**H^p Estimates for Extensions of Holomorphic Functions on Convex Domains**

KENZŌ ADACHI

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki

Abstract

In this paper we prove that any \( f \in H^p(M) \) (\( 1 \leq p < \infty \)) can be extended to a function in \( H^p(D) \) when \( D \) is some convex domain with real analytic boundary and \( M \) is a submanifold in general position in \( D \).

1. Introduction. Let \( G \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-boundary and \( \tilde{M} \) be a submanifold in a neighborhood of \( \tilde{G} \) which intersects \( \partial G \) transversally. Let \( M = \tilde{M} \cap G \). Henkin [7] proved that any bounded holomorphic function in \( M \) can be extended to a bounded holomorphic function in \( G \). Recently, Cumenge [6] and Beatrous [2, 3] studied certain norm estimates for extensions of holomorphic functions on \( M \) to \( G \). On the other hand, Bruna and Castillo [5] proved the fundamental inequality for some convex domain \( D \) with real analytic boundary, and they obtained Hölder and \( L^p \) estimates for the \( \bar{\partial} \)-equation. In the previous paper [1], the author studied \( L^p \) extensions of holomorphic functions in \( M \) to \( D \). In the present paper, we shall show that any function \( f \) in \( H^p(M) \), \( 1 \leq p < \infty \), can be extended to a function \( H \) in \( H^p(D) \). Moreover we give some estimates for extensions of bounded holomorphic functions in \( M \). Finally, we will adopt the convention of denoting by \( c \) any positive constant which does not depend on the relevant parameters in the estimate in which it occurs.

2. \( H^p \) estimates. Let \( D \) be a bounded domain in \( \mathbb{C}^n \) of the type

\[
D = \{ z : \rho(z) < 0 \}
\]

where

\[
\rho(z) = \sum_{k=1}^{n} s_i(\|z^i\|) - 1.
\]
We set $\rho_0(w) = s_0(\lvert w \rvert^2)$ for one complex variable $w$. We assume $s_0$ is real analytic in an interval $[0, a_1]$ such that

(i) $s_0(t) \geq 0$, $s_0(t) + 2ts_1(t) \geq 0$ for $0 \leq t < a_1$,
(ii) $s_0(0) = 0$, $s_0(a_1) > 1$.

For example, $D_r^n = \{ z : \sum_{i=1}^{r} \lvert z_i \rvert^2 < 1 \}$ is one of the above domains, where $m_1$'s are positive integers.

Let

$$F(\zeta, z) = \sum_{i=1}^{m} \frac{\partial \rho}{\partial \zeta_i}(\zeta) \left( \zeta_i - z_i \right)$$

Let $\bar{M}$ be a submanifold of dimension $k$ in a neighborhood of $\bar{D}$ which intersects $\partial D$ transversally. Let $M = M \cap D$, and $\delta(z) = \text{dist}(z, \partial D)$. For $\epsilon > 0$ sufficiently small, we set $D_\epsilon = \{ z : \rho(z) < -\epsilon \}$. For an open set $\Omega$ in a complex manifold, we denote by $H^s(\Omega)$ the usual Hardy class, and by $L^1(\Omega)$ the space of all integrable functions in $\Omega$. By applying the theorem of Berndtsson [4], we have the following. (cf. Adachi [1]).

**Proposition 1.** Let $f \in L^1(M) \cap O(M)$. Then

$$H(z) = c_1 \int_{\partial \Omega} f(\zeta) \rho(\zeta)^* \left( \partial \bar{\partial} \log \left( -\frac{1}{\rho(\zeta)} \right) \right)^* \wedge \mu$$

is holomorphic in $D$ and satisfies $H \mid_M = f$, where $\mu$ is a $(n-k, n-k)$-current in $\zeta$ whose coefficients are measures supported in $M$, depending holomorphically on $z$.

Now we prove the following theorem. The proof is based on the techniques of Range [8].

**Theorem 1.** Let $f \in H^1(M)$. Then $H \in H^1(D)$.

**Proof.** By the estimates of Adachi [1], if we set

$$a_i(\zeta) = \frac{\partial^2 \rho}{\partial \zeta_i \partial \bar{\zeta}_j}(\zeta)$$

then

$$|H(z)| \leq c \int_{\partial \Omega} \frac{|f(\zeta)|}{(1 - \rho(\zeta), z - \zeta)^*} dV_M(\zeta).$$

In the above integral, $i_1, \ldots, i_k$ are mutually distinct integers. For a small neighborhood $U$ of a point in $\partial D$, we can choose local coordinates $(t_1, t_2, \ldots, t_n)$ in $U$ such that

$t_1 = \rho(\zeta) + |\rho(z)|$, $t_s = \text{Im} F(\zeta, z)$, and

$t_{s+1} + it_s = \zeta_s - z_s$ (s = 2, ..., k).
We set \( t' = (t_{n+1}, \ldots, t_n) \). Then we have for \( \epsilon > 0 \) sufficiently small

\[
|H(z)| \leq c \int_{|t_1| < \delta_1, |t_1| < \delta_2} \frac{dt_1 \ldots dt_n}{(\epsilon + |t_1| + |t'|^2)^s} \prod_{i=1}^n (\epsilon + t_{n-i+1} + t_i')
\]

\[
\leq c \cdot \rho(\xi) - 1 + \frac{1}{m} - \delta(k-1)
\]

We choose \( \delta > 0 \) such that \( \eta = \frac{1}{m} - \delta(k-1) > 0 \). Then we have

\[
I \leq c \cdot \rho(\xi)^{-1+s}
\]

By Fubini's theorem and the partition of unity argument, we have

\[
\int_{\partial D_\epsilon} |H(z)| \, d\sigma(z) \leq c \int_M |f(\xi)| \cdot |\rho(\xi)^{-1+s} \cdot dV_M(\xi)
\]

\[
\leq c \int_0^\delta (\int_{\partial M_i} |f(\xi)| \cdot t^{-1+s} \cdot d\sigma_M(\xi)) \, dt \leq c \int_0^\delta t^{-1+s} \, dt \leq c.
\]

Therefore \( H \in H'(D) \). This completes the proof of theorem 1.

3. \( H^p \) estimates\( (1 < p \leq \infty) \). For \( z \in M \), we may assume that

\[
\left( \frac{\partial \rho}{\partial x_1}(z), \frac{\partial \rho}{\partial y_1}(z), \ldots, \frac{\partial \rho}{\partial y_n}(z) \right) = (1, 0, \ldots, 0).
\]

By the transversality of \( M \), we can choose local coordinates \( (w_1, \ldots, w_n) \) for \( M \) in a neighborhood \( U \) of \( z \) such that

\[
w_i = \rho(\xi) + i\tau_i(\xi), \quad w_i = \xi_i - z_i \quad (i = 2, \ldots, k).
\]

We set \( w_i = t_{i-1} + i t_i \quad (j = 1, \ldots, k) \). Then we prove the following:

**Theorem 2.** Let \( f \in H^p(M) \quad (1 < p < \infty) \). Then \( H \in H^p(D) \).

**Proof.** We set

\[
K(\xi, z) \, dV_M(\xi) = \frac{c_1 \rho^{s+1}(\partial \overline{\partial} \log(1/\rho(\xi)))^s \wedge \mu}{<\rho(\xi), z - \xi> + \rho(\xi)^{s+1}}
\]

where \( dV_M(\xi) \) is the Lebesgue measure on \( M \). Then we have

\[
H(z) = \int_M f(\xi) \cdot K(\xi, z) \, dV_M(\xi).
\]
Let \( q \) be a positive number such that \( \frac{1}{p} + \frac{1}{q} = 1 \). We choose \( \varepsilon \) such that \( 0 < \varepsilon p < \frac{1}{2} \).

By Hölder's inequality, we obtain

\[
| H(z) |^q \leq \left( \int_{D} | f(\xi) |^p \, d\sigma(\xi) - t^{\varepsilon} | K(\xi, z) | \, dV_{H}(\xi) \right) \left( \int_{D} | K(\xi, z) |^{q} \, dV_{H}(\xi) \right)^{p/q}
\]

Let \( V \) be a small neighborhood of a point in \( M \). Let \( V \subset U \), and \( U \) be an open set in which we can choose local coordinates as above. We fix \( z \) in \( V \). Then

\[
\int_{M \cap U} | K(\xi, z) | \, dV_{H}(\xi)
\]

\[
\leq c \int_{t \leq \delta_{\varepsilon}} \prod_{i=2}^{m} \int_{|w_{i}| \leq \delta_{\varepsilon}} \frac{dt_{j}}{|w_{j}|^{1/q}} \leq c,
\]

provided that we choose \( \delta > 0 \) such that \( \varepsilon \delta > \delta^{(k-1)} \). The partition of unity arguments yields

\[
\int_{M} | K(\xi, z) | \, dV_{H}(\xi) \leq c.
\]

Now we choose local coordinates \((u, \ldots, u_{n})\) in a neighborhood \( V \) such that \( u_{i} = -\rho(z), u_{i} = \text{Im} F(\xi, z), \) and \((u, \ldots, u_{n})\) form local coordinates of \( M \cap V \). We set \( u = (u_{n+1}, \ldots, u_{n}) \). Then by Fubini's theorem we obtain

\[
\int_{\partial D \cap V} | H(z) | \, d\sigma(z) \leq c \int_{W} | f(\xi) | \, dV_{H}(\xi) \int_{\partial D \cap V} | K(\xi, z) | \, dV_{H}(\xi)
\]

\[
\leq c \int_{W} | f(\xi) | \, dV_{H}(\xi) \int_{u_{j} \leq \delta_{\varepsilon}} \frac{du_{j}}{\delta_{\varepsilon}^{1/q}} \delta(\xi)^{-\delta \varepsilon - (k-1)p} \, dV_{H}(\xi)
\]

\[
\leq c \int_{W} | f(\xi) | \, dV_{H}(\xi) \int_{u_{j} \leq \delta_{\varepsilon}} \delta(\xi)^{-\delta \varepsilon - (k-1)p} \, dV_{H}(\xi).
\]

We choose \( \varepsilon \) and \( \delta \) so that \( \varepsilon \delta + \delta \leq (k-1) < \frac{1}{m} \). Then

\[
\sup_{\xi \in D} \int_{\partial D \cap V} | H(z) | \, d\sigma(z) < \infty.
\]

The partition of unity arguments yields \( H \in H^{p}(D) \). This completes the proof of theorem 2.

**Theorem 3.** Let \( f \in H^{p}(M) \). Then for any \( \varepsilon > 0 \), \( \delta(z)H(z) \) is bounded in \( D \).
PROOF. By the same method as proofs of the above two theorems, we have
\[ |\delta(z)^\prime H(z)| \]
\[ \leq c \int \frac{\delta(z)^\prime dt_1 \ldots dt_{2k}}{|t_1| < \delta_s (\delta(z) + |t_1| + |t_2| + \ldots + |t_s| + |t_s'|)^{s}} \prod_{j=2}^{s} (\delta(z) + t'_{s-1} + t'_{s}) \]
\[ \leq c \int \frac{\delta(z)^{s-(s-1)} dt_1 \int \frac{dt_2 \ldots dt_{2k}}{|t_2| < \delta_s \prod_{j=2}^{s} (t'_{s-1} + t'_{s})^{s-\sigma} \prod_{j=2}^{s} |t_{s-1}| < \delta_s}}{\prod_{j=2}^{s} |t_{s-1}| < \delta_s} \]

provided that \( \epsilon > \delta(k-1) \). This completes the proof of theorem 3.

REFERENCES