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On the Multiplicative Cousin Problem for
Strictly Pseudoconvex Domains

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Abstract

Let D be a strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^4 \)-boundary. Let the multiplicative presheaf \( N^p_\Omega, 1 < p < \infty \), on \( D \) be defined by
\[
U \rightarrow \{ f \in \mathcal{F} (U \cap D, 0^*) : \text{both } (\log^+ |f|)^p \text{ and } (\log^- |f|)^p \text{ have harmonic majorants on } U \cap D \}.
\]

The purpose of this paper is to show that the multiplicative Cousin problems for the presheaf \( N^p_\Omega \) are solvable.

1. Introduction. Let D be a strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^4 \)-boundary. Let \( \mathcal{O} (0^*) \) be the sheaf of germs of holomorphic (nonzero holomorphic) functions on \( \mathbb{C}^n \). Let the multiplicative presheaf \( N^p_\Omega, 1 < p < \infty \), be defined by
\[
U \rightarrow \{ f \in \mathcal{F} (U \cap D, 0^*) : \text{both } (\log^+ |f|)^p \text{ and } (\log^- |f|)^p \text{ have harmonic majorants on } U \cap D \}.
\]

We denote the associated sheaf by \( N^p_\Omega \). The author [1] has proved that the multiplicative Cousin problems for the presheaf \( N^p_\Omega \) are solvable in case D is a strictly convex domain in \( \mathbb{C}^n \) with \( C^2 \)-boundary. In the present paper we prove that the multiplicative Cousin problems for the presheaf \( N^p_\Omega \) are solvable in case D is a strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^4 \)-boundary.

2. \( \mathcal{L}^p \)-functions. Let D be a domain in \( \mathbb{C}^n \) with \( C^4 \)-boundary, i.e., \( D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \), where \( \rho \) is a \( C^4 \)-function in \( \mathbb{C}^n \) and \( d\rho \neq 0 \) on \( \partial D \). Let \( \mathcal{L}^p (\partial D), 1 \leq p < \infty \), be the space of all measurable functions on \( D \) which satisfy
sup_{\varepsilon < 0} \int_{\partial D_\varepsilon} |f(z)|^p \, ds(z) < \infty,

where $D_\varepsilon = \{ z : \rho(z) < \varepsilon \}$ and $ds(z)$ is the element of surface area on $\partial D_\varepsilon$.

Let $L^\infty(\partial D)$ be the space of all bounded measurable functions on $\partial D$. Let $L^p_{(0,1)}(\partial D)$ and $C^k(\partial D)$ be the space of all $(0,1)$-forms on $D$ whose coefficients belong to $L^p(\partial D)$ and $C^k(D)$, respectively. By E. M. Stein [7], the following (1) and (2) are equivalent for harmonic functions $f$ in $D$ and $1 \leq p < \infty$;

(1) \[ \sup_{\varepsilon < 0} \left( \int_{\partial D_\varepsilon} |f(z)|^p \, ds(z) \right)^{1/p} \leq \infty \]

(2) $|f(z)|^p$ has a harmonic majorant.

**Lemma 1.** Let $1 \leq p < \infty$. Let $f$ be a measurable function in $D$ and let $U=\{U_i\}$ be an open covering of $D$ such that for any $i$ and any domain $W \subset U_i$ with $C^1$-boundary, $f \in L^p(\partial W)$. Then $f \in L^p(\partial D)$.

**Proof.** Let $M=\max \{ x_{zn} : \text{for some } z \in D, z=(z_1, \ldots, z_n), \ x_{zn}=\Im z_n \}$, and let $m$ be the corresponding minimum. Let $\varepsilon_0$ satisfy $0 < \varepsilon_0 < (M-m)/12$. Let $\eta_i, i=1,2$, be real valued functions of a real variable such that

1. $\eta_i$ is of class $C^1$,
2. $\eta_i(t) = 0$ if $t \leq \frac{1}{2}(M+m) - \frac{5}{2} \varepsilon_0$
3. $\eta_i(t) \geq 2$ if $t \geq \frac{1}{2}(M+m) + 3 \varepsilon_0$
4. $\eta_i(t) > 0$ if $t > \frac{1}{2}(M+m) + \frac{5}{2} \varepsilon_0$

Let $D_1 = \{ z : \rho(z) + \eta_1(x_{zn}) < 0 \}, \ D_2 = \{ z : \rho(z) + \eta_2(x_{zn}) < 0 \}$. Suppose $f \in L^p(\partial D_1).$ Then $f \in L^p(\partial D_2)$ or $f \in L^p(\partial D_2)$. Say $f \in L^p(\partial D_1)$. The $x_{zn}$-width of $D_1$, i.e., the number $\max |x'_{zn} - x''_{zn}|$, the maximum taken over all pairs of points $z', z''$ in $D_1$, is not more than three fourths of the $x_{zn}$-width of $D$. We now treat $D_1$ as we treated $D$, using the coordinate $x_{zn-1}$ rather than $x_{zn}$, and we find a smaller set $D_{11} \subset D_1$ for which $f \in L^p(\partial D_{11})$. We iterate this process, running cyclically through the real coordinate of $C^n$, and we obtain a shrinking sequence of sets $\{ D_{ij} \}$ for which $f \in L^p(\partial D_{ij})$. One of the domains $\{ D_{ij} \}$ will fall inside some $U_i$, which is a contradiction. Therefore lemma 1 is proved.
Let \( D \) be a strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^k \)-boundary, i.e., 
\[ D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \]
\( \rho \) is a \( C^k \)-strictly plurisubharmonic function in \( \mathbb{D} \), \( \mathbb{D} = \overline{D} \)
and \( d\rho \neq 0 \) on \( \partial D \).

Let
\[ F(\xi, z) = \sum_{i=1}^{n} \frac{\partial \rho}{\partial \xi_i}(\xi)(z_i - \xi_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial \xi_i \partial \overline{\xi}_j}(\xi)(z_i - \xi_i)(z_j - \xi_j). \]

By the results of E. Ramirez [5] and G. M. Henkin [4], (after shrinking \( \mathbb{D} \)) we obtain a \( C^{k-\varepsilon} \)-function \( \Phi(\xi, z) \) on \( \mathbb{D} \times \mathbb{D} \) holomorphic in \( z \) with the following properties:

1. \( \Phi(\xi, z) \neq 0 \) for \( \xi, z \in \mathbb{D} \) with \( \rho(\xi) > \rho(z) \),
2. for \( \xi^o \in \partial D \) there exist an open neighborhood \( U \) of \( \xi^o \) in \( \mathbb{D} \) and a nowhere vanishing \( C^{k-\varepsilon} \)-function \( H(\xi, z) \) on \( U \times U \) holomorphic in \( z \) such that \( \Phi(\xi, z) = H(\xi, z) F(\xi, z) \) on \( U \times U \),
3. there exist \( C^{k-\varepsilon} \)-functions \( P_i(\xi, z) \) on \( \mathbb{D} \times \mathbb{D} \) holomorphic in \( z \) such that
\[ \Phi(\xi, z) = \sum_{i=1}^{n} (z_i - \xi_i) P_i(\xi, z). \]

Let
\[ \omega'(Z_1, \ldots, Z_n) = \sum_{i=1}^{n} (-1)^{i-1} Z_i \wedge (\wedge dZ_i), \]
where \( Z_1, \ldots, Z_n \) are intermediates. Define
\[ K'(\xi, z, \lambda) = c_n \omega'(\lambda \frac{Z_1 - \xi_1}{|Z_1 - \xi_1|^2}, \ldots, \frac{Z_n - \xi_n}{|Z_n - \xi_n|^2}) + (1 - \lambda) \frac{P_i(\xi, z)}{\Phi(\xi, z)} \wedge \omega(\xi) \]
and
\[ L(\xi, z) = c_n \omega'(\frac{Z_1 - \xi_1}{|Z_1 - \xi_1|^2}, \ldots, \frac{Z_n - \xi_n}{|Z_n - \xi_n|^2}) + (1 - \lambda) \frac{P_n(\xi, z)}{\Phi(\xi, z)} \wedge \omega(\xi), \]
where \( \omega(\xi) = d\xi_1 \wedge \ldots \wedge d\xi_n \). By integrating over \( \lambda \in [0, 1] \) terms of \( K'(\xi, z, \lambda) \) containing \( d\lambda \), we obtain a form \( K(\xi, z) \). For a \( \partial \)-closed \( C^\infty(0 1) \)-form defined on an open neighborhood of \( \mathbb{D} \), we define

\[ (1) \quad T_D(f) = \int_{\xi \in \partial D} f(\xi) \wedge K(\xi, z) - \int_{\xi \in \partial D} f(\xi) \wedge L(\xi, z). \]

Then \( \partial T_D(f) = f \) on \( D \). Let \( \widetilde{f}(\xi), \widetilde{K}(\xi, z) \) and \( \widetilde{L}(\xi, z) \) be one of the coefficients of \( f(\xi) \), \( K(\xi, z) \) and \( L(\xi, z) \), respectively.

Let \( \varepsilon < 0 \) be sufficiently near zero. Then
\[ \int_{D/\partial D} \widetilde{f}(\xi) \widetilde{L}(\xi, z) d\mu(\xi) = \int_{D} \frac{1}{\varepsilon} dr \int_{\partial D} \widetilde{f}(\xi) \widetilde{L}(\xi, z) ds(\xi). \]

Since \( \widetilde{L}(\xi, z) \) and \( \widetilde{K}(\xi, z) \) are bounded when \( z \in D \) is fixed, the integrals in (1) have meaning for \( f \in L^1((\varepsilon, 1)) (\partial D) \cap C^\infty((\varepsilon, 1))(D) \).

Let \( f \in L^1((\varepsilon, 1)) (\partial D) \cap C^\infty((\varepsilon, 1))(D) \). Let
\[ T_D(f) = \int_{\xi \in \partial D} f(\xi) K(\xi, z) d\lambda_r(z) - \int_{\xi \in D} f(\xi) L(\xi, z). \]

Then \( \overline{\partial} T_D(f) = f \) on \( D \). Therefore \( \overline{\partial} T_D(f) = f \) on \( D \).

**Lemma 2.** Let \( f \in L^p(\partial D) \cap C^\infty(\partial D) \), \( 1 \leq p \leq \infty \), and \( U \) be a subdomain of \( D \) with \( C^1 \)-boundary. Then there exists a function \( v \in L^p(\partial U) \cap C^\infty \) (\( D \)) such that \( \overline{\partial} v = f \) on \( D \).

**Proof.** Let \( T^{(\tau)} : L^1(\partial D) \to C^1(\rho_i = r) \) be defined by \( T^{(\tau)}(f) = T_D(f)/((\rho_i = r)) \), where \( \rho_i \) is a defining function of \( U \). We now prove:

(a) \( \| T^{(\tau)}(f) \| L^\infty(\rho_i = r) \leq C \| f \| L^\infty(\partial D) \)

(b) \( \| T^{(\tau)}(f) \| L^1(\rho_i = r) \leq C \| f \| L^1(\partial D) \).

(a) is a result of H. Grauert and I. Lieb [3]. We have

\[
\int_{\partial U_r} |T_D(f)(z)| d\lambda_r(z) \leq \int_{\partial U_r} \int_{\xi \in \partial D} f(\xi) K(\xi, z) d\lambda_r(z)
+ \int_{\partial U_r} \int_{\xi \in D} f(\xi) L(\xi, z) d\lambda_r(z),
\]

where \( d\lambda_r \) is Lebesgue measure on \( \partial U_r \). By Fubini's theorem and the fact that

\[
\int_{\partial U_r} \tilde{K}(\xi, z) d\lambda_r(z) \quad \text{and} \quad \int_{\partial U_r} \tilde{L}(\xi, z) d\lambda_r(z)
\]

are bounded uniformly in \( r \) and \( \xi \), (see R. M. Range and Y. T. siu [6]), we have

\[
\| T^{(\tau)}(f) \| L^1(\rho_i = r) \leq C \| f \| L^1(\partial D). \]

By Riesz-Thorin theorem, (a) and (b) imply that

\[
\| T^{(\tau)}(f) \| L^p(\rho_i = r) \leq C \| f \| L^p(\partial D)
\]

for \( 1 \leq p \leq \infty \). Therefore \( T_D(f) \in L^p(\partial U) \). This completes the proof.

It is well known (H. Grauert [2]) that for \( \delta > 0 \), \( \delta \) sufficiently near zero, the restriction map \( \Gamma(D_\delta, O) \to \Gamma(D, O) \) induces isomorphisms

\[
r_\delta : H^q(D_\delta, O) \sim H^q(D, O)
\]

and

\[
r^*_\delta : H^q(D_\delta, O^*) \sim H^q(D, O^*)
\]

for \( q \geq 1 \). Now we prove the following:

**Theorem 1.** Let \( D \) be a strictly pseudoconvex domain in \( C^\infty \) with \( C^1 \)-boundary. The natural homomorphism

\[
i^* : H^q(\overline{D}, N^p) \to H^q(D, O^*)
\]

is an isomorphism for \( 1 \leq p \leq \infty \).
On the multiplicative Cousin problem for strictly pseudoconvex domains

**PROOF.** Since the isomorphism $H^1(D, O^*) \to H^1(D, O^*)$ factors through $H^1(D, N^*)$, $i^*$ is surjective. To prove injectivity, let $c \in H^1(D, N^*)$ with $i^*c = 0$. It follows that for a suitable locally finite covering $U = \{U_i\}$ of $D$, $c$ is represented by $\{c_{i,is} \} = i^* \{b_{i,i} \} \in Z^1(U, N^*)$, where $\{b_{i,i} \} \in C^0(U \cap D, O^*)$. We may assume that each set $U_{i,io} = U_{i,io} \cap D$ is simply connected and has $C^1$-boundary, and hence after choosing a fixed branch for the logarithm, $\{b_{i,i} \}$ is a well defined cochain in $C^0(U \cap D, O)$.

Let $\{a_{i,is} \} = \delta \{\log b_{i,i} \} \in Z^1(U \cap D, O)$. Since $\exp a_{i,is} = \exp c_{i,is}$, $(\log^* \exp a_{i,is})^p$ and $(\log^* \exp a_{i,is})^p$ have both harmonic majorants on $U_{i,io}$. Let $\{\phi_i \}$ be a $C^\infty$ partition of unity subordinate to the covering $U$. Let $c_{i,io} = \sum \rho_i a_{i,io}$ and let $c^i = i^* c$. Therefore $c^i$ is a $\bar{\partial}$-closed $C^\infty(0, 1)$-form in $D$. If we write $c^i$ in the form $c^i = \sum_{j=1}^n c_j^i dz_j$, then $c_j^i = \sum_{i=1}^{r_j} \frac{\partial \phi_i}{\partial z_j} a_{i,io}$ on $U_{i,io}$.

Let $W = \{z : \rho(z) < 0\}$ be a subdomain of $U_{i,io}$ with $C^2$-boundary and $A_1, A_2, A_3, A_4$ be constants depending only on $p$ and $W$.

Since $|c_j^i|^p \leq A_1 \left\{ \sum_{i=1}^{r_j} \frac{\partial \phi_i}{\partial z_j} |\Re a_{i,io}|^p + \sum_{i=1}^{r_j} \frac{\partial \phi_i}{\partial z_j} |\Im a_{i,io}|^p \right\}$.

We have

$$\int_{\partial W} |c_j^i|^p d\lambda \leq A_1 \int_{\partial W} \sum_{i=1}^{r_j} \frac{\partial \phi_i}{\partial z_j} |\Re a_{i,io}|^p d\lambda + \int_{\partial W} \sum_{i=1}^{r_j} \frac{\partial \phi_i}{\partial z_j} |\Im a_{i,io}|^p d\lambda.$$

Let $\sup \phi_i = K_i$. Let $U_1, U_2$ be open sets such that $W \cap K_i \subset U_1 \setminus (U_1 \cup U_i)$. Let $X_i(z)$ be a real valued non-negative function such that $X_i(z) = 0$ on $U_1$, $X_i(z) = N+1$ on $C^n - U_1$. Where $N = \sup |\rho(z)|$. Let $D_i = \{z : \rho(z) + X_i(z) < 0\}$. Then $D_i$ is a domain with $C^1$-boundary having following properties : $D_i \subset W \cap U_1$, $\partial W \cap K_i \subset \partial (D_i)$, for $\varepsilon < 0$ sufficiently near zero. Hence we have

$$\int_{\partial W} |c_j^i|^p d\lambda \leq A_1 \left( \int_{D_i} |\frac{\partial \phi_i}{\partial z_j}|^p |\Re a_{i,io}|^p d\lambda + \sum_i \int_{\partial (D_i)} |\frac{\partial \phi_i}{\partial z_j}|^p |\Im a_{i,io}|^p d\lambda.\right.$$

By the Riesz type theorem obtained by E. L. Stout [8], we have

$$\int_{\partial (D_i)} |\Im a_{i,io}|^p d\lambda \leq A_2 \left( \int_{\partial (D_i)} |\Re a_{i,io}|^p d\lambda + A_3 \right).$$

Since $|\Re a_{i,io}|^p$ has a harmonic majorant in $D_i$, $c_j^i \in L^p(\partial W)$. By lemma 1, $c_j^i \in L^p(\partial D)$. By lemma 2, there exists a function $v \in L^p(\partial U_{i,io} \cap C^\infty(D))$ such that $\bar{\partial}v = f$ on $D$. Let $b = c^i - v$. Then $\partial b = \bar{\partial} c^i = a$, $\bar{\partial} b = \bar{\partial} c^i - \bar{\partial} v = 0$. Therefore $b \in C^\infty(U \cap D, O)$ and
\[
\int_{\partial(U_{io})_e} |\text{Re } b_{io}|^p \, d\lambda \leq A, \quad (\int_{\partial(U_{io})_e} |\text{Re } c_{io}|^p \, d\lambda + \int_{\partial(U_{io})} |v|^p \, d\lambda).
\]
Therefore \(\text{Re } b_{io} \in L^p(\partial U_{io})\). Hence \(\text{Re } b_{io} \|^p\) has a harmonic majorant in \(U_{io}\).
This implies that \(\{\exp b_{io}\} \in C^p(U \cap D, N^p)\). Since \(\partial (\exp b_{io}) = \{c_{io}\}\), this completes the proof of theorem 1.

The proof of theorem 1 implies:

**Theorem 2.** Let \(D\) be a strictly pseudoconvex domain in \(\mathbb{C}^n\) with \(C^1\)-boundary and suppose that \(H^1(D,O^p) = 0\). Let \(U\) be a finite covering of \(\overline{D}\). Then \(H^1(U, N^p) = 0\) for \(1 < p < \infty\), i.e., the multiplicative Cousin problems for \(N^p\) are solvable.

**References**


