Extension of Holomorphic Functions from Subvarieties to Convex Domains

Kenzō ADACHI

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki
(Received Oct. 31, 1987)

Abstract

In this paper we shall prove that any holomorphic $L^p$ function on $V$, $(1 \leq p < \infty)$, can be extended to a holomorphic $L^p$ function on $D$ when $D$ is the real ellipsoid and $M$ is a submanifold in general position in $D$. We also study the $H^\omega$ case.

1. Introduction. Let $D$ be the domain such that

$$D = \left\{ x + iy \in \mathbb{C}^N : \sum_{j=1}^{N}(x_j^{2n_j} + y_j^{2m_j}) < 1 \right\}$$

where $n_j, m_j$ are positive integers. We set

$$\rho(z) = \sum_{j=1}^{N}(x_j^{2n_j} + y_j^{2m_j}) - 1 \quad \text{for } z = x + iy.$$ 

Let $\tilde{V}$ be a subvariety in a neighborhood $\tilde{D}$ of $\tilde{D}$ which intersects $\partial D$ transversally. Suppose that $\tilde{V}$ is written in the form

$$\tilde{V} = \{ z \in \tilde{D} : h_1(z) = \ldots = h_m(z) = 0 \} \quad (m < n)$$

where $h_1(z), \ldots, h_m(z)$ are holomorphic in $\tilde{D}$ which satisfy $\partial h_1 \wedge \ldots \wedge \partial h_m \wedge \partial \rho \neq 0$ on $\tilde{V} \cap \partial D$. Let $V = \tilde{V} \cap D$. Under the above assumption concerning $V$, we shall show that

**THEOREM 1.** Suppose that $f$ is a bounded holomorphic function in $V$ and $0 < \varepsilon < 1$. Then there exist a holomorphic function $F$ in $D$ such that $F \restriction V = f$, $\rho(z)^\varepsilon F(z) \in \Delta_\alpha(D)$ for any $0 < \alpha < \varepsilon$.

Let $\tilde{W}$ be a submanifold of dimension $k$ in a neighborhood of $\tilde{D}$ which intersects $\partial D$ transversally. Let $W = \tilde{W} \cap D$. Then we have

**THEOREM 2.** Let $f$ be a holomorphic function in $W$ satisfying $\int_{\tilde{W}} |f|^\rho d\sigma < \infty$, $(1 \leq p < \infty)$. Then there exists a holomorphic function $F$ in $D$ satisfying

$$\int_{\tilde{D}} |F|^\rho \, dm \leq C(D) \int_{\tilde{W}} |f|^\rho \, d\sigma,$$
where \( dm \) and \( d\sigma \) are Lebesgue measures on \( D \) and \( W \), respectively.

To prove the above theorems, we use the techniques of Diederich, Fornaess and Wiegerinck [2]. They constructed the support function \( \Phi(\zeta, z) \), holomorphic in \( z \), and proved the Hölder estimates for \( \overline{\partial} \) equation on \( D \). Finally, we will adopt the convention of denoting by \( c \) any positive constant which does not depend on the relevant parameters in the estimate.

2. Preliminaries. Let \( f^*(z) \) be the boundary value of \( f \in H^\infty(V) \), where \( H^\infty(V) \) is the space of all bounded holomorphic functions in \( V \). Since \( D \) is convex, \( f^*(z) \) exists almost everywhere on \( \partial V \). Let

\[
\gamma = (\gamma_1, \ldots, \gamma_N) : \partial D \times D \rightarrow \mathbb{C}^N
\]

be a smooth function such that

\[
(\zeta - z, \gamma(\zeta, z)) = \sum_{j=1}^{N} (\xi_j - z_j) \gamma_j(\zeta, z) \neq 0 \text{ for } (\zeta, z) \in \partial D \times D.
\]

Using the theorem of Hatziafratis [3], we have

**Proposition 1.** For \( f \in H^\infty(V) \), and \( z \in V \), we have

\[
f(z) = \int_v f^*(\xi) K(\xi, z)
\]

where

1. \( K(\xi, z) \) is written as a sum of terms

\[
\alpha(\xi, z) = \frac{n-m-1}{\frac{\partial}{\partial \gamma_{k_j}} \gamma_{k_j}} \frac{n-m}{\frac{d}{dz_i} \xi_{i_j}}
\]

\[
(\zeta - z, \gamma(\zeta, z))^{n-m}
\]

2. \( \alpha(\xi, z) \) is smooth on \( \partial D \times \overline{D} \)

3. if \( \gamma_j(\zeta, z) \) is holomorphic in \( z \), then \( \alpha(\xi, z) \) is also homomorphic in \( z \).

Definition. We denote by \( \Lambda_\alpha(D) \), \( (0 < \alpha < 1) \), the space of all functions on \( D \) which satisfy

\[
|f(z) - f(w)| \leq c_\alpha |z - w|^{\alpha}
\]

for any \( z, w \in \partial D \).

Now we shall state some results proved by Diederich, Fornaess and Wiegerinck [2]. Let \( z = x + iy \in \partial \overline{D} \), \( \zeta = \xi + i\eta \in \partial \overline{D} \). We set

\[
\gamma_j(\zeta, z) = \rho_j(\zeta) - c_i[(\eta_j^{2m_j-2} - \xi_j^{2m_j-2})(z_j - \zeta_j) + (z_j - \zeta_j)^{2m_j-1}]
\]

where we have used the notation \( \frac{\partial \rho_j}{\partial \zeta} = \rho_j \) and \( \frac{\partial \rho_j}{\partial \overline{z}_j} = \overline{\rho_j} \). We may assume \( n_i \geq m_j \). Then if we choose \( c_i > 0 \) small enough, there exists \( c_j > 0 \) such that

1. \( \text{Re}(\zeta - z, \gamma(\zeta, z)) \geq -\rho(z) + \rho(\zeta) \)

\[
+ \sum_{k=1}^{N} [(\xi_k^{2m_k-2} + \eta_k^{2m_k-2}) |z_k - \zeta_k|^2 + |z_k - \zeta_k|^{2m_k}]
\]

for \( (\zeta, z) \in \partial \overline{D} \times \overline{D} \). Moreover, they obtained the following lemmas:
LEMMA 1. For \( q > 0, s = 0 \) or \( 1, j = s, s + 1, \ldots, \) and \( A \) positive, close to 0,
\[
\int_{|z| < R} \frac{|t + x|^{j-s}|x|^s}{(A + |t + x|^{2}(x^2+y^2))^{q}} = \begin{cases} O(A^{1-q}) & \text{if } q \neq 1 \\ O(\log A) & \text{if } q = 1 \end{cases}
\]
independent of \( t \in (-R, R) \).

LEMMA 2. For \( q > 0, j \geq 1, \) and \( A \) positive, close to 0,
\[
\int_{|z| < R} \frac{|t + x|^{j-1}|y|}{(A + |t + x|^{2} + r^{2})^{q}} = \begin{cases} O(A^{1-q}) & \text{if } q \neq 1 \\ O(\log A) & \text{if } q = 1 \end{cases}
\]
independent of \( t \in (-R, R) \), where \( r = |z| = (x^2+y^2)\frac{1}{2} \).

We set
\[
Q = \sum_{k=1}^{N} \frac{n}{\rho(\zeta)} d\zeta.
\]
Then by Berndtsson [1], we have the following:

PROPOSITION 2. Let \( f \) be a holomorphic function in \( W \) satisfying \( \int_{W} |f| d\sigma < \infty \).
Then
\[
F(z) = c_{N, k} \int_{W} \frac{f(\zeta) \rho(\zeta)^{1+k} (\bar{\partial}Q)^{k} \wedge \mu}{(\gamma(\zeta, z) - \zeta - \rho(\zeta))^{2}}
\]
is holomorphic in \( D \) and satisfies \( F|_{W} = f \), where \( \mu \) is a \((N-k)\) current in \( \zeta \) whose coefficients are smooth functions in \( \zeta \in \overline{D} \), depending holomorphically on \( z \in D \), and \( k \) is the dimension of \( W \).

3. Proof of theorem 1. Let \( k \) be the dimension of \( V \). Let \( B_{i} (i = 0, 1, \ldots, N_{0}) \) be balls with centers on \( \partial V \) and radius \( r_{0} \) which form a cover of \( \partial V \). Let \( B_{i} \) be the ball with the same center as \( B_{i} \) and radius \( 2r_{0} \). Since
\[
\partial h_{1} \wedge \ldots \wedge \partial h_{m} \wedge \partial \rho \neq 0 \text{ on } \partial V,
\]
we may assume that
\[
\left| \frac{\partial \rho}{\partial z_{k}} (z) \right| \geq c > 0 \text{ in } B_{0}.
\]
Then
\[
L_{j} = \rho^{k} \frac{\partial}{\partial z_{j}} - \rho^{k} \frac{\partial}{\partial z_{k}} \quad (j = 1, \ldots, k-1)
\]
form a base for the \((0,1)\) tangential vector fields on \( \partial V \cap B_{0} \). For \( i \neq k \)
(2) \( |L_{j} \gamma_{i}| \leq c(|\xi_{i} |^{2n-2} + |\eta_{i} |^{2n-2} + |z_{i} - \zeta_{i}| (\mu(n_{i}) |\xi_{i}|^{2n-3} + \mu(m_{i}) |\eta_{i}|^{2m-3})) \),
\[
|L_{j} \gamma_{k}| \leq c(|\xi_{j}|^{2n-1} + |\eta_{j}|^{2m-1}),
\]
where \( \mu(j) = 0 \) for \( j = 1 \), \( \mu(j) = 1 \) for \( j = 2, 3 \), \ldots.
We can introduce new real coordinates on $\hat{B}_0$ as follows: For $\zeta \in \hat{B}_0 \cap D$ fixed, if we set $\tau_j = \text{Re}(z_j - \zeta_j)$, $\sigma_j = \text{Im}(z_j - \zeta_j)$, $\lambda = \text{Im} \Phi(\zeta, z)$, $\rho = \rho(\zeta) - \rho(z)$, then $\tau_j, \sigma_j (j = 1, \ldots, k-1, k+1, \ldots, N)$, $\lambda, \rho$ form coordinates on $\hat{B}_0$ in such a way that $\tau_j, \sigma_j (j = 1, \ldots, k-1)$, form coordinates of $\partial V \cap \hat{B}_0$. Let $\epsilon > 0$ and

$$F(z) = \int_{\partial V} f^*(\zeta)K(\zeta, z) \quad \text{for } z \in D.$$ 

Then $F(z)$ is holomorphic in $D$. Let $z = x + iy \in \hat{B}_0$. Then

$$(3) \quad \frac{\partial}{\partial x_j} (\rho(z)^s F(z)) = \varepsilon \rho(z)^{s-1} \frac{\partial \rho}{\partial x_j}(z) F(z) + \rho(z)^s \frac{\partial F}{\partial x_j}(z).$$

Since $\frac{\partial F}{\partial x_j}$ is a sum of terms

$$\int_{\partial V} f^*(\zeta) \beta_l(\zeta, z) \wedge \delta \gamma_j \wedge \iota \xi_j \wedge \delta \xi_j, \quad \int_{\partial V} f^*(\zeta) \beta_l(\zeta, z) \wedge \delta \gamma_j \wedge \delta \xi_j,$$

where $\beta_l(\zeta, z)$ is a smooth $(0, 1)$ form and $\beta_l(\zeta, z)$ is a smooth function. Since $\gamma_j, (j = 1, \ldots, k-1)$, form a base for the $(0, 1)$ tangential vector fields, we have to estimate the following integrals:

$$\int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j|$$

By applying lemmas 1, 2, and inequalities (1), (2), we have

$$\int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j| \int_{\partial V \cap B_1} |\gamma_j|$$

From the equality (3), we have

$$\left| \frac{\partial}{\partial x_j} (\rho(z)^s F(z)) \right| \leq c \left( |\rho(z)^{s-1}| \log |\rho(z)| + |\rho(z)^{s-1}| \right)$$

Therefore we obtain

$$(4) \quad |\nabla(\rho(z)^s F(z))| \leq c |\text{dist}(z, \partial D)|^{s-1}.$$

Therefore we obtain

$$(4) \quad |\nabla(\rho(z)^s F(z))| \leq c |\text{dist}(z, \partial D)|^{s-1}.$$
Extension of Holomorphic Functions from Subvarieties to Convex Domains

where ∇ denotes the real gradient. From (4), we have

\[ |\rho(z)^{p}F(z) - \rho(w)^{p}F(w)| \leq c ||z - w|| \quad \text{for } z, w \in D.\]

This completes the proof of theorem 1.

4. Proof of theorem 2. Since

\[ \frac{1}{\rho^{k}} \frac{\partial \gamma_{j}}{\partial \zeta_{j}} \cdots \frac{\partial \gamma_{j_k}}{\partial \zeta_{j_k}} = \cdots \frac{\partial \gamma_{j}}{\partial \zeta_{j}} \cdots \frac{\partial \gamma_{j_k}}{\partial \zeta_{j_k}}, \]

and \( \partial \rho \wedge \bar{\partial} = 0 \), coefficients of \( (\overline{\partial} Q)^{k} \) consist of the following:

\[ \frac{1}{\rho^{k}} \frac{\partial \gamma_{j}}{\partial \zeta_{j}} \cdots \frac{\partial \gamma_{j_k}}{\partial \zeta_{j_k}} \]

where \( j_1, j_2, \ldots, j_k \) are integers such that \( j_s \neq j_t \) if \( s \neq t \). We may assume that \( j_1 = 1, \ldots, j_k = k \). Now we shall show that

\[ I_1 = \int_{D} \left| \frac{\partial \gamma_{1}}{\partial \zeta_{1}} \cdots \frac{\partial \gamma_{k}}{\partial \zeta_{k}} \rho(\zeta) \right| \mathrm{d}m(z) \]

\[ \leq c \quad \text{for } \zeta \in U_{\epsilon}. \]

(6) Since the integrand of \( I_1 \) is less singular than that of \( I_2 \), we shall show that \( I_2 \leq c \). For \( \epsilon > 0 \) sufficiently small, we set \( U_{\epsilon} = \{ \zeta \in D : |\rho(\zeta)| < \epsilon \} \). Let \( \zeta \in U_{\epsilon} \). To prove the inequality (6), it is sufficient to show that

\[ I_2 = \int_{U \cap B(\zeta, \epsilon)} \left| \frac{\partial \gamma_{1}}{\partial \zeta_{1}} \cdots \frac{\partial \gamma_{k-1}}{\partial \zeta_{k-1}} \gamma_k \right| \mathrm{d}m(z) \]

\[ \leq c. \]

By the same method as the proof of theorem 1, we obtain

\[ I_2 \leq c \int_{U \cap B(\zeta, \epsilon)} \left| \log \left( |\rho(\zeta)| + \sum_{j=1}^{k} (\zeta^{j_{m_j} - 2} + \eta^{j_{m_j} - 2}) |z_j - \zeta_j|^2 \right) \right| \mathrm{d} \sigma(\zeta), \]

we set \( \lambda = \max m_j \), and we introduce polar coordinates. Then we have

\[ I_2 \leq c \int_{0}^{2\pi} \left| \log \left( |\rho(\zeta)| + r^\lambda \right) \right| r \mathrm{d}r \leq c |\rho(\zeta)|^{1/\lambda}. \]

Therefore we have

\[ \int_{D} |F(z)| \mathrm{d}m(z) \leq c \int_{w} |f(\zeta)| \mathrm{d} \sigma(\zeta). \]

In case \( p > 1 \), we write \( F(z) \) in the following form

\[ F(z) = \int_{w} f(\zeta)T(\zeta, z) \mathrm{d} \sigma(\zeta). \]

Let \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, by applying Hölder's inequality, we have

\[ |F(z)|^{p} \leq \left( \int_{w} |f(\zeta)|^{p} T(\zeta, z) \mathrm{d} \sigma(\zeta) \right)^{1/\lambda} \left( \int_{w} |T(\zeta, z)| \mathrm{d} \sigma(\zeta) \right)^{p/q}. \]

By the same method as the case \( p = 1 \), we obtain
\[ \int_{\partial} |F|^p \, dm \leq c \int_{\omega} |f|^p \, d\sigma, \]

which completes the proof of theorem 2.

References

