsilon \).

Let \( \mathring{W} \) be a submanifold of dimension \( k \) in a neighborhood of \( \mathring{D} \) which intersects \( \partial D \) transversally. Let \( W = \mathring{W} \cap D \). Then we have

**THEOREM 2.** Let \( f \) be a holomorphic function in \( W \) satisfying \( \int_W |f|^p \, d\sigma < \infty \), \( (1 \leq p < \infty) \). Then there exists a holomorphic function \( F \) in \( D \) satisfying

\[
\int_D |F|^p \, d\mu \leq C(D) \int_W |f|^p \, d\sigma,
\]
where \( \text{d}m \) and \( \text{d}\sigma \) are Lebesgue measures on \( D \) and \( W \), respectively.

To prove the above theorems, we use the techniques of Diederich, Fornaess and Wiegerinck [2]. They constructed the support function \( \Phi(\xi, z) \), holomorphic in \( z \), and proved the Hölder estimates for \( \overline{\partial} \) equation on \( D \). Finally, we will adopt the convention of denoting by \( c \) any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries. Let \( f^*(z) \) be the boundary value of \( f \in H^\infty(V) \), where \( H^\infty(V) \) is the space of all bounded holomorphic functions in \( V \). Since \( D \) is convex, \( f^*(z) \) exists almost everywhere on \( \partial V \). Let

\[
\gamma = (\gamma_1, \ldots, \gamma_N) : \partial D \times D \rightarrow \mathbb{C}^N
\]

be a smooth function such that

\[
(\zeta - z, \gamma(\zeta, z)) = \sum_{j=1}^N (\xi_j - z_j) \gamma_j(\zeta, z) \neq 0 \text{ for } (\zeta, z) \in \partial D \times D.
\]

Using the theorem of Hatziafratis [3], we have

**Proposition 1.** For \( f \in H^\infty(V) \), and \( z \in V \), we have

\[
f(z) = \int_{\partial V} f^*(\xi) K(\xi, z)
\]

where

1. \( K(\xi, z) \) is written as a sum of terms

\[
a(\xi, z) \frac{n-m-1}{\partial_1 \gamma_{k_j} \wedge d \gamma_{s_j}}
\]

2. \( a(\xi, z) \) is smooth on \( \partial D \times \overline{D} \)
3. if \( \gamma_j(\xi, z) \) is holomorphic in \( z \), then \( a(\xi, z) \) is also homomorphic in \( z \).

**Definition.** We denote by \( \Lambda_\alpha(D) \), \( (0 < \alpha < 1) \), the space of all functions on \( D \) which satisfy

\[
|f(z) - f(w)| \leq c_\alpha |z - w|^\alpha \quad \text{for any } z, w \in \overline{D}.
\]

Now we shall state some results proved by Diederich, Fornaess and Wiegerinck [2]. Let \( z = x + iy \in \overline{D}, \xi = \xi + i\eta \in \overline{D} \). We set

\[
\gamma(\xi, z) = \rho_j(\xi) - c_i [(\eta^{2m_j-2} - \xi^{2m_j-2})(z_j - \xi_j) + (z_j - \xi_j)^{2m_j-1}]
\]

where we have used the notation \( \frac{\partial \rho}{\partial z_j} = \rho \) and \( \frac{\partial \rho}{\partial \bar{z}_j} = \bar{\rho} \). We may assume \( n_j \geq m_j \). Then if we choose \( c_1 > 0 \) small enough, there exists \( c_2 > 0 \) such that

1. \( 2 \text{Re}(\xi - z, \gamma(\xi, z)) \geq -\rho(\xi) + \rho(z) \)

\[
+ c_2 \sum_{k=1}^N [(\xi_k^{2m_k-2} + \eta_k^{2m_k-2}) |z_k - \xi_k|^2 + |z_k - \xi_k|^{2m_k}]
\]

for \( (\xi, z) \in \overline{D} \times \overline{D} \). Moreover, they obtained the following lemmas:
LEMMA 1. For $q > 0, s = 0$ or $1, j = s, s + 1, \ldots$, and $A$ positive, close to 0,
\[
\int_{x^1 < r} (A + |x|^2(x^2 + y^2)^q)^s = \begin{cases} O(A^{1-q}) & \text{if } q \neq 1 \\
O(\log A) & \text{if } q = 1 \end{cases}
\]

independent of $t \in (-R, R)$.

LEMMA 2. For $q > 0, j \geq 1$, and $A$ positive, close to 0,
\[
\int_{x^1 < r} |y| |x|^{j-1} |x|^2(x^2 + y^2)^q = \begin{cases} O(A^{1-q}) & \text{if } q \neq 1 \\
O(\log A) & \text{if } q = 1 \end{cases}
\]
independent of $t \in (-R, R)$, where $r = |z| = (x^2 + y^2)^{1/2}$.

We set
\[
Q = \sum_{j=1}^n \gamma(\xi, z, \rho(\xi)) dz_j.
\]

Then by Berndtsson [1], we have the following:

PROPOSITION 2. Let $f$ be a holomorphic function in $W$ satisfying $\int_W |f| \, d\sigma < \infty$. Then
\[
F(z) = c_{k, h} \int_W \frac{f(\xi) \rho(\xi)^{1+k} (\overline{\partial}Q)^{k+\mu}}{\rho(\xi)^{k+\mu}}
\]
is holomorphic in $D$ and satisfies $F \mid_w = f$, where $\mu$ is a $(N-k, N-k)$ current in $\xi$ whose coefficients are smooth functions in $\xi \in \overline{D}$, depending holomorphically on $z \in D$, and $k$ is the dimension of $W$.

3. Proof of theorem 1. Let $k$ be the dimension of $V$. Let $B_i (i=0, 1, \ldots, N_0)$ be balls with centers on $\partial V$ and radius $r_0$ which form a cover of $\partial V$. Let $\overline{B}_i$ be the ball with the same center as $B_i$ and radius $2r_0$. Since
\[
\partial B_1 \wedge \ldots \wedge \partial B_{m} \wedge (\omega = 0) \text{ on } \partial V,
\]
we may assume that
\[
\left| \frac{\partial \rho}{\partial z} (z) \right| \geq c > 0 \text{ in } \overline{B}_0.
\]

Then
\[
L_j = \rho^{s_j} \frac{\partial}{\partial z_j} - \rho_j \frac{\partial}{\partial z}, \quad (j = 1, \ldots, k-1)
\]
form a base for the $(0, 1)$ tangential vector fields on $\partial V \cap \overline{B}_0$. For $i \neq k$
\[
(2) \quad |L_j \gamma_i| \leq c (\xi_i |2^{n-2} + | \eta_i |2^{m_2} + | z_i - \xi_i | (\mu(n_i) | \xi_i |2^{m_2} + \mu(m_i) | \eta_i |2^{m_2})),
\]
where $\mu(j) = 0$ for $j = 1, \mu(j) = 1$ for $j = 2, 3, \ldots$. 

We can introduce new real coordinates on $\tilde{B}_0$ as follows: For $\zeta \in \bar{B}_0 \cap D$ fixed, if we set $r_j = \text{Re}(z_j - \zeta_j)$, $s_j = \text{Im}(z_j - \zeta_j)$, $\lambda = \text{Im} \Phi(z, z)$, $\rho = \rho(z) - \rho(z)$, then $r_j, s_j(j=1, \ldots, k-1, k+1, \ldots, N), \lambda, \rho$ form coordinates on $\tilde{B}_0$ in such a way that $r_j, s_j(j=1, \ldots, k-1)$, form coordinates of $\partial V \cap \tilde{B}_0$. Let $\varepsilon > 0$ and

$$F(z) = \int_{\partial V} f^*(\zeta) K(\zeta, z) \quad \text{for } z \in D.$$ 

Then $F(z)$ is holomorphic in $D$. Let $z = x + iy \in \tilde{B}_0$. Then

$$\frac{\partial}{\partial x_j}(\rho(z)^*F(z)) = \varepsilon \rho(z)^{r-1} \frac{\partial}{\partial x_j}(z) F(z) + \rho(z)^{r} \frac{\partial F}{\partial x_j}(z).$$

Since $\frac{\partial F}{\partial x_j}$ is a sum of terms

$$\int_{\partial V} f^*(\zeta) \beta_i(\zeta, z) \wedge \delta_i \gamma_i, \wedge d\xi_i, \int_{\partial V} f^*(\zeta) \beta_i(\zeta, z) \wedge \delta_i \gamma_i, \wedge d\xi_i,$$

where $\beta_i(\zeta, z)$ is a smooth $(0,1)$ form and $\beta_i(\zeta, z)$ is a smooth function. Since $L_j, (j=1, \ldots, k-1)$, form a base for the $(0,1)$ tangential vector fields, we have to estimate the following integrals:

$$\int_{\partial V} f_{\zeta} \beta_i(\zeta, z) \wedge \delta_i \gamma_i, \wedge d\xi_i,$$

By applying lemmas 1, 2, and inequalities (1), (2), we have

$$\int_{\partial V} f_{\zeta} \beta_i(\zeta, z) \wedge \delta_i \gamma_i, \wedge d\xi_i \leq c \int_{|z| < R} \sum_{|z| < R} \frac{dz \wedge dk_{i=1} \wedge dk_{i=1}}{\rho(z) + (d\alpha_i + d\alpha_i)^{r-1}} \leq \frac{c}{|\rho(z)|}.$$ 

From the equality (3), we have

$$\left| \frac{\partial}{\partial x_j}(\rho(z)^*F(z)) \right| \leq c \left( |\rho(z)|^{-1} |\log |\rho(z)| || + |\rho(z)|^a |\rho(z)|^{-1} \right) \leq c |\rho(z)|^{a-1}, (0 < a < \varepsilon).$$

Therefore we obtain

$$|\nabla(\rho(z)^*F(z))| \leq c[\text{dist}(z, \partial D)]^{a-1}.$$
where $\nabla$ denotes the real gradient. From (4), we have
$$|\rho(z)F(z) - \rho(w)F(w)| \leq c \left| |z - w| \right|$$
for $z, w \in \Omega$.

This completes the proof of theorem 1.

4. Proof of theorem 2. Since
$$
\begin{align*}
\frac{1}{\rho^k} \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \gamma_k & = \sum_{j=1}^k \frac{\partial^j}{\partial \xi_1^j} \cdots \frac{\partial^j}{\partial \xi_n^j} \rho(\xi) \\
\sum_{j=1}^k \frac{\partial^j}{\partial \xi_1^j} \cdots \frac{\partial^j}{\partial \xi_n^j} \gamma_k & = c_k
\end{align*}
$$
and $\beta \rho \wedge \beta \overline{\rho} = 0$, coefficients of $(\beta Q)^k$ consist of the following:
$$
\frac{1}{\rho^k} \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \gamma_k
$$
where $j_1, j_2, \ldots, j_k$ are integers such that $j_s \neq j_t$ if $s \neq t$. We may assume that $j_1 = 1, \ldots, j_k = k$. Now we shall show that
$$
I_1 = \int_{\Omega} \left| \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \rho(\xi) \right| \, dm(z) \leq c
$$
and
$$
I_2 = \int_{\Omega} \left| \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \gamma_k \right| \, dm(z) \leq c.
$$
Since the integrand of $I_1$ is less singular than that of $I_2$, we shall show that $I_2 \leq c$. For $\varepsilon > 0$ sufficiently small, we set $U_\varepsilon = \{ \xi \in \Omega : |\rho(\xi)| < \varepsilon \}$. Let $\xi \in U_\varepsilon$. To prove the inequality (6), it is sufficient to show that
$$
I_2 \leq c \int_{U_\varepsilon \cap B(\xi, \varepsilon)} \left| \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \gamma_k \right| \, dm(z) \leq c.
$$
By the same method as the proof of theorem 1, we obtain
$$
I_2 \leq c \int_{B(\xi, \varepsilon)} \left| \log \left( \frac{|\rho(\xi)|}{1 + r^2} \right) \right| \, dr \leq c.
$$
Since the integrand of $I_1$ is less singular than that of $I_2$, we shall show that $I_2 \leq c$. For $\varepsilon > 0$ sufficiently small, we set $U_\varepsilon = \{ \xi \in \Omega : |\rho(\xi)| < \varepsilon \}$. Let $\xi \in U_\varepsilon$. To prove the inequality (6), it is sufficient to show that
$$
I_2 \leq c \int_{U_\varepsilon \cap B(\xi, \varepsilon)} \left| \frac{\partial^k}{\partial \xi_1^k} \cdots \frac{\partial^k}{\partial \xi_n^k} \gamma_k \right| \, dm(z) \leq c.
$$
By the same method as the proof of theorem 1, we obtain
$$
I_2 \leq c \int_{B(\xi, \varepsilon)} \left| \log \left( \frac{|\rho(\xi)|}{1 + r^2} \right) \right| \, dr \leq c.
$$
Therefore we have
$$
\int_{\Omega} |F(z)| \, dm(z) \leq c \int_{\Omega} |f(\xi)| \, d\sigma(\xi).
$$
In case $p > 1$, we write $F(z)$ in the following form
$$
F(z) = \int_{\Omega} f(z)T(\xi, z) \, d\sigma(\xi).
$$
Let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, by applying Hölder's inequality, we have
$$
|F(z)|^p \leq \left( \int_{\Omega} |f(\xi)|^p T(\xi, z) \, d\sigma(\xi) \right)^{p/q} \left( \int_{\Omega} T(\xi, z) \, d\sigma(\xi) \right)^{p/q}
$$
By the same method as the case $p=1$, we obtain
$$
\int_{\Omega} |F(z)|^p \, dm(z) \leq c \int_{\Omega} |f(\xi)|^p \, d\sigma(\xi).
$$
\[ \int_{\partial} |F|^p dm \leq c \int_{\omega} |f|^p d\sigma, \]

which completes the proof of theorem 2.

References

