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<th>Title</th>
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</tr>
</thead>
<tbody>
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</tr>
</tbody>
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長崎大学学術研究成績リポジトリ

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Hⁿ Estimates for Extensions of Holomorphic Functions on Convex Domains

Kenzō ADACHI

Department of Mathematics, Faculty of Education
Nagasaki University, Nagasaki

Abstract

In this paper we prove that any \( f \in H^n (M) \) \((1 \leq p < \infty)\) can be extended to a function in \( H^n (D) \) when \( D \) is some convex domain with real analytic boundary and \( M \) is a submanifold in general position in \( D \).

1. Introduction. Let \( G \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-boundary and \( \tilde{M} \) be a submanifold in a neighborhood of \( \tilde{G} \) which intersects \( \partial G \) transversally. Let \( M = \tilde{M} \cap G \). Henkin [7] proved that any bounded holomorphic function in \( M \) can be extended to a bounded holomorphic function in \( G \). Recently, Cumenge [6] and Beatrous [2], [3] studied certain norm estimates for extensions of holomorphic functions on \( M \) to \( G \). On the other hand, Bruna and Castillo [5] proved the fundamental inequality for some convex domain \( D \) with real analytic boundary, and they obtained H"{o}lder and \( L^p \) estimates for the \( \bar{\partial} \)-equation. In the previous paper [1], the author studied \( L^p \) extensions of holomorphic functions in \( M \) to \( D \). In the present paper, we shall show that any function \( f \) in \( H^n (M) \), \( 1 \leq p < \infty \), can be extended to a function \( H \) in \( H^n (D) \). Moreover we give some estimates for extensions of bounded holomorphic functions in \( M \). Finally, we will adopt the convention of denoting by \( c \) any positive constant which does not depend on the relevant parameters in the estimate in which it occurs.

2. \( H^n \) estimates. Let \( D \) be a bounded domain in \( \mathbb{C}^n \) of the type

\[
D = \{ z : \rho (z) < 0 \}
\]

where

\[
\rho (z) = \sum_{i=1}^{n} s_i (|z|^2 - 1).
\]
We set $\rho, (w) = s, (|w|')$ for one complex variable $w$. We assume $s, \text{ is real analytic in an interval } [0, a]$ such that

(i) $s; (t) \geq 0, s; (t) + 2ts; (t) \geq 0 \text{ for } 0 \leq t < a,$
(ii) $s, (0) = 0, s, (a) > 1.$

For example, $D'' = \{z : \sum_{i=1}^{m^n} |z,|^n < 1\}$ is one of the above domains, where $m, n$'s are positive integers.

Let $F (\xi, z) = \sum_{i=1}^{\infty} \frac{\partial \rho}{\partial \xi_i} (\xi, \xi_i) - z_i.$

Let $\overline{M}$ be a submanifold of dimension $k$ in a neighborhood of $\overline{D}$ which intersects $\partial D$ transversally. Let $M = \overline{M} \cap D$, and $\delta (z) = \text{dist} (z, \partial D).$ For $\varepsilon > 0$ sufficiently small, we set $D_\varepsilon = \{z : \rho (z) < -\varepsilon\}.$ For an open set $\Omega$ in a complex manifold, we denote by $H^* (\Omega)$ the usual Hardy class, and by $L^* (\Omega)$ the space of all integrable functions in $\Omega$. By applying the theorem of Berndtsson [4], we have the following. (cf. Adachi [1]).

**Proposition 1.** Let $f \in L^* (M) \cap O (M).$ Then

$$H (z) = c \int_M \frac{f (\xi) \rho (\xi, \xi_i) \left( \partial \overline{\partial} \log \left( \frac{1}{\rho (\xi)} \right) \right) \wedge \mu}{\left( \frac{\partial \rho (\xi)}{\partial (\partial (\xi), z - \xi) + \rho (\xi)} \right) \wedge \xi_i}$$

is holomorphic in $D$ and satisfies $H \mid M = f$, where $\mu$ is a $(n-k, n-k)$-current in $\xi$ whose coefficients are measures supported in $M$, depending holomorphically on $z$.

Now we prove the following theorem. The proof is based on the techniques of Range [8].

**Theorem 1.** Let $f \in H^* (M).$ Then $H \in H^* (D).$

**Proof.** By the estimates of Adachi [1], if we set

$$a_i (\xi_i) = \frac{\partial \rho}{\partial \xi_j, \partial \xi_i} (\xi_i)$$

then

$$|H (z)| \leq c \int_M \frac{|f (\xi)| \prod_i a_i (\xi_i)}{\left( \frac{\partial \rho (\xi_i)}{\partial (\partial (\xi), z - \xi) + \rho (\xi)} \right) \wedge \xi_i} \ dV_M (\xi)$$

In the above integral, $i, \ldots, i$ are mutually distinct integers. For a small neighborhood $U$ of a point in $\partial D,$ we can choose local coordinates $(t, t, \ldots, t_m)$ in $U$ such that $t, = \rho (\xi) | + | \rho (z)|$, $t, = \text{Im} F (\xi, z)$, and $t, = \xi_i - z_i \quad (s = 2, \ldots, k).$
We set \( t' = (t_{n+1}, \ldots, t_n) \). Then we have for \( \varepsilon > 0 \) sufficiently small

\[
H(z) \leq c \int_{|t_i| < \delta_i, |t_{n+1}| < \delta_n} \frac{dt_i \ldots dt_n}{(\varepsilon + |t_i| + |t'|)^2} \prod_{j=1}^n (\varepsilon + t_{n-j+1}^* + t_j^*)
\]

\[
\leq c |\rho(\xi)| -1 + \frac{1}{m} - \delta(k-1)
\]

We choose \( \delta > 0 \) such that \( \eta = \frac{1}{m} - \delta(k-1) > 0 \). Then we have

\[
I \leq c |\rho(\xi)|^{-1+\tau}.
\]

By Fubini's theorem and the partition of unity argument, we have

\[
\int \partial\sigma \leq c \int_{M} |f(\xi)| |\rho(\xi)|^{-1+\tau} d\sigma(\xi)
\]

\[
\leq c \int_{0}^{\delta} (\int \partial\sigma(\xi) |f(\xi)| |t|^{-1+\tau} d\sigma(\xi)) dt \leq c \int_{0}^{\delta} |t|^{-1+\tau} dt \leq c.
\]

Therefore \( H \in H'(D) \). This completes the proof of theorem 1.

3. \( \mathcal{H}' \) estimates \( (1 < p \leq \infty) \). For \( z \in M \), we may assume that

\[
(\frac{\partial \rho}{\partial x_1}(z), \ldots, \frac{\partial \rho}{\partial x_n}(z)) = (1,0,\ldots,0).
\]

If we set \( \tau_z(\xi) = \text{Im} F(\xi, z) \), then

\[
\frac{\partial \tau_z(\xi)}{\partial y_i}(z) = \frac{1}{2} \frac{\partial \rho}{\partial x_i}(z).
\]

By the transversality of \( M \), we can choose local coordinates \((w_1, \ldots, w_n)\) for \( M \) in a neighborhood \( U \) of \( z \) such that

\[
w_i = \rho(\xi) + i\tau_z(\xi), \quad w_i = \xi - z_i \quad (i = 2, \ldots, k).
\]

We set \( w_i = t_i, + it_j \quad (j = 1, \ldots, k) \). Then we prove the following:

**Theorem 2.** Let \( f \in \mathcal{H}'(M) \) \( (1 < p < \infty) \). Then \( H \in \mathcal{H}'(D) \).

**Proof.** We set

\[
K(\xi, z) d\sigma(\xi) = \frac{c_{\tau} \rho(\xi)^{-1+\tau} (\partial \log (\frac{1}{\rho(\xi)})^\tau \wedge \mu}{<\rho(\xi), z - \xi + \rho(\xi) > ^{\tau+\tau}}
\]

where \( d\sigma(\xi) \) is the Lebesgue measure on \( M \). Then we have

\[
H(z) = \int_M f(\xi) K(\xi, z) d\sigma(\xi).
\]
Let $q$ be a positive number such that $\frac{1}{p} + \frac{1}{q} = 1$. We choose $\epsilon$ such that $0 < \epsilon p < \frac{1}{2}$.

By Hölder's inequality, we obtain

\[ |H(z)|^q \leq \left( \int_M |f(\xi)|^q \delta(\xi)^{-\epsilon q} |K(\xi, z)| dV_M(\xi) \right) \left( \int_M |K(\xi, z)| \delta(\xi)^{-\epsilon q} dV_M(\xi) \right)^{\frac{q}{2}} \]

Let $V$ be a small neighborhood of a point in $M$. Let $V \subset U$, and $U$ be an open set in which we can choose local coordinates as above. We fix $z$ in $V$. Then

\[ \int_{M \cap U} |K(\xi, z)| \delta(\xi)^{-\epsilon q} dV_M(\xi) \]

\[ \leq c \int_t \frac{t^{\epsilon q-1}}{t^q + \rho(z)^q} \frac{1}{t} \prod_{j=1}^m \int_{|w_j| < \epsilon |\xi|^{-1}} \frac{dt_{i_j-1} dt_{i_j}}{\delta_\epsilon |w_j|^{1-\epsilon}} \leq c, \]

provided that we choose $\delta > 0$ such that $\epsilon q > \delta(k-1)$. The partition of unity arguments yields

\[ \int_M |K(\xi, z)| \delta(\xi)^{-\epsilon q} dV_M(\xi) \leq c. \]

Now we choose local coordinates $(u, \ldots, u_n)$ in a neighborhood $V$ such that $u_1 = -\rho(z)$, $u_2 = \text{Im} F(\xi, z)$, and $(u, \ldots, u_n)$ form local coordinates of $M \cap V$. We set $u = (u_{n+1}, \ldots, u_n)$. Then by Fubini's theorem we obtain

\[ \int_{\partial V \cap V} H(z) |^q d\sigma(z) \leq c \int_M |f(\xi)|^q \delta(\xi)^{-\epsilon q} \int_{\partial V \cap V} |K(\xi, z)| du \cdots du_n dV_M(\xi) \]

\[ \leq c \int_M \frac{du^{\epsilon q}}{u^q \delta(\xi) + u^{1-\epsilon q}} \delta(\xi)^{-q(q-1)} dV_M(\xi) \]

\[ \leq c \int_M |f(\xi)|^q \delta(\xi)^{-q(q-1)q-1} dV_M(\xi). \]

We choose $\epsilon$ and $\delta$ so small that $\epsilon p + \delta(k-1) < \frac{1}{m}$. Then

\[ \sup_{\delta > 0} \int_{\partial V \cap V} |H(z)|^q d\sigma(z) < \infty. \]

The partition of unity arguments yields $H \in H^s(D)$. This completes the proof of theorem 2.

**Theorem 3.** Let $f \in H^s(M)$. Then for any $\delta > 0$, $\delta(z)H(z)$ is bounded in $D$. 
PROOF. By the same method as proofs of the above two theorems, we have
\[ |\delta(z)|^\epsilon |H(z)| \leq \int \frac{c \delta(z)^{\epsilon} dt_1 \cdots dt_n}{|t_1| < \delta, |t_2| < \delta, \ldots, |t_n| < \delta, |t_{n+1}| < \delta} \]

\[ \leq c \int \frac{\delta(z)^{\epsilon-(n-1)}}{|t_1| < \delta, |t_2| < \delta, \ldots, |t_n| < \delta} dt_1 \int \frac{dt_2 \cdots dt_n}{\prod_{j=2}^n (t_{j-1}' + t_j')^{1-\eta}} \leq c, \]

provided that \( \epsilon > \delta(k-1) \). This completes the proof of theorem 3.

REFERENCES