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The Group of Homeomorphisms on a Connected 1-Manifold

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Abstract

The topological types of the spaces of homeomorphisms on paracompact Hausdorff connected 1-manifolds are classified.

1. Introduction.

A paracompact Hausdorff connected 1-manifold is homeomorphic to one of the following four spaces:
- $R$: the real line.
- $R_+$: a real half-line.
- $S$: a circle.
- $I$: a closed interval on $R$.

This is well-known—an easy proof is given in [5]. Let $M$ be any one of the above four spaces with orientation, $H(M)$ the group of all homeomorphisms of $M$ onto itself endowed with the compact open topology, and $H^+(M)$ the subspace of $H(M)$ which consists of orientation-preserving ones. Then

$H(I) = H^+(I) \cong Z_2 \times Z_2$ for $M = R$ (Anderson [1]),

$H(S) = H^+(S)$ for $M = I$ (Karube [4]),

and

$H(R_+) = H^+(R_+) \cong l_2$ (Karube [4]),

where $l_2$ is the Hilbert space of square-summable sequences, $Z_2$ the discrete space consisting of two points, $\cong$ means being homeomorphic, and $\times$ topological product.

In this note we consider $H(S)$.

2. The group of homeomorphisms on a circle.

**Lemma 1** (Karube [4]). Let $[0,1]$ (resp. $(0,1)$) be the closed (resp. open) interval on $R$. Then $H^([0,1])$ and $H^((0,1))$ are isomorphic as topological
groups by the natural map. And so $H^*(R) = l_2$.

Considering the circle $S$ a multiplicative topological group of complex numbers of norm 1, let $T$ denote the subgroup of $H(S)$ consisting of all translations in the group $S$, $P$ the subgroup of $H(S)$ consisting of all homeomorphisms which leave the identity 1 fixed, $P^*$ the subgroup of $P$ consisting of orientation-preserving ones, and $Z_2$ either the subgroup of $H(R)$ or of $H(S)$ consisting of the identity map and the reflexion.

**Lemma 2.** $P^*$ and $H^*(S - \{1\})$ are isomorphic as topological groups by the natural map. And so $P^* = l_4$.

**Proof.** A modification of the proof of Lemma 1 ([4]) ensures that $P^*$ is isomorphic to $H^*(S - \{1\})$ as topological groups. Since $S - \{1\}$ is homeomorphic to $R$, $H^* (S - \{1\}) = H^*(R)$. Hence $P^* = l_4$ by Lemma 1.

**Theorem.** $H(S) = T \times P^* \times Z_2 = T \times P^* \times Z_2$ and $(T, P^*) = (S, l_4)$.

**Proof.** Both $T$ and $P$ are closed subgroups and $H(S) = TP$, $T \cap P = \{1\}$. Whereas $H(S)$ is neither a direct product nor a semidirect product of $T$ and $P$, the correspondence of $u \in H(S)$ to $(t_{u(1)}, t_{u(1)^{-1}})$ gives a homeomorphism between $H(S)$ and the product space $T \times P$ ——this owes to a remark of Keesling ([6], p. 15). The space $T$ is homeomorphic to $S$. Since $P^*$ is an open and closed subgroup of $P$, the space $P$ is homeomorphic to $l_4 \times Z_2$ by Lemma 2. Consequently $H(S) = S \times l_4 \times Z_2$.

**Remark.** Another proof that $P = l_4 \times Z_2$ is obtained by the fact: let $X$ be a locally connected, locally compact Hausdorff space, $X^*$ the compactification of $X$ by adding a point $x_\infty$ to $X$, and $H(X^*, x_\infty)$ the subspace of $H(X^*)$ consisting of the mappings that leave the point $x_\infty$ fixed, then $H(X^*, x_\infty) = H(X)$ ——this owes to Theorem 2 of [2] and Theorems 1, 3, and 4 of [3].

**References**


