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Pluriharmonic Functions on a Domain Over a Product Space

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Abstract

Let D be a domain over a product space of a Stein manifold S and Grassmann manifolds \(G_i (i=1,2,...,N)\) and \(\hat{D}\) be the envelope of holomorphy of D. In this paper we shall show that each real-valued pluriharmonic function on D is the real part of a holomorphic function on D if and only if \(H^1(\hat{D}, \mathbb{Z}) = 0\), provided that \(\hat{D}\) is not holomorphically equivalent to the set \(E \times V_1 \times ... \times V_{i-1} \times G_i \times V_{i+1} \times ... \times V_N (i=1,...,N)\), where E is an open set of S and \(V_i\) is an open set of \(G_i\).

1. Introduction. Let M be a complex manifold. The real part of a holomorphic function on M is a real-valued pluriharmonic function on M. On the other hand, a real-valued pluriharmonic function on M is not always the real part of a holomorphic function on M. Matsugu[5] proved that each real-valued pluriharmonic function on a domain D over a Stein manifold is the real part of a holomorphic function on D if and only if \(H^1(\hat{D}, \mathbb{Z}) = 0\), where \(\hat{D}\) is the envelope of holomorphy of D and \(\mathbb{Z}\) is the constant sheaf of integers. In the previous paper[2] we considered the case of a domain over a Grassmann manifold. In this paper we generalize the above two results.

2. Pluriharmonic function and envelope of pluriharmony. Let M be a complex manifold and \(u\) be a 2 times continuously differentiable complex-valued function on M. \(u\) is said to be pluriharmonic at a point \(p \in M\) if \(\hat{\partial} \hat{\partial} u = 0\) in U, where U is a neighborhood of \(p\). If \(u\) is pluriharmonic at every point of M, \(u\) is said to be pluriharmonic on M. Let \(O\) be the sheaf of germs of holomorphic functions and \(H\) be the sheaf of germs of real-valued pluriharmonic functions. We consider the two sheaf homomorphisms obtained by corresponding a holomorphic function \(f\) to its real part \(\text{Re} f, r : O \rightarrow H\), and obtained

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by corresponding a real number $b$ to a purely imaginary number $b\sqrt{-1}$, $i : \mathbb{R} \to \mathbb{O}$, where $\mathbb{R}$ is the constant sheaf of the real number field. Since $i$ is surjective by [3] (p. 272) and $i$ is injective, we have the following lemma.

**Lemma 1.** Let $M$ be a complex manifold. Then the sequence of sheaves on $M$

$$0 \to \mathbb{R} \to \mathbb{O} \to \mathbb{H} \to 0$$

is exact.

Let $M$ be a complex manifold. If $\phi$ is a locally biholomorphic mapping of a complex manifold $D$ into $M$, $(D, \phi)$ is called an open set over $M$. Moreover, if $D$ is connected, $(D, \phi)$ is called a domain over $M$. If $\phi$ is a biholomorphic mapping of $D$ into $M$, $(D, \phi)$ is called a schlicht open set over $M$ and is identified with the open subset $\phi(D)$ in $M$. Let $(D, \phi)$ and $(D', \phi')$ be open sets over $M$. A holomorphic mapping $\lambda$ of $D$ into $D'$ with $\phi = \phi' \circ \lambda$ is called a mapping of $(D, \phi)$ into $(D', \phi')$. If $\lambda$ is a biholomorphic mapping of $D$ onto $D'$, $(D, \phi)$ and $(D', \phi')$ are identified.

Consider domains $(D, \phi)$ and $(D', \phi')$ over $M$ with a mapping $\lambda$ of $(D, \phi)$ into $(D', \phi')$. Let $f$ be a pluriharmonic (or holomorphic) function in $D$. A pluriharmonic (or holomorphic) function $f'$ in $D'$ with $f = f' \circ \lambda$ is called a pluriharmonic (or holomorphic) continuation of $f$ to $(\lambda, D', \phi')$, or shortly $(D', \phi')$. Let $F$ be a family of pluriharmonic (or holomorphic) functions in $D$. If any pluriharmonic (or holomorphic) function of $F$ has a pluriharmonic (or holomorphic) continuation to $(\lambda, D', \phi')$, $(\lambda, D', \phi')$ or shortly $(D', \phi')$ is called a pluriharmonic (or holomorphic) completion of $(D, \phi)$ with respect to $F$. Let $(\lambda, D', \phi')$ be a pluriharmonic (or holomorphic) completion of $(D, \phi)$ with respect to $F$. Let $(\lambda, D', \phi')$ be any pluriharmonic (or holomorphic) completion of $(D, \phi)$ with respect to $F$ and $F'$ be the set of pluriharmonic (or holomorphic) continuations of all pluriharmonic (or holomorphic) functions of $F$ to $(\lambda, D', \phi')$. Then if there exists a mapping $\mu$ of $(D', \phi')$ into $(D, \phi)$ with $\lambda = \mu \circ \lambda$ such that $(\mu, D, \phi)$ is a pluriharmonic (or holomorphic) completion of $(D', \phi')$ with respect to $F'$, $(D, \phi)$ is called an envelope of pluriharmony (or holomorphy) of $(D, \phi)$ with respect to $F$.

If $F$ is the family of all pluriharmonic (or holomorphic) functions in $D$, an envelope of pluriharmony (or holomorphy) of $(D, \phi)$ with respect to $F$ is called simply an envelope of pluriharmony (or holomorphy) of $(D, \phi)$. If $F$ consists of only a pluriharmonic (or holomorphic) function $f$ in $D$, an envelope of pluriharmony (or holomorphy) of $(D, \phi)$ with respect to $F$ is called simply a domain of pluriharmony (or holomorphy) of $f$. The following lemma is given by Matsugu [5].

**Lemma 2.** Let $(D, \phi)$ be a domain over a complex manifold $M$ and $F$ be a family of pluriharmonic (or holomorphic) functions in $D$. Then there exists uniquely an envelope of pluriharmony of $(D, \phi)$ with respect to $F$. 
A domain \((D, \phi)\) over a complex manifold \(M\) is said to be pseudoconvex if for every point \(p\) of \(M\) there exists a neighborhood \(U\) of \(p\) such that \(\phi^{-1}(U)\) is a Stein manifold.

The following lemma is given in [1].

**Lemma 3.** Let \((D, \phi)\) be a domain over a complex manifold \(M\) and \(F\) be a family of pluriharmonic (or holomorphic) functions in \(D\). Then the envelope of pluriharmony (or holomorphy) \((\hat{D}, \hat{\phi})\) of \((D, \phi)\) with respect to \(F\) is pseudoconvex.

3. Pseudoconvex domain over a product space. Let \(N\) be a positive integer. Let \(n_i\) and \(r_i\) \((i=1,2,...,N)\) be positive integers. Let \(G_{n_i, r_i}\) \((i=1,2,...,N)\) be a Grassmann manifold. Let

\[
G = G_{n_1, r_1} \times G_{n_2, r_2} \times \cdots \times G_{n_N, r_N}
\]

be the product space of \(N\) Grassmann manifolds. Let \(S\) be a connected Stein manifold. Consider the product space \(X = S \times G\). Let \((D, \phi)\) be a domain over \(X\). An open set \(\Omega\) of \(D\) is said to be a univalent open set containing \(G_{n_i, r_i}\) if \(\phi \mid \Omega\) is a biholomorphic mapping of \(\Omega\) onto an open set \(W\) of \(X\), where \(W\) is written in the form

\[
W = E \times V_1 \times \cdots \times V_{i-1} \times G_{n_{i+1}, r_{i+1}} \times \cdots \times V_N,
\]

\(E\) is an open set of \(S\) and \(V_j\) \((j=1,...,i-1, i+1,...,N)\) is an open set of \(G_{n_j, r_j}\), respectively.

**Theorem 4.** Let \((D, \phi)\) be a pseudoconvex domain over \(X\) such that \(D\) does not contain a univalent open set containing \(G_{n_i, r_i}\) for \(i=1,2,...,N\). Then \(D\) is a Stein manifold.

**Proof.** Let \(V_{n_i, r_i}\) be a Stiefel manifold which defines \(G_{n_i, r_i}\) \((i=1,2,...,N)\), respectively. Then there are canonical mappings \(\nu : V_{n_i, r_i} \longrightarrow G_{n_i, r_i}\) \((i=1,2,...,N)\). We set

\[
\tau_i(s, x_1, ..., x_N) = (s, \nu_i(x_1), x_2, ..., x_N)
\]

and

\[
D_i = \{(s, x_1, ..., x_N, y) : \phi S \times V_{n_1, r_1} \times G_{n_2, r_2} \times \cdots \times G_{n_N, r_N} \times D : \tau_i(s, x_1, ..., x_N) = \phi(y)\}
\]

Then we have the following commutative diagram:
We shall show that \((D_1, \phi_1, S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N})\) is a pseudoconvex domain. We set
\[ T = S \times (C^{n_1} \times \ldots \times G_{n_N}) \]
Let \(R\) be the set of all boundary points removable along \(T\). Let \((D_1^*, \phi_1^*, S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N})\) be the extension of \((D_1, \phi_1, S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N})\) along \(T\). Then \((D_1 \cup R, \phi_1^*, D_1 \cup R, S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N})\) is pseudoconvex.

Suppose that \(R\) is not empty. Let \(q \in R\). There exists a point \((s, x_1, \ldots, x_N) \in S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N}\) such that \(\phi_1^*(q) = \tau_1^{-1}(s, x_1, \ldots, x_N)\). We set \(F^* = \phi_1^{-1}(\tau_1^{-1}(s, x_1, \ldots, x_N))\). Let \(F_0^*\) be the connected component of \(F^*\) which contains \(q\). Then \((F_0^*, \phi_1^*, F_0^*, \tau_1^{-1}(s, x_1, \ldots, x_N))\) is a pseudoconvex domain. By using the same method as the proof of Ueda [7], we can prove that \(F_0^*\) is biholomorphic onto \(\tau_1^{-1}(s, x_1, \ldots, x_N)\). There exists a point \(q_0 \in R\) which lies over \((s, 0, x_2, \ldots, x_N)\), where \(0 \in C^{n_1}\). Therefore there exists a neighborhood \(U\) of \(q\) which is mapped biholomorphically onto a neighborhood of \((s, 0, x_2, \ldots, x_N)\). Then \(\tau_1(U \cap D_1)\) is biholomorphic onto an open set \(E \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N}\), where \(E, V_i\) are open sets of \(S, C^{n_i}\), respectively. This is the contradiction. Therefore \((D_1, \phi_1, S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N})\) is pseudoconvex. We define a mapping
\[
\tau_2 : S \times C^{n_1} \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N} \rightarrow S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N}
\]
by \(\tau_2(s, x_1, x_2, \ldots, x_N) = (s, x_1, \nu_2(x_2), x_3, \ldots, x_N)\) and put
\[ D_2 = \{(s, x_1, \ldots, x_N, y) \in S \times C^{n_1} \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N} : \tau_2(s, x_1, \ldots, x_N) = \phi_1(y)\} \]
Then we have the following commutative diagram:

\[
\begin{array}{ccc}
D_2 & \xrightarrow{\tau_2} & D_1 \\
\downarrow{\phi_2} & & \downarrow{\phi_1} \\
S \times C^{n_1} \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N} & \xrightarrow{\tau_2} & S \times C^{n_1} \times G_{n_2} \times \ldots \times G_{n_N}
\end{array}
\]
Then \((D_2, \phi_2, S \times C^{n_1} \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N})\) is pseudoconvex. By using the same process as the preceding proof, we can show that \((D_2, \phi_2, S \times C^{n_1} \times G_{n_2} \times G_{n_3} \times \ldots \times G_{n_N})\) is pseudoconvex. By repeating this process, we arrive at the fact that
(D, φ, S × Cⁿ^1 + n^2 + n^N) is pseudoconvex. Since \( S \times Cⁿ^1 + n^2 + n^N \) is a Stein manifold, \( D \) is a Stein manifold. In view of the theorem of Matsushima-Morimoto [6], \( D \) is a Stein manifold. This completes the proof.

4. Main results. Let \( X \) be the same product space \( S \times G \) as the previous section.

**Lemma 5.** Let \( (D, \phi) \) be a domain over \( X \). Let \( f \) be a real-valued pluriharmonic function in \( D \) and \((\lambda, \tilde{D}, \tilde{\phi})\) be the domain of pluriharmony of \( f \). If \( \tilde{D} \) contains a univalent open set containing \( G_{n_j, r_j} \), then any point of \( \tilde{D} \) is contained in a univalent open set containing \( G_{n_j, r_j} \).

**Proof.** We may assume that \( i = N \). Let \( A \) be the set of all points \( \omega \) of \( \tilde{D} \) such that \( \omega \) is contained in a univalent open set containing \( G_{n_N, r_N} \). Then \( A \) is a non-empty open subset of \( \tilde{D} \). Thus, it is sufficient to show that \( A \) is closed subset in \( D \). Let \( \omega \) be a point of the closure of \( A \). There exist, respectively, open neighborhoods \( W \), \( V \) and \( U \) of \( \omega \), \( \pi(\tilde{\phi}(\omega)) \) and \( \pi_N(\phi(\omega)) \) such that \( \tilde{\phi} \mid W \) is a biholomorphic mapping of \( W \) onto \( V \times U \) and such that \( V \) and \( U \) are coordinate neighborhoods, where \( \pi \) is the projection of \( X \) onto \( S \times G_{n_1, r_1} \times \ldots \times G_{n_{N-1}, r_{N-1}} \) and \( \pi_N \) is the projection of \( X \) onto \( G_{n_N, r_N} \). There exist a point \( z \in V \) and a univalent open subset \( \Omega \) containing \( G_{n_N, r_N} \) such that \( \tilde{\phi} \mid \Omega \) is a biholomorphic mapping of \( \Omega \) onto \( E \times V_1 \times \ldots \times V_{N-1} \times G_{n_{N}, r_N} \), where \( z \in E \times V_1 \times \ldots \times V_{N-1} \), \( E \) is an open set of \( S \) and \( V_j \) \((j = 1, 2, \ldots, N - 1)\) is an open set of \( G_{n_j, r_j} \), respectively. We may assume that there exists a biholomorphic mapping \( \mu \) of \( V \) onto a polydisc \( V' \) such that \( \mu(E \times V_1 \times \ldots \times V_{N-1}) \) and \( V' \) is a polydisc with center the origin. Let \( \tilde{f} \) be the pluriharmonic continuation of \( f \) to \( (\lambda, \tilde{D}, \tilde{\phi}) \). In view of J. Kajiwara and N. Sugawara [4], \( \tilde{f} \circ (\phi \mid W)^{-1} \circ (\mu^{-1} \times 1) \) is a pluriharmonic continuation of \( f \) to \( V \times G_{n_{N}, r_{N}} \). Since \((\lambda, \tilde{D}, \tilde{\phi})\) is the domain of pluriharmony of \( f \), there exists a biholomorphic mapping \( \xi \) of \( V \times G_{n_{N}, r_{N}} \) into \( \tilde{D} \) such that \( \tilde{\phi} \circ \xi \) is the identity of \( V \times G_{n_{N}, r_{N}} \). Since \( \xi(V \times G_{n_{N}, r_{N}}) \supset W \) and \( \xi(V \times G_{n_{N}, r_{N}}) \) is open set in \( \tilde{D} \), \( \omega \) belongs to \( A \). This completes the proof.

**Lemma 6.** Let \( (D, \phi) \) be a domain over \( X \). Let \( f \) be a pluriharmonic function and \((\lambda, \tilde{D}, \tilde{\phi})\) be the domain of pluriharmony of \( f \). Assume that \( \tilde{D} \) contains univalent open sets containing \( G_{n_j, r_j} \) \((j = s, \ldots, N)\) and \( \tilde{D} \) does not contain univalent open sets containing \( G_{n_j, r_j} \) \((j = 1, \ldots, s-1)\). We put \( Y = S \times G_{n_1, r_1} \times \ldots \times G_{n_{s-1}, r_{s-1}} \times G_{n_N, r_N} \times \ldots \times G_{n_{N}, r_{N}} \). Then there exist a Stein manifold \((L, \psi)\) over \( Y \) and a biholomorphic mapping \( \eta \) of \( \tilde{D} \) onto \( L \times G \) such that \( \tilde{\phi} = (\psi \times 1) \circ \eta \).

**Proof.** Let \( \pi_Y \) be the projection of \( X \) onto \( Y \) and \( \pi_G \) be the projection of \( X \) onto \( G \). Let \( x \) be a point of \( D \). We put \( (y, z) = \tilde{\phi}(x) \) where \( y \in Y \) and \( z \in G \). From lemma 5 \( \tilde{\phi}^{-1}((y) \times G) \) is a covering manifold of a simply connected manifold \((y) \times G \). Hence \( \tilde{\phi} \) maps each connected component of \( \tilde{\phi}^{-1}((y) \times G) \) biholomorphically onto \((y) \times G \). We
shall induce in $\tilde{D}$ an equivalence relation $R$ as follows: $x_1 \sim x_2$ if and only if $x_1$ and $x_2$ belong to the same connected component of $\phi^{-1}(\{(y) \times G\})$ for some $y \in Y$. Then $L = \tilde{D}/R$ is a complex manifold such that $(L, \psi)$ is a domain over $Y$ where $\mu$ is the canonical mapping of $\tilde{D}$ onto $L$ and $\psi$ is the canonical mapping $L$ into $Y$ such that $\pi_Y \circ \phi = \psi \circ \mu$. Then the mapping $\eta$ defined by

$$\eta(x) = (\mu(x), \pi_0 \circ \phi(x))$$

is a biholomorphic mapping of $\tilde{D}$ onto $L \times G$ such that $\tilde{\phi} = (\psi \times 1) \circ \eta$. Since $\tilde{D}$ is pseudoconvex and $L$ does not contain univalent open sets containing $G_{n_j, r_j} (j = 1, 2, \ldots, s - 1)$, $(L, \psi)$ is a pseudoconvex domain over $Y$. Hence from theorem 4 $L$ is a Stein manifold. This completes the proof.

Using the above results we prove the following main theorem.

**THEOREM 7.** Let $(D, \phi)$ be a domain over $X$ and $(\lambda, D, \tilde{\phi})$ be the envelope of holomorphy of $(D, \phi)$. If $\tilde{D}$ does not contain univalent open sets containing $G_{n_j r_j} (j = 1, 2, \ldots, N)$, then each real-valued pluriharmonic function on $D$ is the real part of a holomorphic function on $D$ if and only if $H^1(D, Z) = 0$.

**PROOF.** Since $\tilde{D}$ is a Stein manifold from theorem 4, we have $H^1(\tilde{D}, O) = 0$. From lemma 1 we have the exact sequence of cohomologies

$$H^0(\tilde{D}, O) \to H^0(\tilde{D}, H) \to H^1(\tilde{D}, R) \to 0.$$ 

Hence we have that $H^1(\tilde{D}, R) = 0$ if and only if the homomorphism $H^0(\tilde{D}, O) \to H^0(\tilde{D}, H)$ is surjective. Since $(\lambda, D, \tilde{\phi})$ is the envelope of holomorphy of $(D, \phi)$, we have that $\lambda$ induces the isomorphism $\lambda^* : H^0(D, O) \to H^0(D, O)$, where $\lambda^*(\tilde{f}) = \tilde{f} \circ \lambda$ for $\tilde{f} \in H^0(\tilde{D}, O)$. We claim that the induced homomorphism $\mu^* : H^0(D, H) \to H^0(D, H)$ is also an isomorphism, where $\mu^*(\tilde{u}) = \tilde{u} \circ \lambda$ for $\tilde{u} \in H^0(\tilde{D}, H)$. To see this it is sufficient to show that $\mu^*$ is surjective. Suppose $u \in H^0(D, H)$. Let $(\lambda', D', \phi')$ be the domain of pluriharmony of $u$ and $u'$ be the pluriharmonic continuation of $u$ to $(D', \phi')$. From lemma 3 and lemma 6, after permuting $(n_1, n_2, \ldots, n_k)$, if necessary, either $D'$ is a Stein manifold or there exist an integer $s$ with $1 \leq s \leq N$, a Stein manifold $(L, \psi)$ over $Y = S \times G_{n_1, r_1} \times \cdots \times G_{n_{s-1}, r_{s-1}}$ and a biholomorphic mapping $\eta : D' \to L \times G$ such that $\phi' = (\psi \times 1) \circ \eta$ where $G = G_{n_s, r_s} \times \cdots \times G_{n_N, r_N}$. In the former case $D'$ is a domain of holomorphy of a holomorphic function in $D$. Since $(\lambda, D, \tilde{\phi})$ is the envelope of holomorphy of $(D, \phi)$, there exists a holomorphic mapping $\phi : \tilde{D} \to D'$ such that $\lambda' = \phi \circ \lambda$. We put $\tilde{u} = u \circ \phi \in H^0(\tilde{D}, H)$. Then $\mu^*(\tilde{u}) = u' \circ \phi \circ \lambda = u' \circ \lambda' = u$. Therefore $\mu^*$ is surjective. In the latter case, $L \times S$ is a domain of holomorphy of a holomorphic function in $D$ and so is $D'$. Thus by the same argument as the preceding case, we can prove that $\mu^*$ is surjective. From the two isomorphism $H^0(\tilde{D}, O) \cong H^0(D, O)$ and $H^0(\tilde{D}, H) \cong H^0(D, H)$ we see that the homomorphism $H^0(D, O) \to H^0(D, H)$ is surjective if and only if the homomorphism $H^0(\tilde{D}, O) \to H^0(\tilde{D}, H)$ is surjective. From the universal coefficient theorem for cohomology, it follows that
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H^1(\bar{D}, \mathbb{R}) = 0 if and only if H^1(\bar{D}, \mathbb{Z}) = 0.

This completes the proof.

By the same method as the above proof, we have the following corollary.

**Corollary.** Let \((D, \phi)\) be a domain over \(X\) and \((\lambda, \bar{D}, \phi)\) be the envelope of holomorphy of \((D, \phi)\). Then the homomorphism \(H^p(\bar{D}, O) \to H^p(\bar{D}, H)\) is surjective if and only if the homomorphism \(H^p(\bar{D}, O) \to H^p(\bar{D}, H)\) is surjective.

References


