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Bundle Properties of Homeomorphism Groups
Transitive on Coset Spaces

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Abstract

Let $\mathcal{G}$ be any homeomorphism group on a left (or right) coset space $X$ with a local cross section. If $\mathcal{G}$ contains all left (resp. right) translations in $X$ and is endowed with the compact-open topology, then $\mathcal{G}$ is a bundle space over $X$ relative to a natural map: $\mathcal{G} \to X$. Several directions of applicability of this fact are given.

Introduction. Let $X$ be the left (or right) coset space of a Hausdorff topological group $G$, which has a local cross section relative to the natural projection: $G \to X$. Let $\mathcal{G}$ be any homeomorphism group on $X$ which contains the group $\mathcal{T}$ of all left (resp. right) translations in $X$ and is endowed with the compact-open topology, let $\mathcal{G}_a$ be the stability subgroup of $\mathcal{G}$ at an arbitrary point $a$ of $X$, and let $p$ be the map: $\mathcal{G} \to X$ defined by $p(\psi) = \psi(a)$ ($\psi \in \mathcal{G}$). In §1 we show that (i) $\mathcal{G}$ is a bundle space over $X$ relative to $p$ (Theorem 1.2), and (ii) $\mathcal{G}$ is naturally homeomorphic to the product space $\mathcal{T} \times \mathcal{G}_a$ if and only if $X$ is a topological group (Corollary 1.3). The above (i) is a generalization of a result of McCarty [12], that is the case where $X$ is locally compact, locally connected, and $\mathcal{G}$ is Homeo($X$). And the above (ii) is a generalization of a result of Hale [6], that is the case where $\mathcal{G}$ is Homeo($X$), and a result of Karube [10], that is the case where $X$ is locally compact. In §2 several corollaries of our Theorem 1.2 are presented, concerning (1) applicability of theory of fibre spaces, (2) sufficient conditions to be a fibre bundle, and (3) finite product properties. In §3 we show some typical examples.

1. Fundamental results.

Let $X$ be the left coset space of a Hausdorff topological group $G$ by its closed subgroup $H$ (left’ coset space for definiteness), which has a local cross section. Let $\mathcal{G}$ be any homeomorphism group on $X$ which contains the group $\mathcal{T}$ of all left translations in $X$, and let $\mathcal{G}_a$ be the stability subgroup of $\mathcal{G}$ at an arbitrarily fixed point $a$ of $X$. 
Then \( \mathcal{G} = \mathcal{F} \mathcal{G}_a \). Let \( p \) be the map of \( \mathcal{G} \) onto \( X \), defined by \( p(\psi) = \psi(a) \) (\( \psi \in \mathcal{G} \)). Give \( \mathcal{G} \) the compact-open topology. Then the map \( p \) is a continuous surjection. And \( \mathcal{G} \) is a semitopological group (in the sense of Husain ([8], p. 27), but not in the sense of Bourbaki ([2], p. 296), \( \mathcal{G} \) is not necessarily a topological group even if \( X \) is a locally compact topological group (see Remark below). The above notations \( X, G, H, \mathcal{G}, \mathcal{G}_a, a, p \) will keep these meanings throughout the paper.

REMARK. Bracconnier ([3], pp. 56–58) gave an example of a totally disconnected, non-compact, locally compact, abelian topological group \( X' \) whose automorphism group is not a topological group, in fact the inverse operation is not continuous under the compact-open topology. Hence, for example, the full homeomorphism group on the space \( X' \) is not a topological group under the compact open topology, while it is a topological group under the \( g \)-topology of Arens [1].

Let \( \varphi \) be a continuous map of a topological space \( E \) to another topological space \( B \). We say that the space \( B \) has a local cross section \( f \) at a point \( b \) in \( B \) relative to \( \varphi \), if \( f \) is a continuous map of a neighborhood \( U \) of \( b \) into \( E \) such that \( \varphi f(u) = u \) for each \( u \in U \).

The following Theorem 1.1 shows that \( \mathcal{G} \) operates on \( X \) as if it was the group of left translations.

**Theorem 1.1.** Let \( \mathcal{G}^* \) be the left coset space of \( \mathcal{G} \) by \( \mathcal{G}_a \) endowed with the quotient topology relative to the natural projection \( \mathcal{G} \to \mathcal{G}^* \). Then \( \mathcal{G}^* \) is homeomorphic to \( X \) in a natural way.

**Proof.** For any point \( x \) of \( X \), let \( f \) be a local cross section defined on a neighborhood \( U \) of \( x \) relative to the natural projection \( \pi : G \to X \). Let \( \alpha \) be the map of \( G \) onto \( \mathcal{F} \) which carries each element \( g \) of \( G \) to the left translation in \( X \) by \( g \). Then the map \( q \) defined as follows is a local cross section at \( x \) relative to \( \pi \) such that \( q(\alpha(f(u) \cdot g_a^{-1})) = u \) for each \( u \in U \).

Using the existence of a local cross section at any point of \( X \) relative to \( \pi \), we can see that \( p \) is a quotient map. Since the correspondence \( p \circ \pi^*-1 \), where \( \pi^* \) is the natural projection of \( \mathcal{G} \) onto \( \mathcal{G}^* \), is a well-defined map and a continuous bijection of \( \mathcal{G}^* \) onto \( X \). And so the fact that \( p \) is a quotient map implies that \( p \circ \pi^*\) is a homeomorphism.

**Remark.** Ford, Jr. ([5], Theorem 4.1) stated an analogous result to ours, in the case where \( X \) is a uniform space with the so-called "strong local homogeneity" and \( \mathcal{G} \) is \( \text{Homeo}(X) \) with the topology of uniform convergence and transitive on \( X \). But there is a counter example to this. His statement remains true, for example, if \( X \) is compact. Then the result under the compact-open topology is gained also.

Next we will show that \( \mathcal{G} \) has a natural bundle structure defined by Hu ([7], p. 65) as follows.

**Definition.** Let \( p \) be a continuous map of a space \( E \) into another space \( B \). The space \( E \) is called a bundle space over the base space \( B \) relative to the projection \( p \) if there exists a space \( D \) such that for each \( b \in B \), there is an open neighborhood \( V \) of \( b \).
in $B$ together with a homeomorphism
\[ \psi_V: V \times D \to p^{-1}(V) \]
of $V \times D$ onto $p^{-1}(V)$ satisfying the condition
\[ p \psi_V(v, d) = v (v \in V, d \in D). \]

**Theorem 1.2.** $\mathcal{G}$ is a bundle space over $X$ relative to $p$.

**Proof.** For any point $x$ of $X$, let $f, U, \pi, \alpha, g_0$, and $q$ be the same as those in the proof of Theorem 1.1, and put $W = q(U)$. Define the map $\Phi$ of the product space $W \times \mathcal{G}_a$ onto $W \mathcal{G}_a (= p^{-1}(U))$ as follows:
\[ \Phi(w, \psi) = w \circ \psi \quad ((w, \psi) \in W \times \mathcal{G}_a), \]
where $\circ$ means the composition of maps. The map $\Phi$ is bijective. We show that $\Phi$ is bicontinuous. In the following let $w$, $\psi$, $\varphi$, and $g$ represent any element of $W$, $\mathcal{G}_a$, $\mathcal{G}$, and $G$ respectively.

a) The map $w \circ \psi \to w$ of $W \mathcal{G}_a$ to $W$ is continuous, for
\[ [q \circ p] (w \circ \psi) = w. \]
b) Let $\lambda$ be the map of $G \times \mathcal{G}$ to $\mathcal{G}$ defined by
\[ \lambda(g, \varphi) = \alpha(g) \circ \varphi \quad (g \in G, \varphi \in \mathcal{G}). \]
Since the map $(g, x) \to [\alpha(g)](x)$ of $G \times X$ to $X$ is a continuous group action of $G$ on $X$, from Lemma B of Hale [6], $\lambda$ is continuous.

Since
\[ \psi = w^{-1} \circ (w \circ \psi) = \lambda(g_0 \cdot [fp(w)]^{-1} \circ w \circ \psi), \]
the continuity of the map $\lambda$ implies

c) the map $w \circ \psi \to \psi$ of $W \mathcal{G}_a$ to $\mathcal{G}_a$ is continuous.

Therefore from a) and c), $\Phi^{-1}$ is continuous. On the other hand, since

\[ w \circ \psi = \lambda([fp(w)]^{-1} \circ \psi), \]
the continuity of $\lambda$ implies that $\Phi$ is continuous. Consequently $\Phi$ is a homeomorphism.

Put $\Phi' = \Phi \circ (q \times 1)$, where $1$ is the identity map of $\mathcal{G}_a$. Then $\Phi'$ is a homeomorphism of the product space $U \times \mathcal{G}_a$ onto $p^{-1}(U)$. Clearly $p \circ \Phi'(u, \psi) = u$ ($u \in U, \psi \in \mathcal{G}_a$).

**Question.** To apply our results to the more special base spaces, it is desirable to find handy conditions to assure the existence of a local cross section relative to $\pi$. It is one of such condition that $G$ is a finite-dimensional locally compact topological group (see Karube [9] and Nagami [13]). Is there such a handy condition for the case where $G$ is infinite dimensional or not locally compact, and $X$ is not a topological group?

Let $\psi$ be the map of the product set of $\mathcal{F}$ and $\mathcal{G}_a$ onto $\mathcal{G}$ (= $\mathcal{F} \mathcal{G}_a$) such that $(\tau, \varphi)\to \tau \varphi$ ($\tau \in \mathcal{F}, \varphi \in \mathcal{G}_a$). We say that $\mathcal{G}$ is naturally homeomorphic to the product space $\mathcal{F} \times \mathcal{G}_a$ if $\psi$ is a homeomorphism.

**Corollary 1.3.** $\mathcal{G}$ is naturally homeomorphic to the product space $\mathcal{F} \times \mathcal{G}_a$ (and then homeomorphic to the product space $X \times \mathcal{G}_a$ also) if and only if $X$ is a topological group.
PROOF. When $X$ is a topological group, consider $X/{e}$ ($e$ : the identity of $X$) as the base space $X$, then $X/{e}$ has the full cross section relative to the natural projection : $X \rightarrow X/{e}$. Thus our method of proof for Theorem 1.2 implies that $\mathcal{G}$ is naturally homeomorphic to the product space $\mathcal{F} \times \mathcal{G}_a$. For the proof of the converse of this, we need Proposition 1.6 below, which can be proved by the following Lemma 1.4 and Lemma 1.5.

**Lemma 1.4.** $\mathcal{F} \cap \mathcal{G}_a = \alpha(g_aHg_a^{-1})$ for any fixed element $g_a$ of $\pi^{-1}(a)$.

**Lemma 1.5.** $\Ker \alpha = \cap_{g \in G} gHg^{-1}$, i.e. $\Ker \alpha$ is the largest subgroup of $H$ which is invariant in $G$.

**Proposition 1.6.** The following three statements are equivalent : (1) $\mathcal{G}$ is injective, (2) $\mathcal{F} \cap \mathcal{G}_a$ consists of only the identity map, and (3) $H$ is a normal subgroup of $G$.

**Proof.** The equivalence of (1) and (2) is shown directly. Using Lemma 1.4 and Lemma 1.5, we can show the equivalence of (2) and (3).

**Remark.** Even if $X$ is a topological group, $\mathcal{G}$ is not necessarily a direct product nor a semidirect product of $\mathcal{F}$ and $\mathcal{G}_a$. The case where $X$ is a circle group gives a counter example.

2. Other corollaries.

A. By Theorem 4.1 of Hu [7], on p. 65, our Theorem 1.2 implies the following.

**Corollary 2.1.** $\mathcal{G}$ is a fibre space over $X$ relative to $p$.

Thus we can use the theory of fibre spaces, to investigate the homotopy group of $\mathcal{G}$.

B. Conditions for $\mathcal{G}$ to be a fibre bundle.

**Corollary 2.2.** If $X$ is 'compact' or 'locally connected and locally compact', then $\mathcal{G}$ is a principal fibre bundle over $X$ with fibre and group $\mathcal{G}_a$.

**Proof.** The condition on $X$ implies that $\mathcal{G}_a$ is a topological group under the compact-open topology (cf. Arens[1]). Thus Theorem 1.2 implies the above conclusion.

**Corollary 2.3.** If $X$ is 'compact' or 'locally connected and locally compact' topological group, then $\mathcal{G}$ is a product bundle over $X$.

**Remark.** 1) McCarty [12] has shown the same conclusions of these two corollaries in the case where $\mathcal{G} = \text{Homeo}(X)$. 2) If $X$ is only locally compact, these two corollaries are not necessarily true (see Remark before Theorem 1.1).

C. Finite product local properties.

We say a property $P$ is an *FPL property* (finite product local property), if $P$ is a local property such that the product space of spaces $A$ and $B$ has the property $P$ if and only if both $A$ and $B$ have the property $P$. Among those properties studied on full homeomorphism groups, the following are FPL properties ; to be locally connected, locally arcwise connected, $LC^n$, $LC^\omega$, locally contractible, to have arbitrarily small contractible neighborhoods, and to be an ANR.

As a result obtained from Theorem 1.2, $\mathcal{G}$ is locally homeomorphic to the product space $X \times \mathcal{G}_a$. So we have the following.
COROLLARY 2.4. \( \mathcal{G} \) has an FPL property if and only if both \( X \) and \( \G_a \) have the property.

In particular for \( \mathcal{G} \) to have any FPL property, \( X \) must have the property — this is a criterion when we respect \( \mathcal{G} \) to have an FPL property.

A space is called an \( l \)-manifold if it is separable metrizable and locally homeomorphic to \( l \).

COROLLARY 2.5. Let \( X \) be separable metrizable, locally compact, and admits a nontrivial flow, and let \( \mathcal{F} \) be the full homeomorphism group of \( X \) with the compact-open topology. Then \( \mathcal{F} \) is an \( l \)-manifold if and only if \( X \) is an ANR and the stability subgroup \( \mathcal{F}_a \) of \( \mathcal{F} \) at \( aeX \) is an \( l \)-manifold.

PROOF. Since \( X \) is 2nd countable, locally compact Hausdorff space, both \( \mathcal{F} \) and \( \mathcal{F}_a \) are separable and metrizable. So we may only apply the well-known result of Toruńczyk ([14] Corollary 4.8) to our Corollary 2.4 in the case where \( \mathcal{G} \) is \( \mathcal{F} (= \text{Homeo}(X)) \).

COROLLARY 2.6. If \( X \) is a locally connected, compact, metrizable ANR, and admits a nontrivial flow, then the homeomorphism group \( \mathcal{F} \) on \( X \) is an \( l \)-manifold if and only if the homeomorphism group \( \mathcal{F}(X-a) \) on the space \( X-a \) is an \( l \)-manifold.

PROOF. We may only apply a result of Arens, stated in the form of Lemma 4.2 in McCarty [12], to our Corollary 2.5.

COROLLARY 2.7. If \( X \) is compact and locally Euclidean, then the same conclusion as Corollary 2.6 holds.

COROLLARY 2.8. For any coset space of a compact Lie group, the same conclusion as Corollary 2.6 holds.

PROOF. Any coset space of a Lie group has a local cross section with respect to the natural projection (Chevalley [4], p.110).

EXAMPLES. For example, the homeomorphism group on each of the following spaces is an \( l \)-manifold: Euclidean plane, and the punctured space of real projective plane, and the usual torus. This follows from results of Toruńczyk [14], Luke and Mason [11], and Corollary 2.8.

3. Some typical examples.

A. Full homeomorphism groups

In the case where \( \mathcal{G} \) is the hull homeomorphism group on \( X \), and \( X \) has a local cross section, all our results thus far are valid.

B. Diffeomorphism groups

Let \( X \) be the left coset space of a Lie group \( G \) by a closed subgroup \( H \). Give \( X \) the analytic structure induced from that of \( G \), and let Diff\(^r\)(\( X \)) be the group of all \( C^r \) diffeomorphisms on \( X \) with respect to the \( C^r \) differential structure induced from the analytic structure on \( X \) (\( r=0,1,2,\ldots,\infty,\omega \)). Since every left translation on \( X \) is an analytic diffeomorphism, \( \mathcal{F} \) is contained in Diff\(^r\)(\( X \)). Thus in the case where \( \mathcal{G} = \text{Diff}^r(X) \) all conclusions in §1 and §2 are valid.
C. Isometry groups

**Lemma 3.1.** If G has a two-sided invariant metric ρ, then ρ' defined as follows is a left invariant metric on X.

\[ \rho'(x_i, x_2) = \rho(g_iH, g_2H) \text{ for } x_i \in X \text{ and } g_i \in \pi^{-1}(x_i) \text{ (i=1, 2).} \]

The metric ρ' is called the metric induced from ρ.

**Remark.** A T₀-topological group has a two-sided metric if and only if it is 1st countable, and

(∗) it has arbitrarily small invariant neighborhoods of the identity.

The condition (∗) is satisfied for Abelian (topological) groups, compact groups, discrete groups, and is closed under the operations of making subgroups, factor groups, direct product groups, and projective limit groups.

**Lemma 3.2.** If G has a two-sided invariant metric ρ, the induced metric ρ' from ρ is given on X, and let I*(X) be the group of all surjective isometries of X onto itself. Then for \( \mathcal{G} = I^*(X) \), all statements in §1 and §2 are valid.

**Lemma 3.3.** If X is "compact" or "connected, locally Euclidean", and has a left invariant metric, then any isometry of X into itself is surjective.

**Proof.** In the case where X is compact, the conclusion follows from the well known fact that an isometry of a compact metric space into itself is surjective. In the latter case we may use the following facts: (1) if a locally compact left coset space has a left invariant metric, then the metric is complete, (2) in a complete metric space M, for any isometry \( \varphi \) of M into itself, \( \varphi(M) \) is closed, (3) for two locally Euclidean spaces M and N having the same dimension, any continuous injection from M to N is open (and so a local homeomorphism), and (4) for a locally Euclidean, connected, complete metric space M, any isometry of M into itself is surjective.

**Theorem 3.4.** Let G be a group with a two-sided invariant metric ρ, and give X the induced left invariant metric from ρ. If X is "compact" or "connected and locally Euclidean", then the group I(X) of all isometries of X to itself satisfies all statements thus far, considering \( \mathcal{G} = I(X) \).

**Appendix.** All our results are valid also under the g-topology of Arens [1] that is stronger than the compact-open topology. The g-topology is useful to make a homeomorphism group on a locally compact Hausdorff space a topological transformation group. In fact, under the g-topology our Corollary 2.2 and Corollary 2.3 hold only if X is locally compact.

References


