H^p Estimates for Extensions of Holomorphic Functions on Convex Domains

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Abstract

In this paper we prove that any \( f \in H^p(M) \) \((1 \leq p < \infty)\) can be extended to a function in \( H^p(D) \) when \( D \) is some convex domain with real analytic boundary and \( M \) is a submanifold in general position in \( D \).

1. Introduction. Let \( G \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-boundary and \( \tilde{M} \) be a submanifold in a neighborhood of \( \tilde{G} \) which intersects \( \partial G \) transversally. Let \( M = \tilde{M} \cap G \). Henkin [7] proved that any bounded holomorphic function in \( M \) can be extended to a bounded holomorphic function in \( G \). Recently, Cumenge [6] and Beatrous [2], [3] studied certain norm estimates for extensions of holomorphic functions on \( M \) to \( G \). On the other hand, Bruna and Castillo [5] proved the fundamental inequality for some convex domain \( D \) with real analytic boundary, and they obtained Hölder and \( L^p \) estimates for the \( \bar{\partial} \)-equation. In the previous paper [1], the author studied \( L^p \) extensions of holomorphic functions in \( M \) to \( D \). In the present paper, we shall show that any function \( f \) in \( H^p(M) \), \( 1 \leq p < \infty \), can be extended to a function \( H \) in \( H^p(D) \). Moreover we give some estimates for extensions of bounded holomorphic functions in \( M \). Finally, we will adopt the convention of denoting by \( c \) any positive constant which does not depend on the relevant parameters in the estimate in which it occurs.

2. \( H^p \) estimates. Let \( D \) be a bounded domain in \( \mathbb{C}^n \) of the type

\[ D = \{ z : \rho(z) < 0 \} \]

where

\[ \rho(z) = \sum_{i=1}^n s_i(1 |z|^2) - 1. \]
We set \( p, (w) = s, (|w|^2) \) for one complex variable \( w \). We assume \( s, \) is real analytic in an interval \([0, a,]\) such that

(i) \( s, (t) \geq 0, \ s, (t) + 2ts', (t) \geq 0 \) for \( 0 \leq t < a, \)
(ii) \( s, (0) = 0, \ s, (a,) > 1. \)

For example, \( D^m = \{ z : \sum \delta_i |z_i|^2 < 1 \} \) is one of the above domains, where \( m, \)'s are positive integers.

Let

\[
F (\xi, z) = \sum_{\ell = 1}^{n} \frac{\partial p}{\partial \xi_{\ell}} (\xi, (\xi_{\ell} - z_i))
\]

Let \( \bar{M} \) be a submanifold of dimension \( k \) in a neighborhood of \( \bar{D} \) which intersects \( \partial D \) transversally. Let \( M = \bar{M} \cap D, \) and \( \delta (z) = \text{dist} (z, \partial D). \) For \( \epsilon > 0 \) sufficiently small, we set \( D_\epsilon = \{ z : p (z) < -\epsilon \} . \) For an open set \( \Omega \) in a complex manifold, we denote by \( H^p (\Omega) \) the usual Hardy class, and by \( L^p (\Omega) \) the space of all integrable functions in \( \Omega. \) By applying the theorem of Berndtsson [4], we have the following. (cf. Adachi [1]).

**Proposition 1.** Let \( f \in L^p (M) \cap O (M). \) Then

\[
H (z) = c_1 \left( \frac{\partial (\xi, \rho (\xi)))}{\rho (\xi)} \right)^* \wedge \mu
\]

is holomorphic in \( D \) and satisfies \( H |_{M} = f, \) where \( \mu \) is a \((n-k, n-k)\)-current in \( \xi \) whose coefficients are measures supported in \( M, \) depending holomorphically on \( z. \)

Now we prove the following theorem. The proof is based on the techniques of Range [8].

**Theorem 1.** Let \( f \in H^p (M). \) Then \( H \in H^p (D). \)

**Proof.** By the estimates of Adachi [1], if we set

\[
a_{ij} (\xi) = \frac{\partial \rho}{\partial \xi_i \partial \xi_j} (\xi,)
\]

then

\[
| H (z) | \leq c \left( \int_M \frac{|f (\xi)|}{dV_M (\xi)} \right) \left( \sum_{i=1}^n a_{ij} (\xi_{ij}) \right)
\]

In the above integral, \( i_1, \ldots, i_n \) are mutually distinct integers. For a small neighborhood \( U \) of a point in \( \partial D, \) we can choose local coordinates \( (t_1, t_2, \ldots, t_n) \) in \( U \) such that

\[
t_1 = |\rho (\xi)| + |\rho (z)|, \ t_s = \text{Im} F (\xi, z), \) and \( t_s, t_s = \xi_{ij} - z_i, (s=2, \ldots, k). \)
We set $t' = (t_{s+1}, \ldots, t_n)$. Then we have for $\varepsilon > 0$ sufficiently small

$$|H(z)| \leq c \int |t_s| < \delta_s, |t_{s+1}| < \delta_e \frac{dt_s \ldots dt_n}{(\varepsilon + |t_s| + |t'|^s)^s} \prod_{j=1}^s (\varepsilon + t_{s-j}^e + t_j^e)$$

$$\leq c |\rho(\zeta)|^{-1 + \frac{1}{m} - \delta(k-1)}$$

We choose $\delta > 0$ such that $\eta = \frac{1}{m} - \delta(k-1) > 0$. Then we have

$$I \leq c |\rho(\zeta)|^{-1 + \delta(k-1)}$$

By Fubini's theorem and the partition of unity argument, we have

$$\int \partial D_\varepsilon |H(z)| d\sigma(z) \leq c \int_M |f(\xi)| |\rho(\xi)|^{-1 + \delta(k-1)} dV_M(\xi)$$

$$\leq c \int_0^\delta \left( \int \partial M_i |f(\xi)| t^{-1 + \delta(k-1)} d\sigma_M(\xi) \right) dt \leq c \int_0^\delta t^{-1 + \delta(k-1)} dt \leq c.$$ 

Therefore $H \in H^p(D)$. This completes the proof of theorem 1.

3. $H^p$ estimates ($1 < p \leq \infty$). For $z \in M$, we may assume that

$$\left( \frac{\partial \rho}{\partial x_1}(z), \frac{\partial \rho}{\partial y_1}(z), \ldots, \frac{\partial \rho}{\partial y_k}(z) \right) = (1, 0, \ldots, 0).$$

If we set $\tau_z(\zeta) = \text{Im} F(\zeta, z)$, then

$$\frac{\partial \tau_z(\zeta)}{\partial y_i}(z) = \frac{1}{2} \frac{\partial \rho}{\partial x_i}(z)$$

By the transversality of $M$, we can choose local coordinates $(w_1, \ldots, w_s)$ for $M$ in a neighborhood $U$ of $z$ such that

$$w_i = \rho(\zeta) + i\tau_z(\zeta), w_i = \zeta - z_i (i = 2, \ldots, k).$$

We set $w_i = t_{i-j} + it_i (j = 1, \ldots, k)$. Then we prove the following:

THEOREM 2. Let $f \in H^p(M) (1 < p \leq \infty)$. Then $H \in H^p(D)$.

PROOF. We set

$$K(\zeta, z) dV_M(\zeta) = \frac{c_s \rho(\zeta)^{s+1} (\partial \text{log}(\frac{1}{\rho(\zeta)})^s \wedge \mu}{(<\rho(\zeta), z - \zeta> + \rho(\zeta))^{s+1}}$$

where $dV_M(\zeta)$ is the Lebesgue measure on $M$. Then we have

$$H(z) = \int_M f(\zeta) K(\zeta, z) dV_M(\zeta).$$
Let \( q \) be a positive number such that \( \frac{1}{p} + \frac{1}{q} = 1 \). We choose \( \epsilon \) such that \( 0 < \epsilon \rho < \frac{1}{2} \).

By Hölder's inequality, we obtain

\[
| \mathcal{H}(z) | \leq \left( \int_M | f(\xi) | \delta(\xi)^{-\epsilon} | K(\xi, z) | dV_M(\xi) \right) \left( \int_M | K(\xi, z) | \delta(\xi)^{\epsilon} dV_M(\xi) \right)^{\frac{\rho}{\epsilon}}
\]

Let \( V \) be a small neighborhood of a point in \( M \). Let \( V \subset U \), and \( U \) be an open set in which we can choose local coordinates as above. We fix \( z \) in \( V \). Then

\[
\int_{M \cap U} | K(\xi, z) | \delta(\xi)^{\epsilon} dV_M(\xi)
\]

is bounded provided that we choose \( \delta > 0 \) such that \( \epsilon \rho > \delta(k-1) \). The partition of unity arguments yields

\[
\int_M | K(\xi, z) | \delta(\xi)^{\epsilon} dV_M(\xi) \leq c.
\]

Now we choose local coordinates \((u_1, \ldots, u_n)\) in a neighborhood \( V \) such that \( u_i = -\rho(z) \), \( u_i = \text{Im} F(\xi, z) \), and \((u_1, \ldots, u_n)\) form local coordinates of \( M \cap V \). We set \( u = (u_{n+1}, \ldots, u_n) \). Then by Fubini's theorem we obtain

\[
\int_{\mathbb{R}^n \cap V} | \mathcal{H}(z) | \delta(\xi)^{-\epsilon} | K(\xi, z) | d\sigma(z) \leq c \int_M | f(\xi) | \delta(\xi)^{-\epsilon} \int_{\mathbb{R}^n \cap V} | K(\xi, z) | du_1 \ldots du_n dV_M(\xi)
\]

is bounded.

We choose \( \epsilon \) and \( \delta \) so that \( \epsilon \rho + \delta(k-1) < \frac{1}{m} \). Then

\[
\sup_{\xi \in \mathbb{R}^n \cap V} \int_{\mathbb{R}^n \cap V} | \mathcal{H}(z) | \delta(\xi)^{-\epsilon} d\sigma(z) < \infty.
\]

The partition of unity arguments yields \( \mathcal{H} \in H^s(D) \). This completes the proof of theorem 2.

**Theorem 3.** Let \( f \in H^s(M) \). Then for any \( \epsilon > 0 \), \( \delta(z)^s \mathcal{H}(z) \) is bounded in \( D \).
PROOF. By the same method as proofs of the above two theorems, we have

\[
|\delta(z)^{H}(z)| \leq \int \frac{c \delta(z)^{H_1,\ldots,H_n}}{|t_1| < \alpha \delta \delta(z) + |t_1| + |t_2| + \cdots + |t_n|' (\delta(z) + t'_j - t_j' + t_j')} \prod_{j=2}^{n} \left( \delta(z) + t'_j - t_j' + t_j' \right) \\
|t_1| < \delta'_1, \ldots, |t_n| < \delta'_n
\]

provided that \( \epsilon > \delta(k-1) \). This completes the proof of theorem 3.

REFERENCES