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<thead>
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<th>項目</th>
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<tbody>
<tr>
<td>タイトル</td>
<td>ラングリ大学教育学部自然科学研究報告 1983年</td>
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<td>作者</td>
<td>長崎大学教育学部自然科学研究報告 1983年</td>
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<td>キーワード</td>
<td>長崎大学教育学部自然科学研究報告 1983年</td>
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<td>警戒</td>
<td>長崎大学教育学部自然科学研究報告 1983年</td>
</tr>
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Bundle Structure of the Homeomorphism Groups of Locally Compact Homogeneous Spaces

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Abstract

The space $\mathcal{H}(X)$ of homeomorphisms on a locally compact homogeneous space $X$ with a local cross-section is a bundle space over $X$. If $X$ is separable metrizable and admits a nontrivial flow in addition, then $\mathcal{H}(X)$ is an $l_1$-manifold if and only if $X$ is an ANR and $\mathcal{H}(X, a)$ is an $l_1$-manifold, where $\mathcal{H}(X, a)$ is the subspace of $\mathcal{H}(X)$ consisting of all those which leave a point $a$ of $X$ fixed. If $X$ is a locally connected, compact metrizable homogeneous space that is an ANR and admits a local cross-section and a nontrivial flow, then $\mathcal{H}(X)$ is an $l_1$-manifold if and only if $\mathcal{H}(X-a)$ is an $l_1$-manifold, where $\mathcal{H}(X-a)$ is the space of homeomorphisms on $X-a (a \in X)$.

Introduction

McCarty [8] has shown that for a locally connected, locally compact Hausdorff homogeneous space $X$ with a local cross-section, its full homeomorphism group $\mathcal{H}(X)$ with compact-open topology is a principal fiber bundle over $X$, and in particular if the set $X$ is a locally connected, locally compact Hausdorff topological group then $\mathcal{H}(X)$ is a product bundle. And noting the existence of a natural exact homotopy sequence he studied homeotopy groups of several manifolds. On the other hand Keesling ([7], p. 15) has remarked that if $X$ is a locally compact Hausdorff topological group then $\mathcal{H}(X)$ is homeomorphic to the product space $X \times \mathcal{H}(X, e)$ where $e$ is the identity of $X$.

We consider first whether the McCarty's conclusion holds or not without the assumption "local connectedness". The answer is given in §2. In §1 we show that $\mathcal{H}(X)$ is a bundle space over $X$ without the assumption "local connectedness". The same conclusion as this has been obtained in [5] already, and
yet here we try to generalize its premise and to improve the proof. The result
not only contains the Keesling’s remark as a special case but also yields
the natural exact homotopy sequence as in [8]. Next we treat applications of our
Theorem 1 in §3. There our concern now is mainly in several local connec-
tivities of \( \mathcal{H}(X) \), and particularly in local \( l_2 \) property. Our main results are
Theorems 3 and 4. These are slight generalizations of Theorems 2 and 3 in
[5] respectively.

Notations

\( \mathcal{H}(\ast) \) : The group of all homeomorphisms on a topological space \( \ast \), endowed
with the compact-open topology — only in §2 another topology is con-
sidered also.

\( \mathcal{H}(\ast, a) \) : The subspace of \( \mathcal{H}(\ast) \), consisting of all those which leave a point
\( a \) fixed \( (a \in \ast) \).

\( X = G/H \) : The left coset space of a Hausdorff topological group \( G \) by a closed
subgroup \( H \) — in the paper we call such a space a homogeneous space.

\( \pi : \) The natural projection of \( G \) onto \( X \).

These notations will keep these meanings throughout the paper.

1. Bundle structure of \( \mathcal{H}(X) \).

We consider a bundle structure of \( \mathcal{H}(X) \) after clarifying two concepts used
in Theorem 1.

Let \( p \) be a continuous map of a space \( E \) into another space \( B \). We say that
the space \( B \) has a local cross-section \( f \) (at a point \( b \) in \( B \)) relative to \( p \), if \( f \)
is a continuous map from a neighborhood \( U \) of \( b \) into \( E \) such that \( pf(u) = u \) for each
\( u \in U \).

Let \( p, E \), and \( B \) be the same as above. The space \( E \) is called a bundle
space over the base space \( B \) relative to the projection \( p \) if there exists a space \( D \)
such that, for each \( b \in B \), there is an open neighborhood \( V \) of \( b \) in \( B \) together
with a homeomorphism

\[ \phi_V : V \times D \to p^{-1}(V) \]

of \( V \times D \) onto \( p^{-1}(V) \) satisfying the condition

\[ p\phi_V(v, d) = v \quad (v \in V, \quad d \in D) \]

This terminology is the same as in [3].

Theorem 1. Let \( X = G/H \) be a homogeneous space, \( a \) an arbitrary but fixed
point of \( X \), \( p \) the map of \( \mathcal{H}(X) \) to \( X \) defined by \( p(\psi) = \psi(a) \) \( (\psi \in \mathcal{H}(X)) \),
and \( \mathcal{H}^* = \mathcal{H}(X)/\mathcal{H}(X, a) \) the left coset space (with quotient topology) of
Bundle Structure of the Homeomorphism Groups of Locally Compact Homogeneous Spaces

$\mathcal{H}(X)$ by $\mathcal{H}(X, a)$. Then we have the following:

(a) The map $p$ is a continuous surjection. $\mathcal{H}(X) = L \circ \mathcal{H}(X, a)$ where $L$ is the group of all left translations in $X$. And $L \cap \mathcal{H}(X, a)$ consists of just one element if and only if $H$ coincides with the maximal normal subgroup of $G$ which is contained in $H$.

(b) Assume that $X$ has a local cross-section relative to the natural projection $\pi : G \to X$. Then

i) $X$ has a local cross-section relative to $p$,

ii) $X$ is homeomorphic to $\mathcal{H}^*$ in a natural way, and $p$ is a quotient map.

And so we can identify $X$ with $\mathcal{H}^*$.

(c) Assume that $X$ is locally compact and has a local cross-section relative to $\pi$. Then $\mathcal{H}(X)$ is a bundle space over the base space $X$ relative to the projection $p$.

Proof. It is easy to see (a). We give proofs for (b) and (c).

(b), i) : For each element $g$ of $G$, let $\omega(g)$ be the left translation in $X$ by $g$. The map $\omega : g \to \omega(g)$ ($g \in G$) is a continuous (algebraic) homomorphism of $G$ into $\mathcal{H}(X)$. Now let $f$ be a local cross-section from a neighborhood $U$ of a point $x$ in $X$ into $G$. For any fixed point $g_0$ of $\pi^{-1}(a)$, let $q$ be the map of $U$ into $\mathcal{H}(X)$ defined by

$$q(u) = \omega(f(u) \cdot g_0^{-1}) \quad (u \in U).$$

Put $W = q(U)$. Then both maps $q : U \to W$ and $\rho(W) : W \to U$ are homeomorphisms and inverses each other. In particular $q$ is a local cross-section $U \to \mathcal{H}(X)$ relative to $p$.

(b) ii) : Let $\pi^*$ be the natural projection of $\mathcal{H}(X)$ onto $\mathcal{H}^*$, and put $r = p \circ \pi^{-1}$. $r$ is well-defined as a map $\mathcal{H}^* \to X$, and it is a continuous bijection. Now we will show that $p$ is a quotient map. Let $O$ be any nonempty subset of $X$ such that $p^{-1}(O)$ is open in $\mathcal{H}(X)$. For any point $x$ of $O$, take a local cross-section $f$ at $x$ relative to $\pi : G \to X$, which is defined on a neighborhood $U$ of $x$ in $X$. For such $f$ and $U$, take the local cross-section $q : U \to \mathcal{H}(X)$ and the set $W$ as in the proof of (b), i). Let $w = q(x)$ and take a neighborhood $V$ of $w$ in $\mathcal{H}(X)$ such that $V \supset p^{-1}(O)$. Then it is easy to see that $p(V \cap W)$ is a neighborhood of $x$ in $X$, which is contained in $O$. Thus $p$ is a quotient map. Therefore the map $r$ is a homeomorphism of $\mathcal{H}^*$ onto $X$.

(c) : For any point $x$ of $X$, take an open neighborhood $U$ of $x$ and the set $W$ as in the proof of (b), i). Let $\Phi$ be the map of the product space $W \times \mathcal{H}(X, a)$ onto $W \times \mathcal{H}(X, a)$ ($= p^{-1}(U)$) defined by $\Phi(w, \phi) = w \cdot \phi$. It is easy to see that $\Phi$ is a bijection. Since $X$ is locally compact Hausdorff, $\Phi$ is continuous. To show the continuity of $\Phi^{-1}$, in the following let $w$ and $\phi$ be any element of $W$ and $\mathcal{H}(X, a)$ respectively. The map that carries $w \cdot \phi$ to $w$ is continuous, for $w = (q \circ p)(w \cdot \phi)$. The map that carries $w$ to $w^{-1}$ is continuous, for
\[ w^{-1} = \omega \left( [fp(w) \cdot g^{-1}]^{-1} \right). \]
Hence the map that carries \( w \cdot \phi \) to \( \phi \) is continuous, for
\[ \phi = w^{-1} \cdot (w \cdot \phi). \]
Consequently \( \Phi^{-1} \) is continuous. Hence \( \Phi \) is a homeomorphism. From the fact we can show that \( \mathcal{H}(X) \) is a bundle space over the base space \( X \) relative to the projection \( p \).

**Corollary 1** (J. Keesling [7]). If \( X \) is a locally compact Hausdorff topological group, then \( \mathcal{L} \) is isomorphic to \( X \) as topological groups and \( \mathcal{H}(X) \) is homeomorphic to the product space \( X \times \mathcal{H}(X, a) \).

**Proof.** In the case we can consider in the proof of Theorem 1 that
\[ X = G / \{ e \} = U = W = \mathcal{L} = \omega (G), \]
where \( e \) is the identity of \( G \) and \( = \) means "is homeomorphic to". Then \( W \) is a topological group, and the map \( \Phi \) gives a homeomorphism of \( W \times \mathcal{H}(X, a) \) onto \( \mathcal{H}(X) \).

### 2. Fiber bundle structure of \( \mathcal{H}(X) \).

We use the following notations \( \tau_e \) and \( \tau_g \) only in this section.

- \( \tau_e \): The compact-open topology on \( \mathcal{H}(X) \).
- \( \tau_g \): The g-topology, named by R. Arens [1], on \( \mathcal{H}(X) \) as follows. If \( A \) and \( B \) are closed and open subsets, respectively, of \( X \), and either \( A \) or the complement of \( B \) in \( X \) is compact, then let \( \{ A, B \} \) be the set of \( \phi \in \mathcal{H}(X) \) such that \( \phi(A) \subseteq B \). The totality of sets \( \{ A, B \} \) are taken as a subbase for the g-topology.

**Theorem 2.** Let the topology \( \tau_g \) be given on \( \mathcal{H}(X) \) in place of \( \tau_e \). Then Theorem 1 holds, and moreover, under the assumption of (c) in Theorem 1, \( \mathcal{H}(X) \) is a principal fiber bundle over \( X \) with fiber and group \( \mathcal{H}(X, a) \).

**Proof.** For the latter assertion, noting the fact that \( \mathcal{H}(X) \) with \( \tau_g \) becomes a topological group ([1], Th. 3) and (b) in Theorem 1, standard application of the bundle structure theorem (cf. [9]) yields the conclusion.

**Remark 1.** Under the topology \( \tau_e \) the latter assertion in Theorem 2 is not true in general. In fact Braconnier [2] gave an example of a totally disconnected, non-compact, locally compact, abelian topological group \( X \) whose automorphism group \( \mathcal{A} \) is not a topological group under the topology \( \tau_e \). Since \( \mathcal{A} \subseteq \mathcal{H}(X, e) \) where \( e \) is the identity of \( X \), \( \mathcal{H}(X, e) \) is not a topological group under \( \tau_e \).

**Remark 2.** The topology \( \tau_g \) is finer than the topology \( \tau_e \) in general, and if \( X \) is a locally compact homogeneous space then \( \tau_g \) is the coarsest topology for \( \mathcal{H}(X, a) \) to become a topological group. Thus for the latter assertion in Theorem 2, \( \tau_g \) is the most desirable topology on \( \mathcal{H}(X) \).
COROLLARY 2. Let $X$ be a homogeneous space with a local cross-section relative to $\pi$. If i) $X$ is locally connected and locally compact, or ii) $X$ is compact, then $\mathcal{H}(X)$ with topology $\tau_c$ is a principal fiber bundle over $X$ with fiber and group $\mathcal{H}(X, a)$.

Proof. For the case i), by Theorems 3 and 4 of [1] and the fact $\tau_c \subseteq \tau_g$, $\tau_g$ coincides with $\tau_c$ on $\mathcal{H}(X)$. For the case ii) it is seen at once that $\tau_g$ coincides with $\tau_c$. Hence Theorem 2 yields the conclusion.

In [8] the case i) above was used.

3. Some applications.

Hereafter it is assumed again that the compact-open topology is endowed on every set of homeomorphisms.

A. Homotopy property.

Here we follow the terminology of Hu [3]. As corollaries to Theorem 1 we have the following Corollaries 3 and 4 below.

COROLLARY 3. If $X$ is a locally compact homogeneous space with a local cross-section relative to $\pi$, then $\mathcal{H}(X)$ is a fiber space over $X$ relative to $\pi$.

Proof. From (c) in Theorem I and Theorem 4.1 in [3] on p. 65.

Thus the powerful machinery of homotopy theory of fiber spaces is available on such $\{ \mathcal{H}(X), X, \pi \}$.

B. Local property.

DEFINITION. A topological property $P$ is called a finite product local property abbreviated FPL property, if i) a topological space has the property $P$ then every open subspace has the property $P$, and ii) a product space $A \times B$ has the property $P$ if and only if both spaces $A$ and $B$ have the property $P$.

REMARK 3. Among those local properties of $\mathcal{H}(M)$ studied for spaces $M$, for example, the following are FPL properties: locally connected, locally arcwise connected, LC, LCco, locally contractible, ANR. Note that each of these is a kind of property concerning local connectivity. On the other hand though local compactness is also a FPL property, it can be considered on $\mathcal{H}(M)$ only for non-standard spaces $M$. Because for a metric space $M$ if $\mathcal{H}(M)$ is locally compact then $\mathcal{H}(M)$ is zero-dimensional (cf. [6]), while for a Hausdorff space $M$ at least one point of which is locally Euclidean, $\mathcal{H}(M)$ is infinite-dimensional (cf. [4], Th. 1.5).

COROLLARY 4. Let $X$ be a locally compact homogeneous space with a local cross-section relative to $\pi$. Then $\mathcal{H}(X)$ has a FPL property if and only if both $X$ and $\mathcal{H}(X, a)$ have the FPL property.

Proof. From (c) in Theorem 1, $\mathcal{H}(X)$ is locally homeomorphic to the product space $X \times \mathcal{H}(X, a)$. 
DEFINITION. A space is called an \( l_1 \)-manifold if it is separable metrizable space and is locally homeomorphic to \( l_1 \), i.e. the Hilbert space of square-summable sequences.

For about thirteen years now it has been conjectured that \( \mathcal{H}(M) \) is an \( l_1 \)-manifold for a compact metric \( n \)-manifold \( M \), and no affirmative answer has been obtained except the cases where \( n (=\dim M) \) is 1, 2, or \( \infty \) as far as we know.

THEOREM 3. Let \( X \) be a separable metrizable locally compact homogeneous space. Assume that \( X \) has a local cross-section relative to \( \pi \), and admits a nontrivial flow. Then \( \mathcal{H}(X) \) is an \( l_1 \)-manifold if and only if \( X \) is an ANR and \( \mathcal{H}(X, a) \) is an \( l_1 \)-manifold.

(Here "ANR" means absolute neighborhood retract for the class of all metrizable spaces.)

Proof. The same proof for Th. 2 in [5] is valid, though our assumption on local compactness of \( X \) is slightly generalized from Th. 2 in [5]. It is essentially an application of a theorem of Toruńczyk [10] to Corollary 4.

REMARK 4. As partial results of Corollary 4 and Theorem 3, for a locally compact homogeneous space \( X \) with a local cross-section, we get a criterion which local property must \( X \) have when we expect \( \mathcal{H}(X) \) to have the local property as stated in this section.

C. Relations between homeomorphism groups of a space and its punctured space.

The following results are slight generalizations from those in [5].

THEOREM 4. Let \( X \) be a locally connected, compact metrizable homogeneous space. Assume that \( X \) is an ANR and has a local cross-section relative to \( \pi \), and admits a nontrivial flow. Then \( \mathcal{H}(X) \) is an \( l_1 \)-manifold if and only if \( \mathcal{H}(X-a) \) is an \( l_1 \)-manifold.

COROLLARY 5. If \( X \) is a compact (positive dimensional) locally Euclidean homogeneous space with a local cross-section, then the same conclusion as in Theorem 4 holds.

As an application of Corollary 5, for several non-compact manifolds \( M \), we know that \( \mathcal{H}(M) \) are \( l_1 \)-manifolds (see [5]).

References

Bundle Structure of the Homeomorphism Groups of Locally Compact Homogeneous Spaces


