Bundle Properties of Homeomorphism Groups
Transitive on Coset Spaces

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Abstract

Let \( \mathcal{G} \) be any homeomorphism group on a left (or right) coset space \( X \) with a local cross section. If \( \mathcal{G} \) contains all left (resp. right) translations in \( X \) and is endowed with the compact-open topology, then \( \mathcal{G} \) is a bundle space over \( X \) relative to a natural map: \( \mathcal{G} \to X \). Several directions of applicability of this fact are given.

Introduction. Let \( X \) be the left (or right) coset space of a Hausdorff topological group \( G \), which has a local cross section relative to the natural projection: \( G \to X \). Let \( \mathcal{G} \) be any homeomorphism group on \( X \) which contains the group \( \mathcal{F} \) of all left (resp. right) translations in \( X \) and is endowed with the compact-open topology, let \( \mathcal{G}_a \) be the stability subgroup of \( \mathcal{G} \) at an arbitrary point \( a \) of \( X \), and let \( p \) be the map: \( \mathcal{G} \to X \) defined by \( p(\psi) = \psi(a) \) (\( \forall \psi \in \mathcal{G} \)). In §1 we show that i) \( \mathcal{G} \) is a bundle space over \( X \) relative to \( p \) (Theorem 1.2), and ii) \( \mathcal{G} \) is naturally homeomorphic to the product space \( \mathcal{F} \times \mathcal{G}_a \) if and only if \( X \) is a topological group (Corollary 1.3). The above i) is a generalization of a result of McCarty [12], that is the case where \( X \) is locally compact, locally connected, and \( \mathcal{G} \) is Homeo(\( X \)). And the above ii) is a generalization of a result of Hale [6], that is the if part of ii) for the case where \( \mathcal{G} \) is Homeo(\( X \)), and a result of Karube [10], that is the case where \( X \) is locally compact. In §2 several corollaries of our Theorem 1.2 are presented, concerning (1) applicability of theory of fibre spaces, (2) sufficient conditions to be a fibre bundle, and (3) finite product properties. In §3 we show some typical examples.

1. Fundamental results.

Let \( X \) be the left coset space of a Hausdorff topological group \( G \) by its closed subgroup \( H \) (left' coset space for definiteness), which has a local cross section. Let \( \mathcal{G} \) be any homeomorphism group on \( X \) which contains the group \( \mathcal{F} \) of all left translations in \( X \), and let \( \mathcal{G}_a \) be the stability subgroup of \( \mathcal{G} \) at an arbitrarily fixed point \( a \) of \( X \).
Then \( \mathcal{G} = \mathcal{F} \mathfrak{g}_a \). Let \( \psi \) be the map of \( \mathcal{G} \) onto \( X \), defined by \( \psi(g) = \psi(a) \) (\( \psi \in \mathcal{G} \)). Give \( \mathcal{G} \) the compact-open topology. Then the map \( \psi \) is a continuous surjection. And \( \mathcal{G} \) is a semitopological group (in the sense of Husain ([8], p. 27), but not in the sense of Bourbaki ([2], p. 296), \( \mathcal{G} \) is not necessarily a topological group even if \( X \) is a locally compact topological group (see Remark below). The above notations \( X, G, H, \mathcal{G}, \mathfrak{g}_a, a, \psi \) will keep these meanings throughout the paper.

**Remark.** Braconnier ([3], pp. 56–58) gave an example of a totally disconnected, non-compact, locally compact, abelian topological group \( X' \) whose automorphism group is not a topological group, in fact the inverse operation is not continuous under the compact-open topology. Hence, for example, the full homeomorphism group on the space \( X' \) is not a topological group under the compact open topology, while it is a topological group under the \( g \)-topology of Arens [1].

Let \( \varphi \) be a continuous map of a topological space \( E \) to another topological space \( B \). We say that the space \( B \) has a local cross section \( f \) at a point \( b \) in \( B \) relative to \( \varphi \), if \( f \) is a continuous map of a neighborhood \( U \) of \( b \) into \( E \) such that \( \varphi f(u) = u \) for each \( u \in U \).

The following Theorem 1.1 shows that \( \mathcal{G} \) operates on \( X \) as if it was the group of left translations.

**Theorem 1.1.** Let \( \mathcal{G}^* \) be the left coset space of \( \mathcal{G} \) by \( \mathfrak{g}_a \) endowed with the quotient topology relative to the natural projection \( \pi: \mathcal{G} \rightarrow \mathcal{G}^* \). Then \( \mathcal{G}^* \) is homeomorphic to \( X \) in a natural way.

**Proof.** For any point \( x \) of \( X \), let \( f \) be a local cross section defined on a neighborhood \( U \) of \( x \) relative to the natural projection \( \pi: G \rightarrow X \). Let \( \alpha \) be the map of \( G \) onto \( \mathcal{F} \) which carries each element \( g \) of \( G \) to the left translation in \( X \) by \( g \). Then the map \( \varphi \) defined as follows is a local cross section at \( x \) relative to \( \pi \): \( \varphi(u) = \alpha(f(u) \cdot \mathfrak{g}_a^{-1}) \) (\( u \in U \)), where \( \mathfrak{g}_a \) is any fixed point of \( \pi^{-1}(a) \). Using the existence of a local cross section at any point of \( X \) relative to \( \varphi \), we can see that \( \varphi \) is a quotient map. Since the correspondence \( \varphi \circ \pi^{-1} \), where \( \pi^{-1} \) is the natural projection of \( \mathcal{G} \) onto \( \mathcal{G}^* \), is a well-defined map and a continuous bijection of \( \mathcal{G}^* \) onto \( X \). And so the fact that \( \varphi \) is a quotient map implies that \( \varphi \circ \pi^{-1} \) is a homeomorphism.

**Remark.** Ford, Jr. ([5], Theorem 4.1) stated an analogous result to ours, in the case where \( X \) is a uniform space with the so called "strong local homogeneity" and \( \mathcal{G} \) is Homeo(\( X \)) with the topology of uniform convergence and transitive on \( X \). But there is a counter example to this. His statement remains true, for example, if \( X \) is compact. Then the result under the compact-open topology is gained also.

Next we will show that \( \mathcal{G} \) has a natural bundle structure defined by Hu ([7], p. 65) as follows.

**Definition.** Let \( \varphi \) be a continuous map of a space \( E \) into another space \( B \). The space \( E \) is called a bundle space over the base space \( B \) relative to the projection \( \varphi \) if there exists a space \( D \) such that for each \( b \in B \), there is an open neighborhood \( V \) of \( b \)
in \( B \) together with a homeomorphism
\[
\psi_V : V \times D \rightarrow p^{-1}(V)
\]
of \( V \times D \) onto \( p^{-1}(V) \) satisfying the condition
\[
\rho \psi_V(v, d) = v \quad (v \in V, \; d \in D).
\]

**THEOREM 1.2.** \( \mathcal{B} \) is a bundle space over \( X \) relative to \( p \).

**PROOF.** For any point \( x \) of \( X \), let \( f, U, \pi, \alpha, g_0, \) and \( q \) be the same as those in the proof of Theorem 1.1, and put \( W = q(U) \). Define the map \( \Phi \) of the product space \( W \times \mathcal{G}_a \) onto \( W \mathcal{G}_a (= p^{-1}(U)) \) as follows:
\[
\Phi(w, \psi) = w \circ \psi \quad ((w, \psi) \in W \times \mathcal{G}_a),
\]
where \( \circ \) means the composition of maps. The map \( \Phi \) is bijective. We show that \( \Phi \) is bicontinuous. In the following let \( w, \psi, \varphi, \) and \( g \) represents any element of \( W, \mathcal{G}_a, \mathcal{G}, \) and \( G \) respectively.

a) The map \( w \circ \psi \rightarrow w \) of \( W \mathcal{G}_a \) to \( W \) is continuous, for
\[
[q \circ p] (w \circ \psi) = w.
\]
b) Let \( \lambda \) be the map of \( G \times \mathcal{G} \) to \( \mathcal{G} \) defined by
\[
\lambda(g, \varphi) = \alpha(g) \circ \varphi \quad (g \in G, \; \varphi \in \mathcal{G}).
\]
Since the map \( (g, x) \rightarrow [\alpha(g)](x) \) of \( G \times X \) to \( X \) is a continuous group action of \( G \) on \( X \), from Lemma B of Hale [6], \( \lambda \) is continuous.
Since
\[
\psi = w^{-1} \circ (w \circ \psi) = \lambda(g_a, [fp(w)]^{-1}, w \circ \psi),
\]
the continuity of the map \( \lambda \) implies

\[ \text{c) the map } w \circ \psi \rightarrow \psi \text{ of } W \mathcal{G}_a \text{ to } \mathcal{G}_a \text{ is continuous.} \]

Therefore from a) and c), \( \Phi^{-1} \) is continuous. On the other hand, since
\[
w \circ \psi = \lambda([fp(w)] g_a^{-1}, \psi),
\]
the continuity of \( \lambda \) implies that \( \Phi \) is continuous. Consequently \( \Phi \) is a homeomorphism. Put \( \Phi' = \Phi \circ (q \times 1) \), where 1 is the identity map of \( \mathcal{G}_a \). Then \( \Phi' \) is a homeomorphism of the product space \( U \times \mathcal{G}_a \) onto \( p^{-1}(U) \). Clearly \( p \circ \Phi'(u, \psi) = u \) \((u \in U, \psi \in \mathcal{G}_a) \).

**QUESTION.** To apply our results to the more special base spaces, it is desirable to find handy conditions to assure the existence of a local cross section relative to \( \pi \). It is one of such condition that \( G \) is a finite-dimensional locally compact topological group (see Karube [9] and Nagami [13]). Is there such a handy condition for the case where \( G \) is infinite dimensional or not locally compact, and \( X \) is not a topological group?

Let \( \psi \) be the map of the product set of \( \mathcal{T} \) and \( \mathcal{G}_a \) onto \( \mathcal{G} (= \mathcal{T} \mathcal{G}_a) \) such that \( (\tau, \varphi) \rightarrow \tau \varphi \) \((\tau \in \mathcal{T}, \varphi \in \mathcal{G}_a) \). We say that \( \mathcal{G} \) is **naturally homeomorphic** to the product space \( \mathcal{T} \times \mathcal{G}_a \) if \( \psi \) is a homeomorphism.

**COROLLARY 1.3.** \( \mathcal{G} \) is **naturally homeomorphic** to the product space \( \mathcal{T} \times \mathcal{G}_a \) (and then homeomorphic to the product space \( X \times \mathcal{G}_a \) also) if and only if \( X \) is a topological group.
PROOF. When \( X \) is a topological group, consider \( X/(e) \) (\( e \) : the identity of \( X \)) as the base space \( X \), then \( X/(e) \) has the full cross section relative to the natural projection : \( X \to X/(e) \). Thus our method of proof for Theorem 1.2 implies that \( \mathcal{G} \) is naturally homeomorphic to the product space \( \mathcal{F} \times \mathcal{G}_a \). For the proof of the converse of this, we need Proposition 1.6 below, which can be proved by the following Lemma 1.4 and Lemma 1.5.

**Lemma 1.4.** \( \mathcal{F} \cap \mathcal{G}_a = \alpha(g_aHg_a^{-1}) \) for any fixed element \( g_a \) of \( \pi^{-1}(a) \).

**Lemma 1.5.** \( \ker \alpha = \bigcap_{a \in G} gHG^{-1} \), i.e. \( \ker \alpha \) is the largest subgroup of \( H \) which is invariant in \( G \).

**Proposition 1.6.** The following three statements are equivalent : (1) \( \mathcal{G} \) is injective, (2) \( \mathcal{F} \cap \mathcal{G}_a \) consists of only the identity map, and (3) \( H \) is a normal subgroup of \( G \).

**Proof.** The equivalence of (1) and (2) is shown directly. Using Lemma 1.4 and Lemma 1.5, we can show the equivalence of (2) and (3).

**Remark.** Even if \( X \) is a topological group, \( \mathcal{G} \) is not necessarily a direct product nor a semidirect product of \( \mathcal{F} \) and \( \mathcal{G}_a \). The case where \( X \) is a circle group gives a counter example.

2. Other corollaries.

**A.** By Theorem 4.1 of Hu [7], on p. 65, our Theorem 1.2 implies the following. 

**Corollary 2.1.** \( \mathcal{G} \) is a fibre space over \( X \) relative to \( p \).

Thus we can use the theory of fibre spaces, to investigate the homotopy group of \( \mathcal{G} \).

**B.** Conditions for \( \mathcal{G} \) to be a fibre bundle.

**Corollary 2.2.** If \( X \) is 'compact' or 'locally connected and locally compact', then \( \mathcal{G} \) is a principal fibre bundle over \( X \) with fibre and group \( \mathcal{G}_a \).

**Proof.** The condition on \( X \) implies that \( \mathcal{G}_a \) is a topological group under the compact-open topology (cf. Arens[1]). Thus Theorem 1.2 implies the above conclusion.

**Corollary 2.3.** If \( X \) is 'compact' or 'locally connected and locally compact' topological group, then \( \mathcal{G} \) is a product bundle over \( X \).

**Remark.** 1) McCarty [12] has shown the same conclusions of these two corollaries in the case where \( \mathcal{G} = \text{Homeo}(X) \). 2) If \( X \) is only locally compact, these two corollaries are not necessarily true (see Remark before Theorem 1.1).

**C.** Finite product local properties.

We say a property \( P \) is an \( FPL \) property (finite product local property), if \( P \) is a local property such that the product space of spaces \( A \) and \( B \) has the property \( P \) if and only if both \( A \) and \( B \) have the property \( P \). Among those properties studied on full homeomorphism groups, the following are \( FPL \) properties ; to be locally connected, locally arcwise connected, \( LC^* \), \( LC^\omega \), locally contractible, to have arbitrarily small contractible neighborhoods, and to be an ANR.

As a result obtained from Theorem 1.2, \( \mathcal{G} \) is locally homeomorphic to the product space \( X \times \mathcal{G}_a \). So we have the following.
COROLLARY 2.4. \( \mathcal{G} \) has an FPL property if and only if both \( X \) and \( \mathcal{G}_a \) have the property.

In particular for \( \mathcal{G} \) to have any FPL property, \( X \) must have the property — this is a criterion when we respect \( \mathcal{G} \) to have an FPL property.

A space is called an \( l_\ast \)-manifold if it is separable metrizable and locally homeomorphic to \( l_\ast \).

COROLLARY 2.5. Let \( X \) be separable metrizable, locally compact, and admits a nontrivial flow, and let \( \mathcal{H} \) be the full homeomorphism group of \( X \) with the compact-open topology. Then \( \mathcal{H} \) is an \( l_\ast \)-manifold if and only if \( X \) is an ANR and the stability subgroup \( \mathcal{H}_a \) of \( \mathcal{H} \) at \( aeX \) is an \( l_\ast \)-manifold.

PROOF. Since \( X \) is 2nd countable, locally compact Hausdorff space, both \( \mathcal{H} \) and \( \mathcal{H}_a \) are separable and metrizable. So we may only apply the well-known result of Toruńczyk ([14] Corollary 4.8) to our Corollary 2.4 in the case where \( \mathcal{G} = \mathcal{H}(= \text{Homeo}(X)) \).

COROLLARY 2.6. If \( X \) is a locally connected, compact, metrizable ANR, and admits a nontrivial flow, then the homeomorphism group \( \mathcal{H} \) on \( X \) is an \( l_\ast \)-manifold if and only if the homeomorphism group \( \mathcal{H}(X-a) \) on the space \( X-a \) is an \( l_\ast \)-manifold.

PROOF. We may only apply a result of Arens, stated in the form of Lemma 4.2 in McCarty [12], to our Corollary 2.5.

COROLLARY 2.7. If \( X \) is compact and locally Euclidean, then the same conclusion as Corollary 2.6 holds.

COROLLARY 2.8. For any coset space of a compact Lie group, the same conclusion as Corollary 2.6 holds.

PROOF. Any coset space of a Lie group has a local cross section with respect to the natural projection (Chevalley [4], p.110).

EXAMPLES. For example, the homeomorphism group on each of the following spaces is an \( l_\ast \)-manifold: Euclidean plane, and the punctured space of real projective plane, and the usual torus. This follows from results of Toruńczyk [14], Luke and Mason [11], and Corollary 2.8.

3. Some typical examples.
A. Full homeomorphism groups

In the case where \( \mathcal{G} \) is the hull homeomorphism group on \( X \), and \( X \) has a local cross section, all our results thus far are valid.

B. Diffeomorphism groups

Let \( X \) be the left coset space of a Lie group \( G \) by a closed subgroup \( H \). Give \( X \) the analytic structure induced from that of \( G \), and let \( \text{Diff}^r(X) \) be the group of all \( C^r \) diffeomorphisms on \( X \) with respect to the \( C^r \) differential structure induced from the analytic structure on \( X \) \((r = 0, 1, 2, ..., \omega) \). Since every left translation on \( X \) is an analytic diffeomorphism, \( \mathcal{T} \) is contained in \( \text{Diff}^r(X) \). Thus in the case where \( \mathcal{G} = \text{Diff}^r(X) \) all conclusions in §1 and §2 are valid.
C. Isometry groups

**Lemma 3.1.** If $G$ has a two-sided invariant metric $\rho$, then $\rho'$ defined as follows is a left invariant metric on $X$.

$$\rho'(x_1, x_2) = \rho(g_1 H, g_2 H)$$

for $x_i \in X$ and $g_i \in \pi^{-1}(x_i)$ ($i = 1, 2$). The metric $\rho'$ is called the metric induced from $\rho$.

**Remark.** A $T_0$-topological group has a two-sided metric if and only if it is 1st countable, and

(*) it has arbitrarily small invariant neighborhoods of the identity.

The condition (*) is satisfied for Abelian (topological) groups, compact groups, discrete groups, and is closed under the operations of making subgroups, factor groups, direct product groups, and projective limit groups.

**Lemma 3.2.** If $G$ has a two-sided invariant metric $\rho$, the induced metric $\rho'$ from $\rho$ is given on $X$, and let $I^*(X)$ be the group of all surjective isometries of $X$ onto itself. Then for $I^* = I^*(X)$, all statements in §1 and §2 are valid.

**Lemma 3.3.** If $X$ is "compact" or "connected, locally Euclidean", and has a left invariant metric, then any isometry of $X$ into itself is surjective.

**Proof.** In the case where $X$ is compact, the conclusion follows from the well known fact that an isometry of a compact metric space into itself is surjective. In the latter case we may use the following facts: (1) if a locally compact left coset space has a left invariant metric, then the metric is complete, (2) in a complete metric space $M$, for any isometry $\varphi$ of $M$ into itself, $\varphi(M)$ is closed, (3) for two locally Euclidean spaces $M$ and $N$ having the same dimension, any continuous injection from $M$ to $N$ is open (and so a local homeomorphism), and (4) for a locally Euclidean, connected, complete metric space $M$, any isometry of $M$ into itself is surjective.

**Theorem 3.4.** Let $G$ be a group with a two-sided invariant metric $\rho$, and give $X$ the induced left invariant metric from $\rho$. If $X$ is "compact" or "connected and locally Euclidean", then the group $I(X)$ of all isometries of $X$ to itself satisfies all statements thus far, considering $I$ as $I(X)$.

**Appendix.** All our results are valid also under the $g$-topology of Arens [1] that is stronger than the compact-open topology. The $g$-topology is useful to make a homeomorphism group on a locally compact Hausdorff space a topological transformation group. In fact, under the $g$-topology our Corollary 2.2 and Corollary 2.3 hold only if $X$ is locally compact.

**References**


