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The Group of Homeomorphisms on a Connected 1-Manifold

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(Received Oct. 31, 1983)

Abstract

The topological types of the spaces of homeomorphisms on paracompact Hausdorff connected 1-manifolds are classified.

1. Introduction.

A paracompact Hausdorff connected 1-manifold is homeomorphic to one of the following four spaces:

- \( R \): the real line.
- \( R_{+} \): a real half-line.
- \( S \): a circle.
- \( I \): a closed interval on \( R \).

This is well-known—an easy proof is given in [5]. Let \( M \) be any one of the above four spaces with orientation, \( H(M) \) the group of all homeomorphisms of \( M \) onto itself endowed with the compact open topology, and \( H^{+}(M) \) the subspace of \( H(M) \) which consists of orientation-preserving ones. Then

\[
H(I) = H^{+}(I) \times \mathbb{Z}_2 \quad \text{and} \quad H^{+}(I) = I_2 \quad \text{(Anderson [1])},
\]

\[
H(R) = H^{+}(R) \times \mathbb{Z}_2 \quad \text{and} \quad H^{+}(R) = I_2 \quad \text{(Karube [4])},
\]

and \( H(R_{+}) = H^{+}(R_{+}) = I_2 \) (Karube [4]), where \( I_2 \) is the Hilbert space of square-summable sequences, \( \mathbb{Z}_2 \) the discrete space consisting of two points, \( \sim \) means being homeomorphic, and \( \times \) topological product.

In this note we consider \( H(S) \).

2. The group of homeomorphisms on a circle.

**Lemma 1** (Karube [4]). Let \([0,1]\) (resp. \((0,1))\) be the closed (resp. open) interval on \( R \). Then \( H^{+}([0,1]) \) and \( H^{+}((0,1)) \) are isomorphic as topological
groups by the natural map. And so $H'(R) = l_1$.

Considering the circle $S$ a multiplicative topological group of complex numbers of norm 1, let $T$ denote the subgroup of $H(S)$ consisting of all translations in the group $S$, $P$ the subgroup of $H(S)$ consisting of all homeomorphisms which leave the identity fixed, $P^*$ the subgroup of $P$ consisting of orientation-preserving ones, and $Z_2$ either the subgroup of $H(R)$ or of $H(S)$ consisting of the identity map and the reflexion.

**Lemma 2.** $P^*$ and $H^*(S - \{1\})$ are isomorphic as topological groups by the natural map. And so $P^* = l_1$.

**Proof.** A modification of the proof of Lemma 1 ([4]) ensures that $P^*$ is isomorphic to $H^*(S - \{1\})$ as topological groups. Since $S - \{1\}$ is homeomorphic to $R$, $H^*(S - \{1\}) = H'(R)$. Hence $P^* = l_1$ by Lemma 1.

**Theorem.** $H(S) = T \cdot P^* \cdot Z_2 = T \times P^* \times Z_2$ and $(T, P^*) = (S, l_1)$.

**Proof.** Both $T$ and $P$ are closed subgroups and $H(S) = TP$, $T \cap P = \{1\}$. Whereas $H(S)$ is neither a direct product nor a semidirect product of $T$ and $P$, the correspondence of $u \in H(S)$ to $(t_u, t_{u^{-1}}^{-1} \cdot u) \in T \times P$ ($t_a$ : the multiplication by $a$ in $S$) gives a homeomorphism between $H(S)$ and the product space $T \times P$ — this owes to a remark of Keesling ([6], p. 15). The space $T$ is homeomorphic to $S$. Since $P^*$ is an open and closed subgroup of $P$, the space $P$ is homeomorphic to $l_2 \times Z_2$ by Lemma 2. Consequently $H(S) \approx S \times l_2 \times Z_2$.

**Remark.** Another proof that $P = l_2 \times Z_2$ is obtained by the fact : let $X$ be a locally connected, locally compact Hausdorff space, $X^*$ the compactification of $X$ by adding a point $x_\infty$ to $X$, and $H(X^*, x_\infty)$ the subspace of $H(X^*)$ consisting of the mappings that leave the point $x_\infty$ fixed, then $H(X^*, x_\infty) \approx H(X)$ — this owes to Theorem 2 of [2] and Theorems 1, 3, and 4 of [3].

References